19 ottobre

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p \\ \nabla\cdot u = 0 \\ u\big|_{t=0} = u_0 \end{cases}$$

u = velocità fluido
p pressione

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \dots \ \vec{e}_d = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$u\cdot\nabla u = u_k \partial_k u_j \, \vec{e}_j = \partial_k(u_k u_j)\, \vec{e}_j - \cancel{\underbrace{\partial_k u_k}_{0} u_j \vec{e}_j}$$

$$= \operatorname{div}(u\otimes u)$$

$$u\otimes u = \{ u_j u_k \}$$

$$\operatorname{div}(u\otimes u) = \{ \operatorname{div}^j (u\otimes u) \} = \{ \partial_k (u_j u_k) \}$$

$$L^2(\mathbb{R}^d, \mathbb{R}^d) = \underbrace{\nabla \dot{H}^1(\mathbb{R}^d, \mathbb{R})}_{\operatorname{Ker} \mathbb{P}} \oplus \underset{\substack{\shortparallel \\ \operatorname{Range}(\mathbb{P})}}{H(\mathbb{R}^d)}$$

$$\boxed{u_t - \nu\Delta u + u\cdot\nabla u = -\nabla p \qquad \cdot\,\mathbb{P}}$$

$$u\cdot\nabla u = \mathbb{P}\, u\cdot\nabla u + \underbrace{(1-\mathbb{P})\, u\cdot\nabla u}_{\in \operatorname{Ker}\mathbb{P}}$$

$$(1-\mathbb{P})\, u\cdot\nabla u = -\nabla p$$

$$u_t - \nu \Delta u + \operatorname{div}(u \otimes u) = -\nabla p \qquad \mathbb{P}$$

$$\begin{cases} u_t - \nu \Delta u + \mathbb{P} \operatorname{div}(u \otimes u) = 0 \\ u = \mathbb{P} u \\ u|_{t=0} = u_0 \qquad u_0 = \mathbb{P} u_0 \end{cases}$$

$$\langle \, , u \rangle_{L^2(\mathbb{R}^d, \mathbb{R}^d)}$$

$$u_t - \nu \Delta u + \operatorname{div}(u \otimes u) = -\nabla p$$

$$\langle u_t, u \rangle_{L^2} - \nu \langle \Delta u, u \rangle_{L^2} + \cancel{\langle \operatorname{div}(u \otimes u), u \rangle_{L^2}} = -\cancel{\langle \nabla p, u \rangle_{L^2}}$$

$$\langle \nabla p, u \rangle = \langle \partial_j p, u_j \rangle = -\langle p, \partial_j u_j \rangle = 0$$

$$\langle \operatorname{div}(u \otimes u), u \rangle = \langle \partial_j (u_j u_k), u_k \rangle =$$

$$= \langle \cancel{\partial_j u_j u_k} + u_j \partial_j u_k, u_k \rangle = \frac{1}{2} \langle u_j, \partial_j u_k^2 \rangle$$

$$= -\frac{1}{2} \langle \partial_j u_j, u_k u_k \rangle = 0$$

$$\frac{1}{2} \frac{d}{dt} |u|_{L^2}^2 + \nu |\nabla u|_{L^2}^2 = 0$$

$$\boxed{|u(t)|_{L}^2 + 2\nu \int_0^t |\nabla u(s)|_{L^2}^2 \, ds = |u_0|_{L^2}^2}$$

$$\begin{cases} \partial_t u - \nu \Delta u + \mathbb{P}\, \mathrm{div}(u \otimes u) = 0 \\ u_{t=0} = u_0 \qquad \mathbb{P} u \equiv u \end{cases}$$

Def Sia $u_0 \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ un campo $\mathrm{div}\, u_0 = 0$

$u \in L^2_{loc}([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)$ è una soluzione debole se

1) $t \to u(t)$ è in $C^0([0,+\infty), L^2_w(\mathbb{R}^d, \mathbb{R}^d))$

$t \to \langle u(t), \phi \rangle \in C^0([0,+\infty), \mathbb{R}) \quad \forall\ \phi \in L^2(\mathbb{R}^d, \mathbb{R}^d)$

2) $\mathrm{div}\, u \equiv 0$

3) $\forall\ \Phi \in C^\infty_c([0,+\infty) \times \mathbb{R}^d, \mathbb{R}^d)$ con

$\mathrm{div}_x \phi \equiv 0$ si ha $\qquad \phi(t) = \mathbb{P}\phi(t) \quad \forall t$.

$$\langle u(t), \Phi(t) \rangle_{L^2_x} = \int_0^t \Big[\nu \langle u(t'), \Delta \phi(t') \rangle_{L^2_x} + \langle u(t'), \partial_t \phi(t') \rangle_{L^2_x}$$

$$- \langle \mathrm{div}(u \otimes u)(t'), \phi(t') \rangle_{L^2_x} \Big] dt' + \langle u_0, \phi(0) \rangle_{L^2_x}$$

$$\partial_t u - \nu \Delta u + \mathbb{P}\, \mathrm{div}(u \otimes u) = 0 \qquad \langle\ , \phi(t) \rangle_{L^2_x}$$

$$\langle \partial_t u, \phi \rangle_{L^2}$$

Esercizio $t \to \langle u(t), \Phi(t) \rangle_{L^2}$ è continua

Teor (Leray, $d=2,3$) Sia $u_0 \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ $\quad \mathrm{div}\, u_0 = 0$

Allora esiste una soluzione debole

$$u(t) \in L^\infty([0,+\infty), L^2_x), \quad \nabla u(t) \in L^2([0,+\infty), L^2_x)$$

ed inoltre vale

$$|u(t)|^2_{L^2_x} + 2\nu \int_0^t |\nabla u(t')|^2_{L^2_x}\, dt' \le |u_0|^2_{L^2_x} \qquad \forall t \ge 0$$

$f \in L^p(\mathbb{R}, X)$ $\qquad$ Bochner

$$|f|_{L^p(\mathbb{R}, X)} = \Big|\, |f|_X \,\Big|_{L^p(\mathbb{R})}$$

Teor Nel caso $d=2$ la soluzione di Leray è unica, $u \in C^0([0,+\infty), L^2(\mathbb{R}^2,\mathbb{R}^2))$ e vale l'identità dell'energia.

Lemma $\exists\ C_T$ t.c. $\forall\ u \in L^2((0,T), H^1(\mathbb{R}^d)) \cap H^1((0,T), H^{-1}(\mathbb{R}^d))$ si ha $u \in C^0([0,T], L^2(\mathbb{R}^d))$ ed inoltre

$$|u|_{L^\infty((0,T),L^2)} \le C_T \left(|u|_{L^2((0,T),H^1)} + |u|_{H^1((0,T),H^{-1})} \right)$$

Inoltre $|u(t)|^2_{L^2_x} \in AC([0,T])$ con

$$\frac{d}{dt} |u(t)|^2_{L^2} = 2 \langle u(t), \dot{u}(t) \rangle_{L^2_x} \qquad q.o.$$

$$\langle\ ,\ \rangle_{L^2} : H^1_x \times H^{-1}_x \longrightarrow \mathbb{R}$$

Il teorema in dim 2 è basato su

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2) \qquad \frac{1}{4} = \frac{1}{2} - \frac{\frac{1}{2}}{2}$$

Dim Dimostreremo che

$$\partial_t u \in L^2((0,T), H^{-1}(\mathbb{R}^2,\mathbb{R}^2)) \quad \forall\ T>0$$

Siccome $u \in L^2((0,T), H^1)$

(perché $u \in L^\infty((0,T), L^2)$ e $\nabla u \in L^2((0,T), L^2)$)

allora $\Rightarrow u \in C^0([0,T], L^2_x)$

Supponiamo u e v siano soluzioni con $u(0)=v(0)$

$w := u - v$ $\quad w(0)=0$. Supponiamo che possa prendere

come funzione test w.

$$\langle u(t), w(t)\rangle = \int_0^t \left[-\nu\langle \nabla u, \nabla w\rangle + \langle u, \partial_t w\rangle - \langle \operatorname{div}(u\otimes u), w\rangle\right] dt'$$

$$\langle v(t), w(t)\rangle = \int_0^t \left[-\nu\langle \nabla v, \nabla w\rangle + \langle v, \partial_t w\rangle - \langle \operatorname{div}(v\otimes v), w\rangle\right] dt'$$

$$* \quad \langle u(t), \phi_n(t)\rangle = \int_0^t \left[-\nu\langle \nabla u, \nabla \phi_n\rangle + \langle u, \partial_t \phi_n\rangle - \langle \underline{\operatorname{div}(u\otimes u)}, \phi_n\rangle\right] dt' + \langle u_0, \phi_n(0)\rangle$$

$w \in L^2((0,T), H^1) \cap H^1((0,T), H^{-1}) \cap C^0([0,T], L^2)$

$\phi_n \to w$

$$\langle u(t), w(t)\rangle = \int_0^t \left[-\nu\langle \nabla u, \nabla w\rangle + \langle u, \partial_t w\rangle\right] dt' \underbrace{- \lim_{n\to\infty} \int_0^t \langle \operatorname{div}(u\otimes u), \phi_n\rangle\, dt'}_{\int_0^t \langle \operatorname{div}(u\otimes u), w\rangle}$$

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \qquad \frac{1}{p}+\frac{1}{q}=1$$

$$\left| \int_0^t \langle \operatorname{div}\underbrace{(u\otimes u)}_{(u\cdot\nabla)u}, \phi_n - w\rangle\, dt' \right| \le$$

$$\le \int_0^t |u(t')|_{L^4} |\nabla u(t')|_{L^2} |\phi_n - w|_{L^4}\, dt'$$

$$\le \int_0^t |u(t')|_{L^2}^{\frac12} |\nabla u(t')|_{L^2}^{\frac12} |\nabla u(t')|_{L^2} |\phi_n - w|_{L^2}^{\frac12} |\nabla(\phi_n - w)|_{L^2}^{\frac12}$$

$$\le |u|_{L^\infty((0,t),L^2)}^{\frac12} |\phi_n - w|_{L^\infty((0,t);L^2)}^{\frac12} \int_0^t |\nabla u|_{L^2}^{\frac32} |\nabla(\phi_n - w)|_{L^2}^{\frac12}$$

$$1 = \frac14 + \frac34 \qquad \boxed{\frac34 \int_0^t |\nabla u|_{L^2}^{\frac32\cdot\frac43 = 2} + \frac14 \int_0^t |\nabla(\phi_n - w)|_{L^2}^2}$$

$$\langle u(t), w(t)\rangle = \int_0^t \left[-\nu\langle \nabla u, \nabla w\rangle + \langle u, \partial_t w\rangle - \langle \operatorname{div}(u\otimes u), w\rangle\right] dt'$$

$$\langle v(t), w(t)\rangle = \int_0^t \left[-\nu\langle \nabla v, \nabla w\rangle + \langle v, \partial_t w\rangle - \langle \operatorname{div}(v\otimes v), w\rangle\right] dt'$$

$$|w(t)|_{L^2}^2 = \int_0^t \left[-\nu |\nabla w|_{L^2}^2 + \cancel{\langle w, \partial_t w\rangle} + \langle \operatorname{div}(v\otimes v) - \operatorname{div}(u\otimes u), w\rangle\right] dt'$$

$$\int_0^t \langle w, \partial_t w\rangle = \frac{1}{2} |w(t)|_{L^2}^2$$

$$\frac{1}{2}|w(t)|_{L^2}^2 + \nu\int_0^t |\nabla w|_{L^2}^2\, dt' = \int_0^t \langle \operatorname{div}(v\otimes v) - \operatorname{div}(u\otimes u), w\rangle$$

$$\frac{1}{2}\frac{d}{dt}|w(t)|_{L^2}^2 + \nu |\nabla w|_{L^2}^2 = \langle \operatorname{div}(v\otimes v) - \operatorname{div}(u\otimes u), w\rangle$$

$$= \langle \operatorname{div}(v\otimes v) - \operatorname{div}(u\otimes v), w\rangle + \langle \operatorname{div}(u\otimes v) - \operatorname{div}(u\otimes u), w\rangle$$

$$= -\langle \operatorname{div}(w\otimes v), w\rangle - \langle \operatorname{div}(u\otimes w), w\rangle =$$

$$= -\langle \partial_k(w^k v^j), w^j\rangle - \langle \partial_k(u^k w^j), w^j\rangle$$

$$= -\langle w^k \partial_k v^j, w^j\rangle - \langle u^k \partial_k w^j, w^j\rangle$$

$$= -\langle (w\cdot\nabla) v, w\rangle - \frac{1}{2}\underbrace{\langle u^k, \partial_k(w^j w^j)\rangle}_{0}$$

$$\leq |\nabla v|_{L^2} |w^2|_{L^2} = |\nabla v|_{L^2} |w|_{L^4}^2 \leq$$

$$\leq C |\nabla v|_{L^2} |w|_{L^2} |\nabla w|_{L^2}$$

$$\frac{d}{dt}|w|^2 + \cancel{2}\nu |\nabla w|_{L^2}^2 \leq C |\nabla v|_{L^2} |w|_{L^2} |\nabla w|_{L^2}$$

$$\leq \cancel{\nu |\nabla w|_{L^2}^2} + \frac{C^2}{\nu} |\nabla v|_{L^2}^2 |w|_{L^2}^2$$

$$\boxed{|w(t)|_{L^2}^2 \leq \int_0^t \frac{C^2}{\nu} |\nabla v(t')|_{L^2}^2 |w|_{L^2}^2 dt'}$$

$$\frac{d}{dt}|w|_{L^2}^2 \leq \frac{C^2}{\nu} |\nabla v|_{L^2}^2 |w|_{L^2}^2$$

$$\boxed{|w(t)|_{L^2}^2 \leq e^{\frac{C^2}{\nu}\int_0^t |\nabla v(t')|_{L^2}^2 dt'} \underbrace{|w(0)|_{L^2}^2}_{0}} \implies w(t) \equiv 0$$