$\mathbb{N} \longrightarrow \mathbb{Z} \rightsquigarrow \mathbb{Q}$

$\{0, 1, 2, 3, \dots\}$ $\quad\quad$ $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$+, \cdot$ $\quad\quad$ $+, \cdot, -$ $\quad\quad$ Es. $(12 - 20 = -8)$

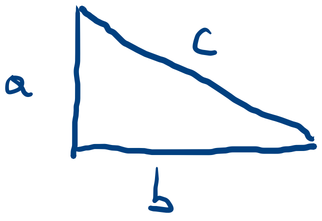
$\frac{m}{n}, \quad n \neq 0$

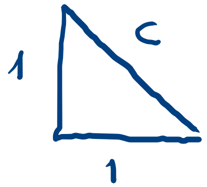
$\frac{m}{n} = \frac{mk}{nk}, \quad k \neq 0, \ n \neq 0$

$\mathbb{Q}: +, \cdot, -, :$

T. di Pitagora

$$\boxed{a^2 + b^2 = c^2}$$





$1 + 1 = c^2$

$$\boxed{c^2 = 2}$$

Esiste $c \in \mathbb{Q}$, $c>0$, t.c. $c^2 = 2$?

$\boxed{\text{NO}}$

$\boxed{\mathbb{N} \subset \mathbb{Z}}$

Supponiamo, per assurdo, che esiste $c \in \mathbb{Q}$ t.c. $c^2 = 2$

$c = \frac{m}{n}$, $m, n \in \mathbb{N}$, $n \neq 0$, $(m,n) = 1$
("MCD")

$\frac{m^2}{n^2} = 2$; $\boxed{m^2 = 2n^2}$

$\Rightarrow 2 \mid m^2 \Rightarrow 2 \mid m$

$\Rightarrow m = 2k$

$4k^2 = 2n^2$

$2k^2 = n^2 \Rightarrow 2 \mid n^2$

$\Rightarrow 2 \mid n$ $\Rightarrow$ m, n non sono coprimi.

Contraddizione

$m \in \mathbb{Z}$ è pari o dispari

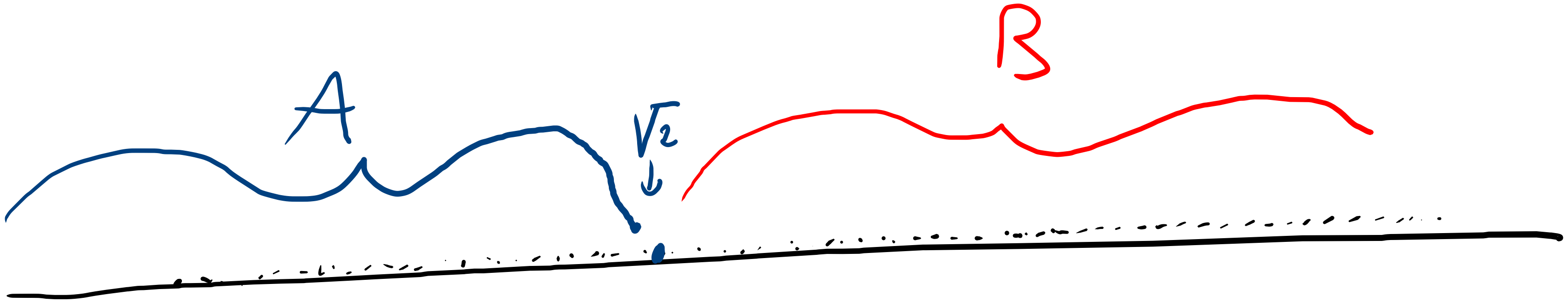
pari : $2 \mid m \iff m = 2k$

dispari : $m = 2k+1$

$(2k+1)^2 = 4k^2 + 4k + 1 =$

$= 2\underline{(2k^2+2k)} + 1$

$\mathbb{Q} \rightsquigarrow \mathbb{R}$



Sezioni di Dedekind di $\mathbb{Q}$

$A = \{ q \in \mathbb{Q} \mid q \leq 0 \text{ oppure } q > 0 \text{ e } q^2 < 2 \}$

$B = \{ q \in \mathbb{Q} \mid q > 0 \text{ e } \underline{q^2 > 2} \}$

$\Rightarrow \sqrt{2} = (A, B)$

$A \cup B = \mathbb{Q}$

$A \neq \emptyset$

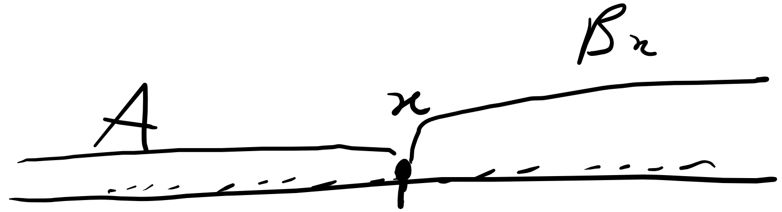
$B \neq \emptyset$

Def Una sez. di Dedekind è una coppia (A,B) di sottoinsiemi non vuoti di $\mathbb{Q}$ t.c.

1) $A \cup B = \mathbb{Q}$
2) $a \in A, b \in B \Rightarrow a < b$ $\quad (\Rightarrow A \cap B = \emptyset)$
3) A non ha massimo

Def $\mathbb{R} = \{ (A,B) \mid (A,B) \text{ è sez. di Ded. di } \mathbb{Q} \}$

OSS. $\exists \; j : \mathbb{Q} \longrightarrow \mathbb{R}$ iniettiva $\rightsquigarrow \mathbb{Q} \subset \mathbb{R}$



$$j(x) = (A_x, B_x)$$

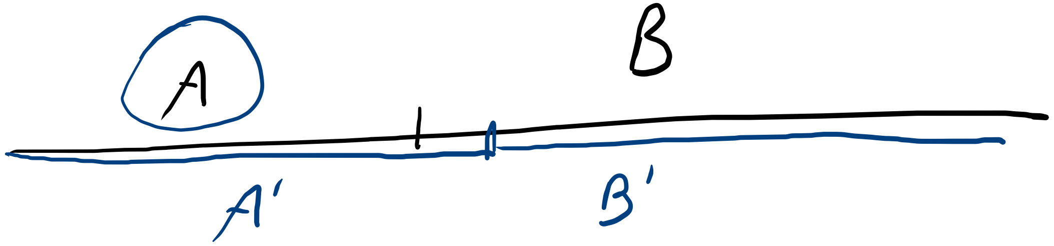
$$\begin{cases} A_x = \{ q \in \mathbb{Q} \mid q < x \} \\ B_x = \{ q \in \mathbb{Q} \mid q \geq x \} \end{cases}$$

$$(A,B) + (A',B') := (\underline{A + A'}, \underline{B + B'})$$

dove $\underline{A + A' := \{ a + a' \mid a \in A, a' \in A' \}}$, $\underline{B + B' := \{ b + b' \mid b \in B, b' \in B' \}}$

$$(A,B) \leq (A',B') \overset{\text{def}}{\Longleftrightarrow} A \subseteq A' \qquad \left(\Longleftrightarrow B' \subseteq B \right)$$

Esercizio



$$0 = (A_0, B_0)$$

(A,B) è positivo $\overset{\text{def}}{\Longleftrightarrow}$ $(A,B) > 0$

$$\left(0 < (A,B) \text{ e } 0 \neq (A,B) \right)$$

$$(A,B) \cdot (A',B') = (\underline{A \cdot A'}, \underline{B \cdot B'})$$

se uno dei due è ≥ 0

$$A \cdot A' = \{ a \cdot a' \mid a \in A, a' \in A' \}$$

$$\left(-(A,B) = (\underline{-B'}, -A') \right.$$

$B' = B$ se B non ha min

altrimenti $B' = B - \{\min B\}$

Es.

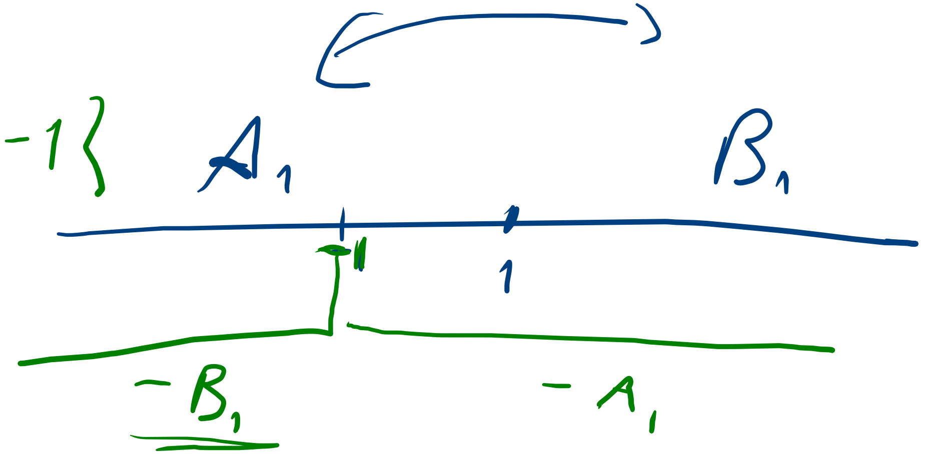
$1 = (A_1, B_1)$ $\qquad\qquad -(A_1, B_1)$

$A_1 = \{q \in \mathbb{Q} \mid q < 1\}$.

$B_1 = \{q \in \mathbb{Q} \mid q \geq 1\}$.

$-B_1 := \{-b \mid b \in B_1\} = \{q \leq -1\}$

$-A_1 := \{-a \mid a \in A_1\}$



$(-B_1, -A_1)$ NO

max $\to$ min

$-(A, B)$ è il numero reale t.c.

$(-(A,B)) + (A,B) = 0 = (A_0, B_0)$

CAMPO

$\mathbb{R}$: $+, -, \cdot, :$

$+, \cdot$ | commutative — $\begin{cases} a+b = b+a \\ ab = ba \end{cases}$ $\forall a,b,c \in \mathbb{R}$

associative — $\begin{cases} (ab)c = a(bc) \\ (a+b)+c = a+(b+c) \end{cases}$

$\cdot$ è distributivo rispetto a $+$

$$a(b+c) = ab + ac = (ab)+(ac)$$

$\forall a \in \mathbb{R}$ ha un opposto : $\exists\, -a \in \mathbb{R}$ t.c. $a+(-a)=0$

$0+a = a \quad \forall a$ $\quad 0$ elemento neutro per $+$

1 el. neutro per $\cdot$ $\quad a \cdot 1 = a \quad \forall a \in \mathbb{R}$

$\forall a \in \mathbb{R}-\{0\} \ \exists\, a' \in \mathbb{R}$ t.c. $a a' = 1$

$a' =: a^{-1}$

Esercizi

$(232, 70) = 2$

$232 = 70 \cdot 3 + 22$

$70 = 22 \cdot 3 + 4$

$22 = 4 \cdot 5 + 2$

$4 = 2 \cdot 2$

$54 = 27 \cdot 2 = 2 \cdot 3^3$

Divisori di 54

1, 2, 3, 6, 9, 18, 27, 54

$\mathbb{R}$, $\mathbb{Q}$ sono campi

$\mathbb{Z}$ non è un campo