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This condition is, as we should expect, the opposite of the condition (45.6) for Boltzmann statistics to be valid. The temperature defined by the relation $T_F \cong \varepsilon_F$ is called the *degeneracy temperature.*

A degenerate electron gas has the peculiar property that it increasingly approaches the ideal gas state as its density increases. This is easily seen as follows.

Let us consider a *plasma*, i.e. a gas consisting of electrons and a corresponding number of positively charged nuclei which balance the charge on the electrons; a gas composed of electrons alone would obviously be entirely unstable, but we have not mentioned the nuclei hitherto, because the assumption of ideal-gas properties means that the presence of the nuclei does not affect the thermodynamic quantities for the electron gas. The energy (per electron) of the Coulomb interaction between the electrons and the nuclei is of the order of Ze^2/a, where Ze is the nuclear charge and $a \sim (ZV/N)^{1/3}$ is the mean distance between the electrons and the nuclei. The condition for an ideal gas is that this energy should be small compared with the mean kinetic energy of the electrons, which in order of magnitude is equal to the limiting energy ε_F. The inequality $Ze^2/a \ll \varepsilon_F$, after the substitution of $a \sim (ZV/N)^{1/3}$ and the expression (57.3) for ε_F, gives the condition

$$N/V \gg (e^2 m/\hbar^2)^3 Z^2. \qquad (57.9)$$

We see that this condition is more nearly met as the density N/V of the gas increases.†

PROBLEM

Determine the number of collisions with a wall in an electron gas at absolute zero.

SOLUTION. The number of electrons per unit volume with momenta in the interval dp at an angle to the normal to the wall in the interval $d\theta$ is $2\cdot 2\pi \sin\theta \, d\theta \, p^2 dp/(2\pi\hbar)^3$. The required number of collisions ν (per unit area of wall) is obtained by multiplying by $v\cos\theta$ $(v = p/m)$ and integrating with respect to θ from 0 to $\frac{1}{2}\pi$ and with respect to p from 0 to p_F. The result is

$$\nu = \frac{3(3\pi^2)^{1/3}}{16}\frac{\hbar}{m}\left(\frac{N}{V}\right)^{4/3}.$$

§ 58. The specific heat of a degenerate electron gas

At temperatures which are low compared with the degeneracy temperature T_F, the distribution function (57.4) has the form shown by the broken line in Fig. 6: it is appreciably different from unity or zero only in a narrow range of

values of the energy ε close to the limiting energy ε_F. The width of this "transition zone" of the Fermi distribution is of the order of T.

The expressions (57.6), (57.7) are the first terms in the expansions of the corresponding quantities in powers of the small ratio T/T_F. Let us now determine the next terms in the expansions.

Formula (56.6) involves an integral of the form

$$I = \int_0^\infty \frac{H(\varepsilon)\, d\varepsilon}{e^{(\varepsilon-\mu)/T}+1},$$

(here: $T \to k_B T$)

where $H(\varepsilon)$ is a function such that the integral converges; in (56.6) $H(\varepsilon) = \varepsilon^{3/2}$. We transform this integral by the substitution $\varepsilon - \mu = Tz$:

$$I = \int_{-\mu/T}^{\infty} \frac{H(\mu+Tz)}{e^z+1} T\, dz$$

$$= T\int_0^{\mu/T} \frac{H(\mu-Tz)}{e^{-z}+1}\, dz + T\int_0^\infty \frac{H(\mu+Tz)}{e^z+1}\, dz.$$

In the first integral we put $1/(e^{-z}+1) = 1 - 1/(e^z+1)$, obtaining

$z = \frac{\mu}{T} \to \varepsilon = 0$
$z = 0 \to \varepsilon = \mu$

$$I = \int_0^\mu H(\varepsilon)\, d\varepsilon - T\int_0^{\mu/T} \frac{H(\mu-Tz)}{e^z+1}\, dz + T\int_0^\infty \frac{H(\mu+Tz)}{e^z+1}\, dz.$$

In the second of these integrals we replace the upper limit by infinity, since $\mu/T \gg 1$ and the integral is rapidly convergent.† This gives

$$I = \int_0^\mu H(\varepsilon)\, d\varepsilon + T\int_0^\infty \frac{H(\mu+Tz) - H(\mu-Tz)}{e^z+1}\, dz.$$

We now expand the numerator of the second integrand as a Taylor series of powers of z and integrate term by term:

$$I = \int_0^\mu H(\varepsilon)\, d\varepsilon + 2T^2 H'(\mu)\int_0^\infty \frac{z\, dz}{e^z+1} = \frac{\pi^2}{12}$$

$$+\frac{1}{3}T^4 H'''(\mu)\int_0^\infty \frac{z^3\, dz}{e^z+1} + \dots .$$