TEORIE di GAUGE NON-ABELIANE

Prendiamo una teoria inv. sotto una trasf. globale
data da un gruppo di Lie G semplice (semi-semplice)

Prendiamo un campo φ $\rightarrow$ $\varphi(x)$ è in una RAPP. R,
che trasf. sotto G cioè è un campo a valori
$$\text{in } V_R \text{ (sezione del fib.}$$
$$\text{vett.}$$
$$\text{con fibra } E_x \cong V_R)$$

Es: $G = SU(2)$ $R = \underline{2}$

$\varphi \in V_R$ $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$

Se φ è "uno scalare
nella rap. $\underline{2}$" vuol
dire che φ_1 e φ_2
sono scalari
(φ prende valori in
$V_R \otimes V_{\Gamma_{Lorentz}}$)

Lagrangiana invariante (φ scalare $\mathbb{C}$)

$$\mathcal{L} = \partial_\mu \varphi_1^* \partial^\mu \varphi_1 + \partial_\mu \varphi_2^* \partial^\mu \varphi_2 - m^2 |\varphi_1|^2 - m^2 |\varphi_2|^2$$

$$= \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi \quad\quad\quad N \otimes \bar{N} = 1 \oplus Adj$$
$$\underset{\bar N}{\nearrow} \quad \underset{N}{\nwarrow}$$

$$\underset{U \in SU(2)}{\longmapsto} \partial_\mu \varphi^\dagger U^\dagger U \partial^\mu \varphi - m^2 \varphi^\dagger U^\dagger U \varphi = \mathcal{L}$$

$\mathcal{L}(\varphi, \partial\varphi)$ invariante sotto $\varphi \mapsto U\varphi$ (U matrice costante)

$$U \in G \quad\quad U = e^{-i\alpha^a t_R^a}$$

φ trasforma in rap. R

Passiamo a trasf. locali (di gauge)
$$U \leadsto U(x) = e^{-i\alpha^a(x)t_R^a}$$

$$\partial_\mu \varphi \longmapsto \partial_\mu (U\varphi) = (\partial_\mu U)\varphi + U \partial_\mu \varphi$$

Introduciamo la DERIVATA COVARIANTE

$$D_\mu \varphi = (\partial_\mu + i A_\mu)\varphi$$

$$D_\mu v^\beta = \partial_\mu v^\beta + \Gamma^\beta_{\mu\nu} v^\nu$$

$\uparrow$ matrice

$$(D_\mu \varphi)_i = \partial_\mu \varphi_i + i (A_\mu)_i{}^j \varphi_j$$

A_μ è definita in modo tale che

$$D_\mu \varphi \longmapsto U D_\mu \varphi .$$

$$D_\mu \varphi \longmapsto (\partial_\mu + i A'_\mu)\varphi' = (\partial_\mu + i A'_\mu) U\varphi =$$

$$= (\partial_\mu U)\varphi + U\partial_\mu \varphi + i A'_\mu U\varphi$$
$$+ i U A_\mu \varphi - i U A_\mu \varphi$$

$$= U D_\mu \varphi + (\partial_\mu U + i A'_\mu U - i U A_\mu)\varphi$$

$\Rightarrow$ Otteniamo $D_\mu \varphi \longmapsto U D_\mu \varphi$ se

$$A'_\mu = U A_\mu U^{-1} + i (\partial_\mu U) U^{-1}$$

$$[= -i U \partial_\mu U^{-1} \quad \text{perchè} \quad U U^{-1} = \mathbb{1}$$
$$\Rightarrow \partial_\mu (U U^{-1}) = 0]$$

$\mathcal{L}(\varphi, D_\mu \varphi)$ è invariante sotto trasf. locali:

$$\begin{cases} \varphi' = U\varphi \\ A'_\mu = U A_\mu U^{-1} + i(\partial_\mu U) U^{-1} \end{cases}$$

Trasf. INFINITESIME $\longleftrightarrow$ $\alpha^a(x) \ll 1$

$$U(x) = e^{-i\alpha(x)^a t^a_R} \simeq 1 - i\alpha^a_{(x)} t^a_R \equiv 1 - i\alpha(x)$$

oggetto nella
RAPP. AGGIUNTA

$$\delta\varphi = -i\alpha(x)\varphi \qquad (\delta\varphi \equiv \varphi' - \varphi)$$

$$\delta A_\mu = A'_\mu - A_\mu = (1 - i\alpha) A_\mu (1 + i\alpha) + i\left[\partial_\mu(1 - i\alpha)\right](1 + i\alpha)$$
$$- A_\mu$$

$$= A_\mu - i\alpha A_\mu + i A_\mu \alpha + \partial_\mu \alpha - A_\mu$$

$$= \partial_\mu \alpha + i[A_\mu, \alpha] \qquad \leftarrow \quad \textcolor{red}{D_\mu \alpha} \quad \textcolor{red}{(*)}$$
vedi sotto

$$= \partial_\mu \alpha^a t^a_R - i\alpha^a[t^a_R, A_\mu]$$

Se prendiamo α^a cost. $\rightsquigarrow$ $\delta A_\mu = -i[\alpha^a t^a_R, A_\mu]$

$\Rightarrow A_\mu$ trasf nella rap Adj

$\rightarrow A_\mu = A^a_\mu t^a_R \qquad (*)$

La connessione che agisce su un vett. nella rapp. R
è data da $(*)$

$(\text{Se abbiamo } \varphi \in R \qquad \subset \mathcal{X} \in \tilde{R}$
$D_\mu \varphi = \partial_\mu \varphi + i A^a_\mu t^a_R \varphi \qquad D_\mu \mathcal{X} = \partial_\mu \mathcal{X} + i A^a_\mu t^a_{\tilde{R}} \mathcal{X})$

$$\delta A_\mu = \partial_\mu \alpha^a t_R^a - i\alpha^a A_\mu^b [t_R^a, t_R^b] =$$

$$= \partial_\mu \alpha^c t_R^c - i\alpha^a A_\mu^b if^{abc} t_R^c$$

$$\delta A_\mu^c t_R^c = (\partial_\mu \alpha^c + f^{cab} \alpha^a A_\mu^b) t_R^c$$

$$\color{red}{\delta A_\mu^a = \partial_\mu \alpha^a - f^{abc} A_\mu^b \alpha^c}$$

(A) $\quad D_\mu \alpha = \partial_\mu \alpha + i A_\mu^a t_{Ad}^a \alpha = \partial_\mu \alpha + i A_\mu^a [t^a, \alpha] =$

$$= \partial_\mu \alpha + i [A_\mu, \alpha]$$

TERMINE CINETICO per i bosoni di gauge $A_\mu^a(x)$

$$\Bigg[\quad \text{In QED} \quad \text{avevamo} \quad -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Bigg)$$

Partiamo calcolandoci $[D_\mu, D_\nu]$

$$[(\partial_\mu + i A_\mu), (\partial_\nu + i A_\nu)] \varphi =$$

$$= \underbrace{[\partial_\mu, \partial_\nu]}_{=0} \varphi + i[\partial_\mu, A_\nu] \varphi - i[\partial_\nu, A_\mu] \varphi$$

$$\underbrace{- [A_\mu, A_\nu] \varphi}_{\neq 0 \text{ in teorie di gauge non-ab}}$$

$$= i \partial_\mu (A_\nu \varphi) - i A_\nu \partial_\mu \varphi \qquad - (\mu \leftrightarrow \nu)$$

$$- [A_\mu, A_\nu] \varphi$$

$$= i(\partial_\mu A_\nu)\varphi + iA_\nu \partial_\mu \varphi - iA_\nu \partial_\mu \varphi - (\mu \leftrightarrow \nu)$$
$$- [A_\mu, A_\nu]\varphi$$

$$= i\left[\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\right]\varphi$$

$$\equiv i \, \mathbb{F}_{\mu\nu}\varphi \qquad\qquad \mathbb{F}_{\mu\nu} \equiv -i[D_\mu, D_\nu]$$

Vediamo come trasforma $\mathbb{F}_{\mu\nu}$

$$\mathbb{F}_{\mu\nu}\varphi = [D_\mu, D_\nu]\varphi \longmapsto U[D_\mu, D_\nu]\varphi = U\mathbb{F}_{\mu\nu}\varphi =$$
$$= U\mathbb{F}_{\mu\nu} U^{-1}(U\varphi)$$

$$\longmapsto \mathbb{F}'_{\mu\nu}\varphi' = \mathbb{F}'_{\mu\nu}(U\varphi)$$

$$\color{red}{\mathbb{F}'_{\mu\nu} = U\mathbb{F}_{\mu\nu}U^{-1}} \qquad \text{trasforma in maniera covariante (come un off. nella rap. Adj)}$$

$$\mathbb{F}_{\mu\nu} = F_{\mu\nu}^a \, t_R^a \qquad \color{red}{\leftarrow \text{CURVATURA (della connessione } A_\mu)}$$

$$\mathbb{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = \partial_\mu A_\nu^a \, t_R^a - \partial_\nu A_\mu^a \, t_R^a +$$
$$+ iA_\mu^b A_\nu^c [t_R^b, t_R^c] =$$

$$= \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c\right) t_R^a$$

$$\Rightarrow \quad \color{red}{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c}$$

Il termine cinetico dev'essere Lorentz inv. e gauge inv.

$$= - k \, tr \left(F_{\mu\nu} F^{\mu\nu} \right)$$

tr $(FF) \mapsto$ tr $\left(U F \mu^{-1} U F U^{-1} \right) =$
$$= tr \, (FF)$$
ciclicità delle traccia

$$= - k \, F_{\mu\nu}^a \, F^{b\,\mu\nu} \, tr \left(t_R^a \, t_R^b \right) =$$

$$= - k \, F_{\mu\nu}^a \, F^{b\,\mu\nu} \, C(R) \, \delta^{ab} \circ - k \, C(R) \, F_{\mu\nu}^a \, F^{a\,\mu\nu}$$

$$= - \frac{1}{4} F_{\mu\nu}^a \, F^{a\,\mu\nu} \qquad \Leftarrow \qquad k = \frac{1}{4 \, C(R)}$$

termine
cinetico normalizzato
canonican.

$$G = SU(N) \quad e \quad R = N \qquad C(N) = \frac{1}{2}$$

$$\to k = \frac{1}{2}$$

[Se G non è compatto, allora tr $\left(t_R^a \, t_R^b \right) \neq \delta^{ab}$,
ma sulla diagonale ci saranno $+1$ e -1
$\Rightarrow$ avremmo dei termini cinetici negativi.]

$- \frac{1}{4} F_{\mu\nu}^a \, F^{a\,\mu\nu}$ contiene termini CUBICI e QUARTICI in A_μ^a
$\rightarrow$ termini di interazione tra i
bosoni vettori

La lagrangiana $\mathcal{L}(\varphi, A)$ inv. sotto trasf. gauge

$$\mathcal{L}(\varphi, A) = \mathcal{L}(\varphi, D_\mu \varphi)_{before} - \frac{1}{4g^2} F_{\mu\nu}^a \, F^{a\,\mu\nu}$$

Per i calcoli perturbativi, spesso è più utile riscalare

$$A_\mu^a \mapsto g A_\mu^a \qquad (\text{ridef. di campo})$$

$$F_{\mu\nu}^a \mapsto g\,\partial_\mu A_\nu^a - g\,\partial_\nu A_\mu^a - g^2 f^{abc} A_\mu^b A_\nu^c$$

$$= g\,F_{\mu\nu}^a$$

↗ g^2 sparisce dal termine cin.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\, f^{abc} A_\mu^b A_\nu^c$$

$$D_\mu \varphi \mapsto (\partial_\mu + i g A_\mu)\varphi$$

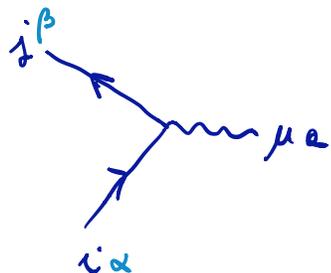$$\mathcal{L} = \mathcal{L}_{before}(\varphi, D_\mu \varphi) - \frac{1}{4} F_{\mu\nu}^a F^{a\,\mu\nu}$$

g è la **COSTANTE DI ACCOPIAMENTO**

REGOLE DI FEYNMAN (massless)

Scegliamo φ spinore e lo chiamiamo ψ

$$\mathcal{L}_{before}(\psi, D_\mu \psi) = i\,\bar\psi\,\gamma^\mu D_\mu \psi = \qquad \text{propagatore} \longrightarrow \frac{i}{\not p}$$

$$= i\,\bar\psi\,\gamma^\mu \partial_\mu \psi \;\underbrace{-g\,\bar\psi\,\gamma^\mu A_\mu \psi}$$

$$-g\,\bar\psi\,\gamma^\mu A_\mu^a t_R^a \psi = -g\,\bar\psi_{ia}\,\gamma_{\alpha\beta}^\mu (t_R^a)_{ij}\,\psi_{j\beta}\,A_\mu^a$$

$$-i g\,\gamma_{\alpha\beta}^\mu (t_R^a)_{ij}$$

$$\mathcal{L}_{kin}(A) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\right).$$

$$\cdot \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} - g f^{ade} A^{d\mu} A^{e\nu}\right)$$

$$= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \frac{1}{2} g f^{abc} A^b_\mu A^c_\nu \left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right)$$

$$- \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}$$

$$\underbrace{2 \partial^\mu A^{a\nu}}$$

propagator
(?)

vertex