4

Stress and strain

Theory: Deformation and stresses. Definition of stress, strain and strain-rate tensors. Deviatoric stresses. Mean stress as a dynamic (non-lithostatic) pressure. Symmetry of stress tensor. Stress and strain rate invariants.
Exercises: Computing the strain rate tensor components in 2D from the material velocity fields.

4.1 Stress

Tensors are field variables which characterise the internal state of a continuum and are, perhaps, the most difficult quantities to intuitively understand. Indeed, at least three of them have to be used in the following and these are the *stress, strain* and *strain rate* tensors.

Stress is the *internal distribution and intensity of force* acting at any point within a continuum in response to various internal and external loads applied to the continuum. Stress is defined as a *force per unit area* and we can easily 'apprehend' its effect by pressing two fingers against each other – equal *force is applied from both sides and therefore nothing moves*, but we have a *feeling of pressure* between the fingers, which is a sign of the presence of stress. This stress is directly proportional to the applied force – the stronger we press the stronger the feeling is. On the other hand, the stress is inversely proportional to the contact surface between the fingers – if we press one finger with the nail of the other the feeling is much stronger because the same force is applied to a much smaller area. This is why pricking a finger with a needle is so painful – the force applied to the needle is not big but the contact surface of the needle with the finger is very small and the resulting stress is consequently very big.

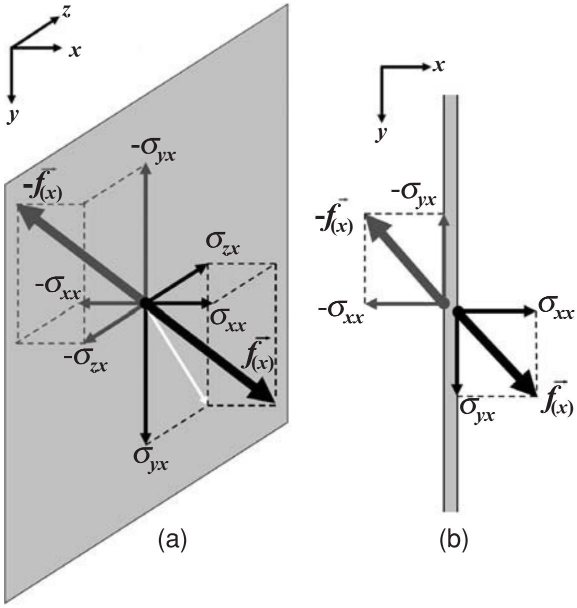

Fig. 4.1 Components of stress tensor defined from the force balance on a surface. (a) relationship between the stress components (thin arrows σ_{xx}, σ_{yx}, σ_{zx} and $-\sigma_{xx}$, $-\sigma_{yx}$, $-\sigma_{zx}$) and force vectors (thick arrows $\vec{f}_{(x)}$ and $-\vec{f}_{(x)}$) acting on the two sides of the unit element (grey) of a Lagrangian surface orthogonal to x-axis (i.e. x-surface). White arrow in (a) shows the direction of shear along the surface. (b) physical analogy: normal and shear stress components acting on a thin plate (cross-section of the plate in x-y-plane is shown).

In order to characterise the stress tensor, let us consider the force $\vec{f}_{(x)}$ acting on a unit element of a Lagrangian x-surface (i.e. surface orthogonal to the x-axis) (Fig. 4.1(a)). First of all, we need to understand that the force vector $\vec{f}_{(x)}$ acting on one side of the surface element is balanced by the counterforce vector $-\vec{f}_{(x)}$ which acts on the other side, and therefore this *stressed surface element* does not move. Thus, in order to characterise the *force balance state* of the stressed surface element, one needs to characterise the magnitude and direction of the force (balanced by the counterforce) acting on this element. Let us adopt a convention that the characterisation will be based on the force vector $\vec{f}_{(x)}$, applied to the side of the x-surface *from which the x-axis is exiting*. As we will see in the following, according to this convention, *extensional stresses are positive* as is usually assumed in continuum mechanics (e.g., Ranalli, 1995). Notice that this *usual continuum mechanics convention* is opposite to that used in the book of Turcotte and Schubert (2002), where stresses are taken positive under compression (which geoscientists find more intuitive since pressure is also positive under compression).

The force vector $\vec{f}_{(x)}$ can obviously be decomposed into three components (σ_{xx}, σ_{xy}, σ_{xz}) parallel to each coordinate axis (Fig. 4.1(a)). These are the components of the stress tensor since force $\vec{f}_{(x)}$ is acting on the unit element. According to the common continuum mechanics convention (e.g. Ranalli, 1995), which is again opposite to that used in the book of Turcotte and Schubert (2002), the first index (i) of a stress component σ_{ij} denotes the axis along which this stress component is taken (i.e. $i = z$ for the component parallel to the z axis) and the second index (j) indicates the surface on which force balance is considered (i.e. $j = x$ for the surface orthogonal to x axis). It should be pointed out that our 'hard choice' of a stress definition and notation is, indeed, very convenient for formulating several crucial equations, such as the momentum equation and the rheological constitutive relationships, which is the main reason why we deviated from the 'geological convention'. On the other hand, our vertical axis y, is always pointing down, thus preserving common 'geological logic' that the vertical coordinate is depth (and not height as in continuum mechanics) and increases downward rather than upward. A stress component that is orthogonal to the surface (cf. σ_{xx} in Fig. 4.1(a)) is called a *normal stress component* and the components which are parallel to the surface are called *shear stress components* (cf. σ_{yx} and σ_{zx} in Fig. 4.1(a)). The normal stress component characterises the magnitude of extension/compression *across the surface*. The two shear stress components characterise the magnitude and direction (cf. white arrow in Fig. 4.1) of shearing applied *along the considered surface.* A useful physical analogy (Fig. 4.1(b)) – if one imagines that the force and counterforce are applied on two sides of a very thin plate, then the normal component defines how strongly two opposite surfaces of the plate are forced to be shifted from/toward each other and the shear stress components define where and how strong these surfaces are forced to be shifted parallel to each other.

In order to fully characterise the force balance at a point (a small material volume), it is convenient to represent the stress tensor as a $N \times N$ matrix where N is the dimension of the problem such that in one, two and three dimensions we will have one, four and nine stress components respectively (Fig. 4.2)

$$\text{1D stress tensor,} \quad N = 1\ (\text{Fig. 4.2(a)}): \sigma_{ij} = (\sigma_{xx}),$$

$$\text{2D stress tensor,} \quad N = 2\ (\text{Fig. 4.2(b)}): \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix},$$

$$\text{3D stress tensor,} \quad N = 3\ (\text{Fig. 4.2(c)}): \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix},$$

where i and j *are symbolic coordinate indices* (x, y, z) which vary in vertical and horizontal directions, respectively. In continuum mechanics books a numerical

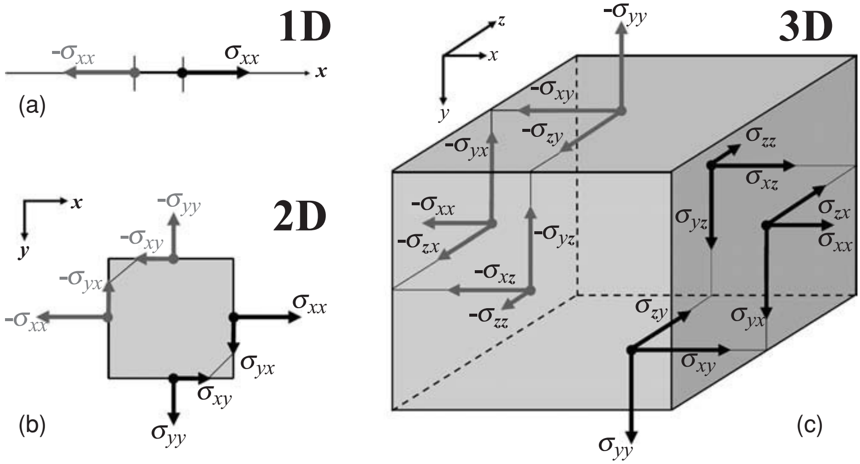

Fig. 4.2 Components of the stress tensor (black and grey arrows) defined in one- (a) two- (b) and three- (c) dimensions on faces of a small interval, square and cube, respectively. The faces are always oriented orthogonal to the main axis. Thin lines in (b) and (c) connect pairs of shear stress components which should be equal to each other in the absence of internal sources of angular momentum.

(1, 2, 3) notation for the coordinate indices i and j and stresses (σ_{11}, σ_{12}, σ_{32}, etc.) is commonly used as well (e.g. Ranalli, 1995). *Note that i and j are indices and not spatial coordinates of geometric points.*

Normal stresses are always located on the main diagonal of the matrix. Due to the *condition of force balance in the absence of internal sources of angular momentum*, this matrix is symmetric relative to the main diagonal so that

$$\sigma_{ij} = \sigma_{ji},$$

i.e. (Fig. 4.2(b), (c))

$$\sigma_{xy} = \sigma_{yx},$$
$$\sigma_{xz} = \sigma_{zx},$$
$$\sigma_{yz} = \sigma_{zy}.$$

Like components of a vector, components of the stress tensor at a point depend on the orientation of the coordinate system. We will discuss this in more detail later in relation to elasticity (Chapter 12).

In continuum mechanics, *pressure* is defined as the mean normal stress:

$$P = -(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 \quad (4.1)$$

where the negative sign on the right-hand side of Eq. (4.1) reflects another convention according to which pressure is positive under compression. Pressure is an *invariant* and, thus, *does not change with changing the coordinate system*. In the case of a *hydrostatic stress state* (which is the state of a fluid *at rest*) all shear stresses are zero and all normal stresses are equal to each other

$$\sigma_{xy} = \sigma_{yx} = \sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0 \tag{4.2a}$$
$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P. \tag{4.2b}$$

In geosciences, pressure is often considered as corresponding to the hydrostatic condition everywhere and it is computed as a function of depth y and vertical density profile $\rho(y)$

$$P(y) = P_0 + g\int_0^y \rho(y)\mathrm{d}y, \tag{4.3}$$

where $P_0 = 0.1$ MPa is pressure on the Earth's surface and g is the gravitational acceleration.

This simplification does not hold when deformations of geological media occur and real *dynamic* pressure may notably deviate from the lithostatic value given by Eq. (4.3).

It is often convenient to define the *deviatoric* stresses σ'_{ij}, which are deviations of stresses from the hydrostatic stress state (i.e., deviations from conditions (4.2))

$$\sigma'_{ij} = \sigma_{ij} + P\delta_{ij}, \tag{4.4}$$

where δ_{ij} is the *Kronecker delta*: $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$, i and j are coordinate indices (x, y, z). The Kronecker delta is a peculiar abbreviation used in the mechanics of continuum. It only takes values of either 1 or 0 and is analogous to the logical operator 'if' used in many programming languages. Any equation with δ_{ij} represents a *group of equations*. For example, Eq. (4.4) in 3D represents the following equations:

Normal deviatoric stresses

$$\sigma'_{xx} = \sigma_{xx} + P,$$
$$\sigma'_{yy} = \sigma_{yy} + P,$$
$$\sigma'_{zz} = \sigma_{zz} + P,$$

and shear stresses which are entirely deviatoric

$$\sigma'_{xy} = \sigma'_{yx} = \sigma_{xy} = \sigma_{yx},$$
$$\sigma'_{xz} = \sigma'_{zx} = \sigma_{xz} = \sigma_{zx},$$
$$\sigma'_{yz} = \sigma'_{zy} = \sigma_{yz} = \sigma_{zy}.$$

It is worth mentioning that the sum of the normal deviatoric stresses is zero by definition (Eq. (4.4))

$$\sigma'_{xx} + \sigma'_{yy} + \sigma'_{zz} = 0,$$

since

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3P.$$

The *second invariant* of the deviatoric stress tensor can be calculated as follows:

$$\sigma_{II} = \sqrt{1/2\,\sigma'^2_{ij}}, \tag{4.5}$$

where the indices *ij* imply a *summation*! This is another abbreviation that is commonly used in continuum mechanics and makes equations shorter (but, indeed, not easier to understand for inexperienced readers). The spelled-out form of Eq. (4.5) is much longer

$$\sigma_{II} = \sqrt{1/2\left(\sigma'^2_{xx} + \sigma'^2_{yy} + \sigma'^2_{zz} + \sigma^2_{xy} + \sigma^2_{yx} + \sigma^2_{xz} + \sigma^2_{zx} + \sigma^2_{yz} + \sigma^2_{zy}\right)}, \tag{4.6a}$$

or, using the condition of force balance $\sigma_{ij} = \sigma_{ji}$

$$\sigma_{II} = \sqrt{1/2\left(\sigma'^2_{xx} + \sigma'^2_{yy} + \sigma'^2_{zz}\right) + \sigma^2_{xy} + \sigma^2_{xz} + \sigma^2_{yz}}. \tag{4.6b}$$

The second stress invariant σ_{II} does not depend on the coordinate system and characterises the local deviation of stresses in the medium from the hydrostatic state.

4.2 Strain and strain rate

Another important quantity is the *strain* γ, that characterises the amount of deformation. Strain is dimensionless and is computed as the ratio of displacement ΔL to the initial length of deforming body L (Fig. 4.3)

$$\gamma = \frac{\Delta L}{L}. \tag{4.7}$$

By analogy with stress, one can discriminate normal and shear strain corresponding to axial and shear deformation, respectively (Fig. 4.3(a) and (b)).

The definition of strain given by Eq. (4.7) can only be applied in cases of relatively simple axial and shear deformations. In case of more complex deformation, the *strain tensor* ε_{ij}, is defined as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4.8}$$

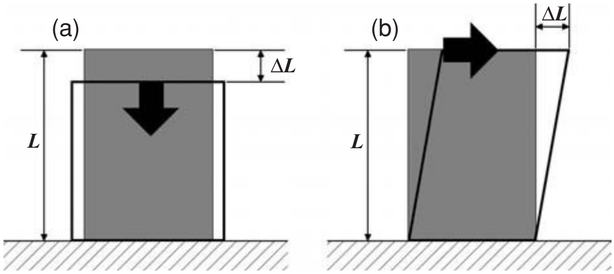

Fig. 4.3 Axial (a) and shear (b) deformation corresponding to normal and shear strain components. The strain in both cases is estimated as $\gamma = \frac{\Delta L}{L}$. Note that in case of shear deformation length L is measured orthogonal to the displacement direction.

where i and j *are coordinate indices* (x, y, z) and x_i and x_j *are spatial coordinates* (i.e., x_x, x_y and x_z are x-, y- and z-coordinates respectively). *Note that in contrast to symbolic i- and j-indices x_i and x_j are physical coordinates of geometrical points.* Do not confuse them with each other! In 3D, we can define nine tensor components:

three normal strain components

$$\varepsilon_{xx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right) = \frac{\partial u_x}{\partial x},$$
$$\varepsilon_{yy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right) = \frac{\partial u_y}{\partial y},$$
$$\varepsilon_{zz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z}\right) = \frac{\partial u_z}{\partial z}$$

and six shear strain components

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),$$
$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right),$$
$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right).$$

Note that stress and strain tensors are very different physical quantities (although they can be strongly correlated in case of reversible elastic deformation, Chapter 12): stress characterises the distribution of forces acting in a continuum at a given moment of time, while strain quantifies in an integrated way the entire *deformation history* of the continuum from the initial state, up until this given moment

(Fig. 4.3). The symmetric form of the strain tensor subtracts the rotational component of the velocity field which does not contribute to the deformation (rotation of a rigid body has gradients in material displacement, but does not produce any internal deformation). In Eq. (4.8), u_i and u_j are components of *material displacement* vector, $\bar{u} = (u_x, u_y, u_z)$ which characterise the displacement of a material point relative to its original position (i.e. before deformation). The time derivative of the displacement vector is the velocity vector $\vec{v} = (v_x, v_y, v_z)$ so that

$$v_i = \frac{Du_i}{Dt} \tag{4.9}$$

and, in 3D deformation

$$v_x = \frac{Du_x}{Dt},$$
$$v_y = \frac{Du_y}{Dt},$$
$$v_z = \frac{Du_z}{Dt}.$$

The strain tensor is widely used when elastic deformation is considered (Chapter 12).

In numerical geodynamic modelling, it is convenient to use the *strain rate*, which characterises the dynamics of changes in the internal deformation rather then the strain which characterises the total amount of deformation compared to the initial state. The strain rate tensor $\dot{\varepsilon}_{ij}$ is the time derivative (indicated by the dot on top of the strain symbol) of the strain tensor ε_{ij}. Components of the strain rate tensor are defined via spatial gradients of the velocity as follows

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{4.10}$$

where i and j are coordinate indices and x_i and x_j are spatial coordinates such that in 3D we can define nine tensor components:

three normal strain rate components

$$\dot{\varepsilon}_{xx} = \frac{1}{2}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial x}\right) = \frac{\partial v_x}{\partial x},$$
$$\dot{\varepsilon}_{yy} = \frac{1}{2}\left(\frac{\partial v_y}{\partial y} + \frac{\partial v_y}{\partial y}\right) = \frac{\partial v_y}{\partial y},$$
$$\dot{\varepsilon}_{zz} = \frac{1}{2}\left(\frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial z}\right) = \frac{\partial v_z}{\partial z}$$

and six shear strain rate components

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right),$$
$$\dot{\varepsilon}_{xz} = \dot{\varepsilon}_{zx} = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right),$$
$$\dot{\varepsilon}_{yz} = \dot{\varepsilon}_{zy} = \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right).$$

Similarly to the strain tensor, the symmetric form of the strain rate tensor is obtained by subtracting the rotational component of the velocity field: it is easy to check that rigid body rotation in 2D has gradients in the velocity field which do not produce any internal deformation, i.e.

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = 0.$$

By analogy to the stress tensor, the strain rate tensor can also be subdivided to isotropic $\dot{\varepsilon}_{kk}$ (which is an invariant) and deviatoric $\dot{\varepsilon}'_{ij}$ components

$$\dot{\varepsilon}_{kk} = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \mathrm{div}(\bar{v}), \tag{4.11}$$

$$\dot{\varepsilon}'_{ij} = \dot{\varepsilon}_{ij} - \delta_{ij}\frac{1}{3}\dot{\varepsilon}_{kk}, \tag{4.12}$$

where i, j and k are coordinate indices.

According to Eqs. (4.11), (4.12) the sum of normal deviatoric strain rate components is zero

$$\dot{\varepsilon}'_{xx} + \dot{\varepsilon}'_{yy} + \dot{\varepsilon}'_{zz} = 0. \tag{4.13}$$

Like the second stress invariant, the second invariant of the deviatoric strain rate tensor is calculated as follows

$$\dot{\varepsilon}_{\mathrm{II}} = \sqrt{1/2\,\dot{\varepsilon}'^{2}_{ij}}. \tag{4.14}$$

Analytical exercise

Exercise 4.1

Show that the symmetric form of the strain rate tensor satisfies the condition

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = 0,$$

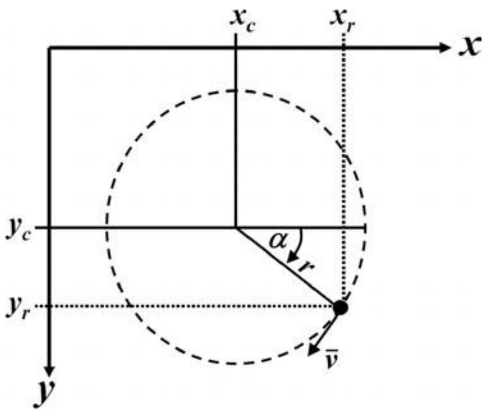

Fig. 4.4 Geometrical relationship in the case of 2D rigid body rotation.

in the case of rigid body rotation with constant angular velocity. Use the fact that coordinates of a rotating point in 2D are given by (Fig. 4.4)

$$x_r = x_c + r\cos(\alpha), \ y_r = y_c + r\sin(\alpha),$$

where r is the distance to the centre of rotation, x_c and y_c are the coordinates of the centre and α is the clockwise rotation angle taken from the horizontal axis.

Programming exercise and homework

Exercise 4.2

Compute and visualise the deviatoric strain rate tensor components and invariants for the model described in the Exercise 1.2. An example is in **Strain_rate.m**.