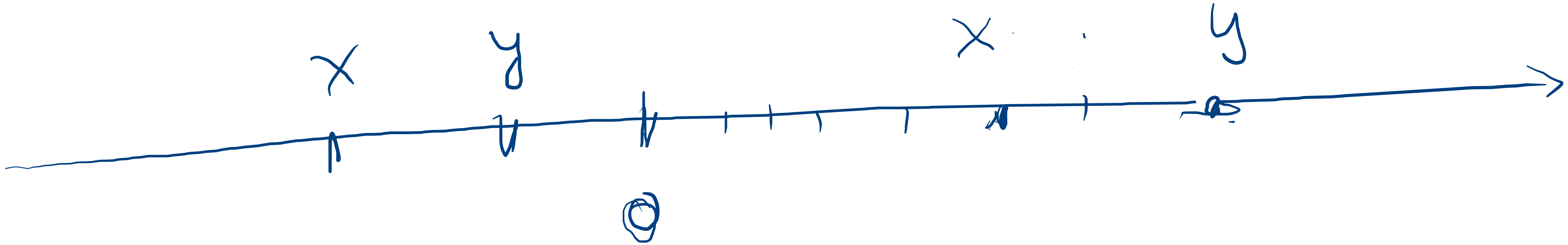
Ven 6/11

$\boxed{11^{30} - 13}$



Presi due numeri reali x, y allora risulta o $x=y$ o $x<y$ o $x>y$

$$|x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$$

Valore assoluto

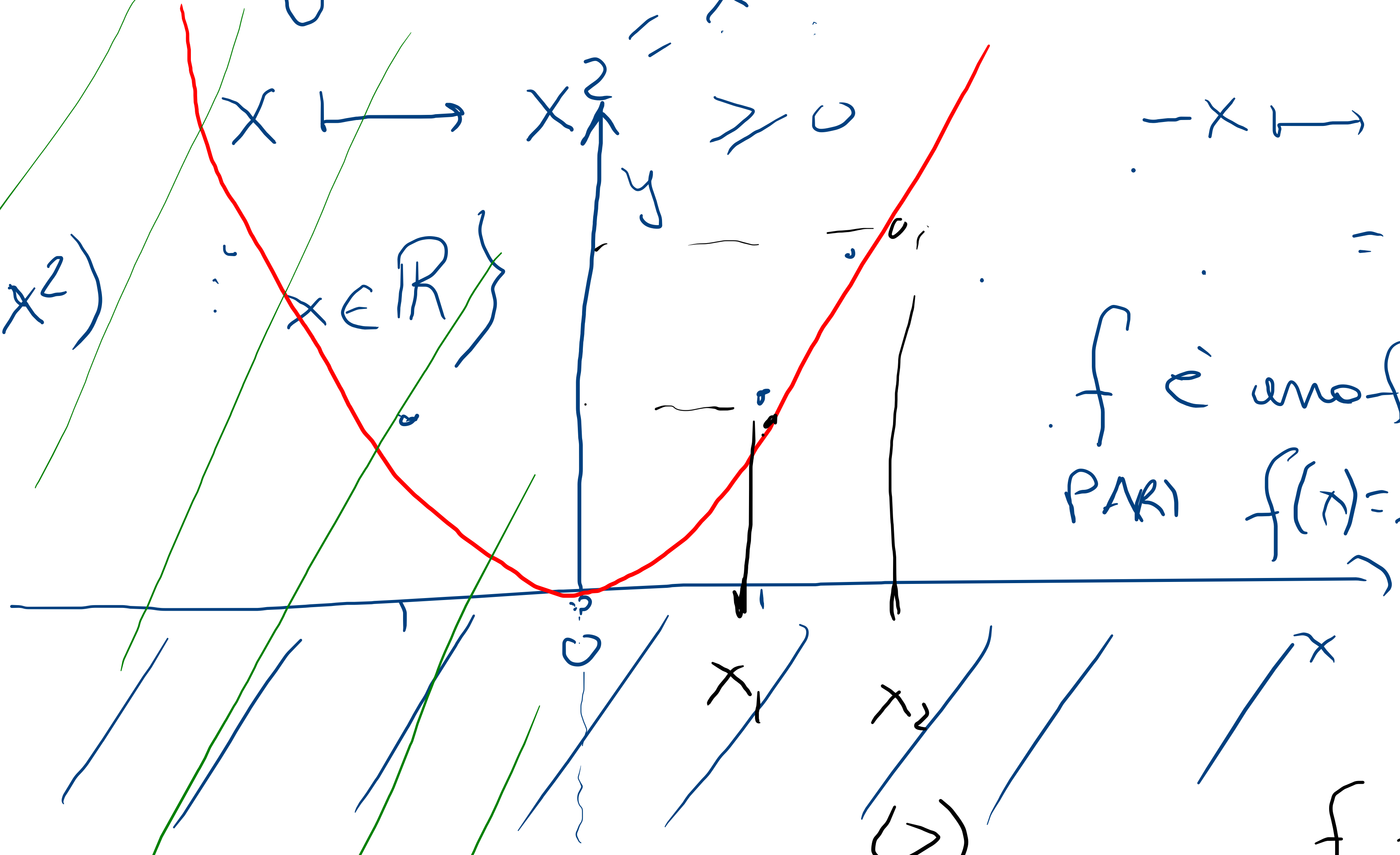
$$|x| = \sqrt{x^2}$$

$$y = x^2$$

$$x \longmapsto x^2 = x \cdot x \geq 0$$

$$\{(x, x^2) : x \in \mathbb{R}\}$$

$$-x \longmapsto (-x)^2 = x^2$$



f è una funzione PARI $f(x) = f(-x)\ \forall x$

$x_1 < x_2$

$$(>)$$

$$f(x_1) < f(x_2) \iff$$

f funzione CRESCENTE

(DECRESCENTE

Def Una funzione crescente o decrescente sull'intervallo $(a,b) \subseteq \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ è detta MONOTÓNA.

Es $x \longmapsto x^2$ è monotono crescente in $(0, +\infty)$

monotono decrescente in $(-\infty, 0)$

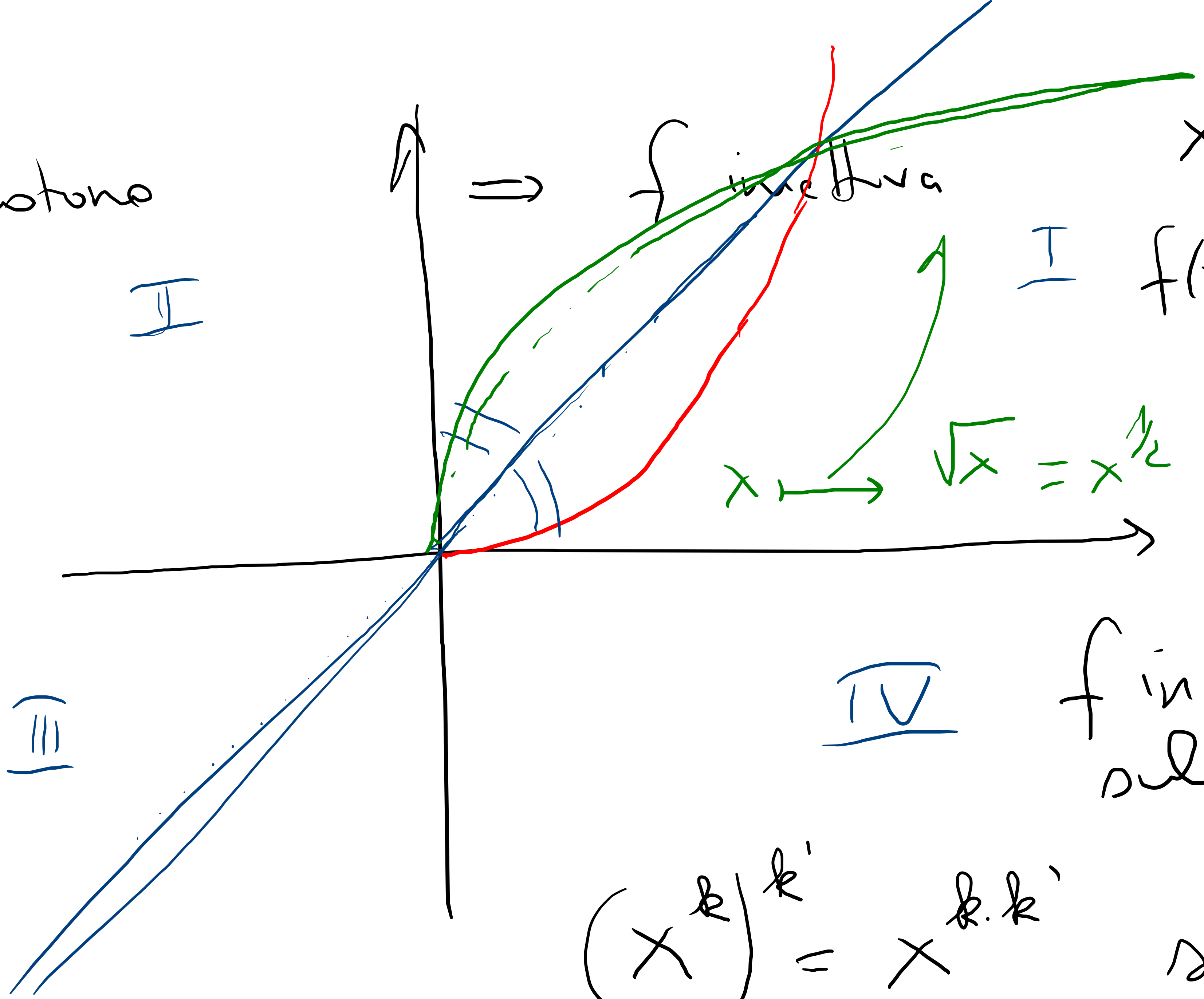
f monotono $\Rightarrow$ f iniettiva

$x_1 \neq x_2$

$f(x_1) \neq f(x_2)$

$\Downarrow$

f invertibile sull'immagine

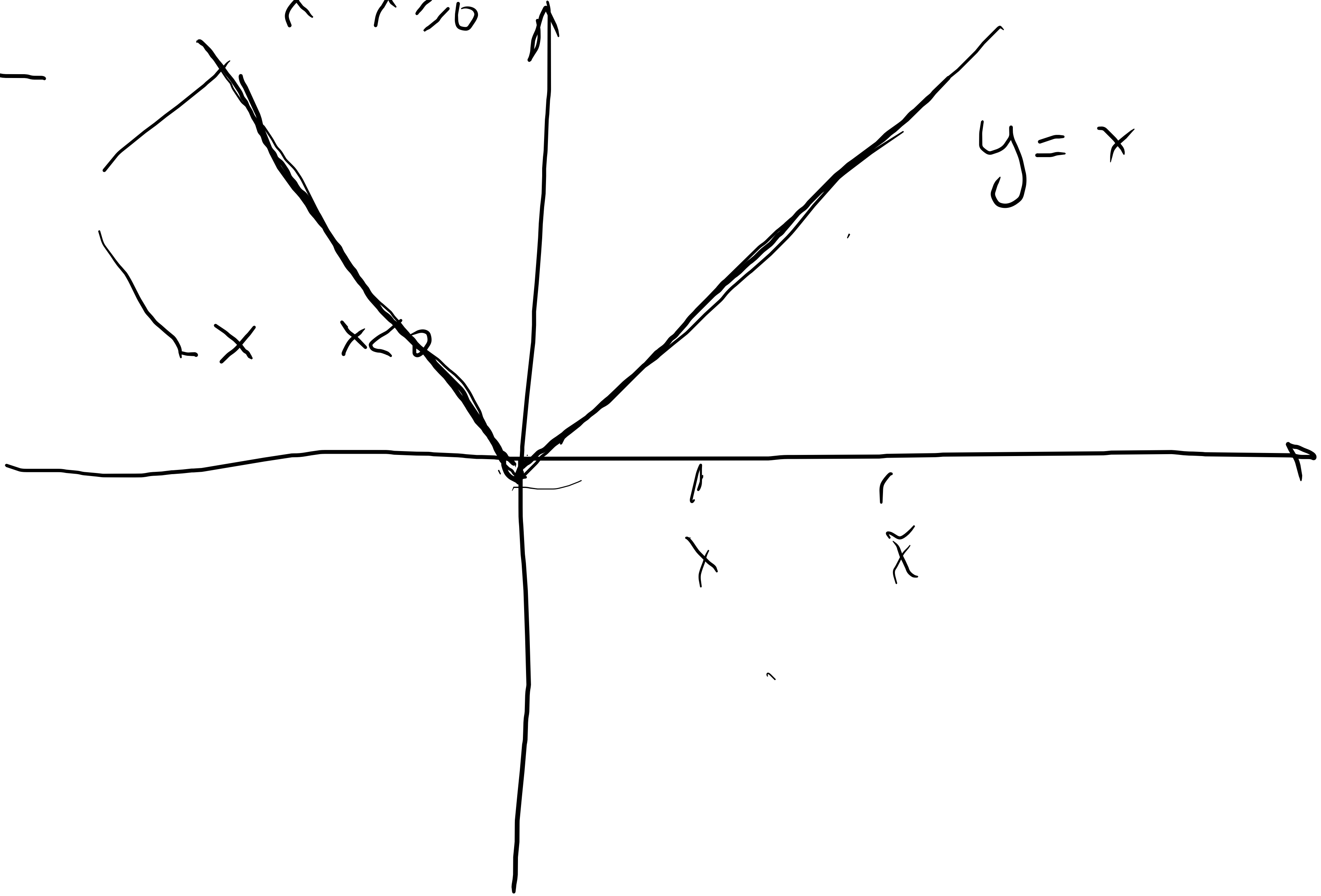


$$(x^k)^{k'} = x^{k \cdot k'}$$

se $k \neq 0$

$\frac{1}{k}$ ha senso

$$|x| = \sqrt{x^2} = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



$$\boxed{\lim_{x \to x_0} f(x) = \underline{\underline{l}}}$$

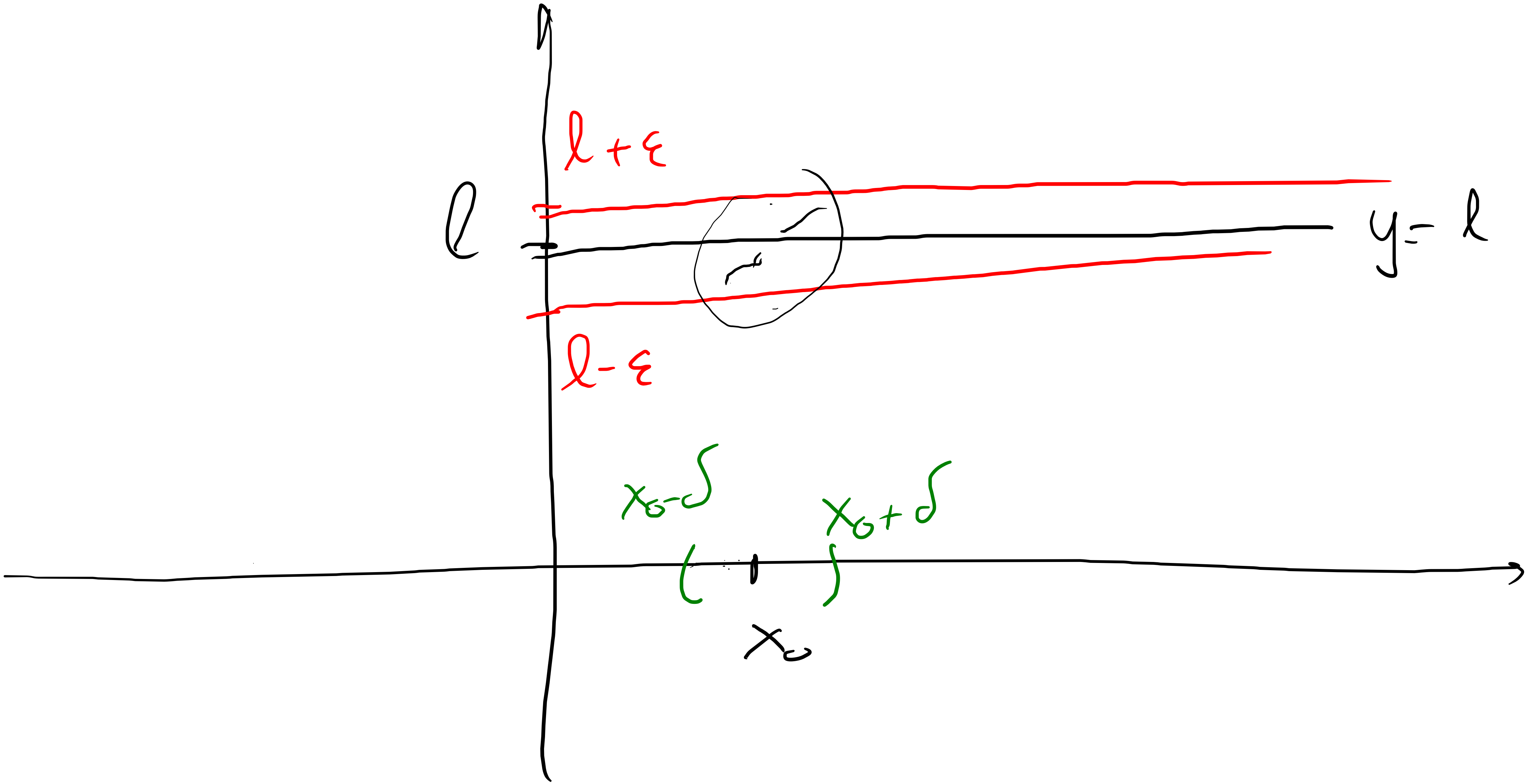
$\underline{\varepsilon > 0}$ $\quad \delta > 0$

$\underline{x_0 \in \mathbb{R}}$ $\quad \underline{\underline{l \in \mathbb{R}}}$ $\quad I_l = \{ |y - l| < \varepsilon \}$

$l - \varepsilon < y < l + \varepsilon$

$I_{x_0 \delta} = \{ \quad |x - x_0| < \delta \}$

$-\delta < x - x_0 < \delta \qquad x_0 - \delta < x < x_0 + \delta$



l+ε
l
y=l
l−ε
x0−δ
x0+δ
x0

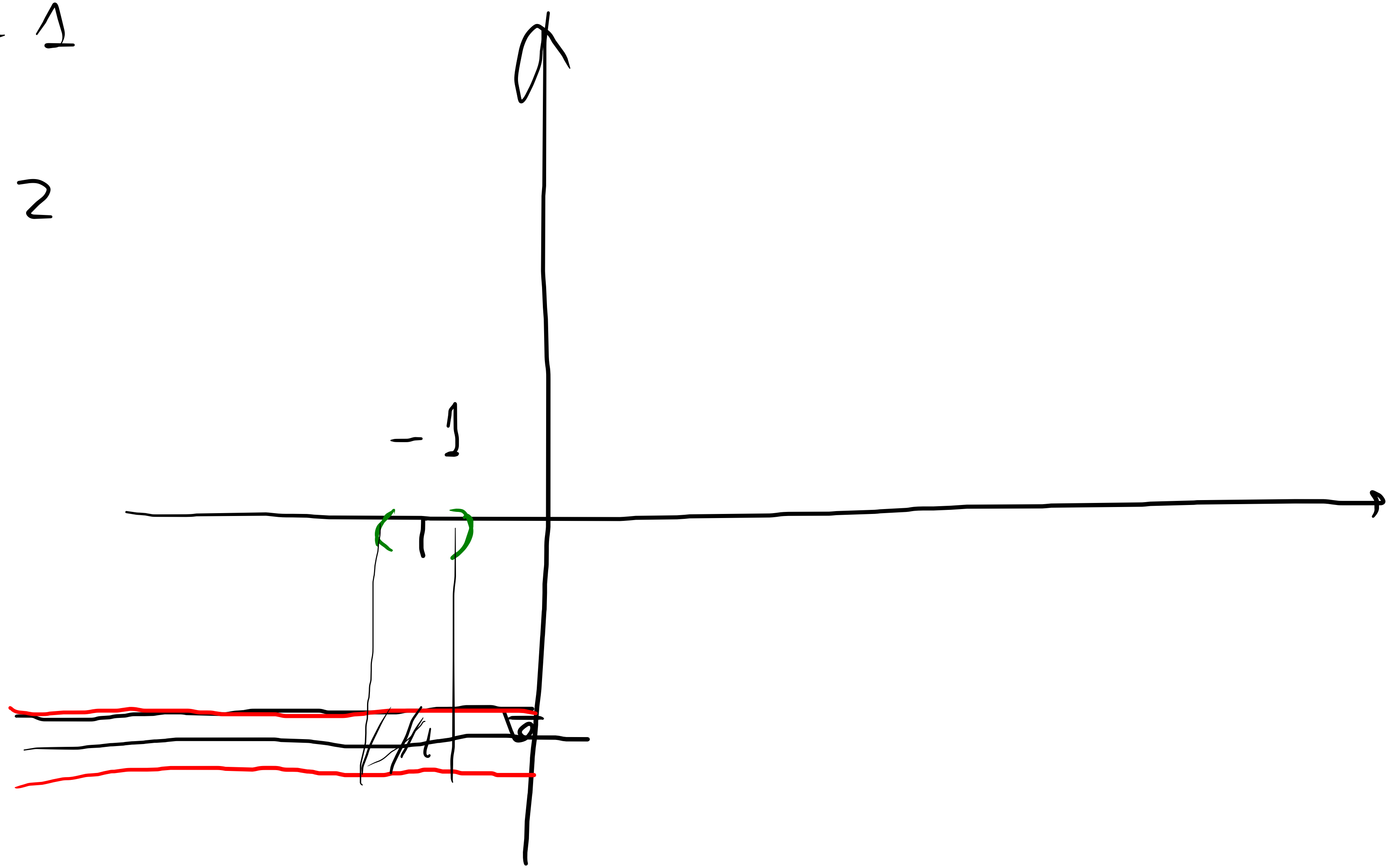
$$\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} \frac{(x-1)\cdot(x+1)}{x+1} = -2$$

$$f(x) = \frac{x^2+1}{x+1}$$ ha per dominio $\mathbb{R} \setminus \{-1\} =$

$$= (-\infty, -1) \cup (-1, +\infty)$$

$x_0 = -1$

$l = -2$



$$\lim_{\substack{x \to +\infty \\ (-\infty)}} f(x) = l \qquad l \in \mathbb{R}$$

$\forall \varepsilon > 0$ $\quad I_l = \{ |y - l| < \varepsilon \}$

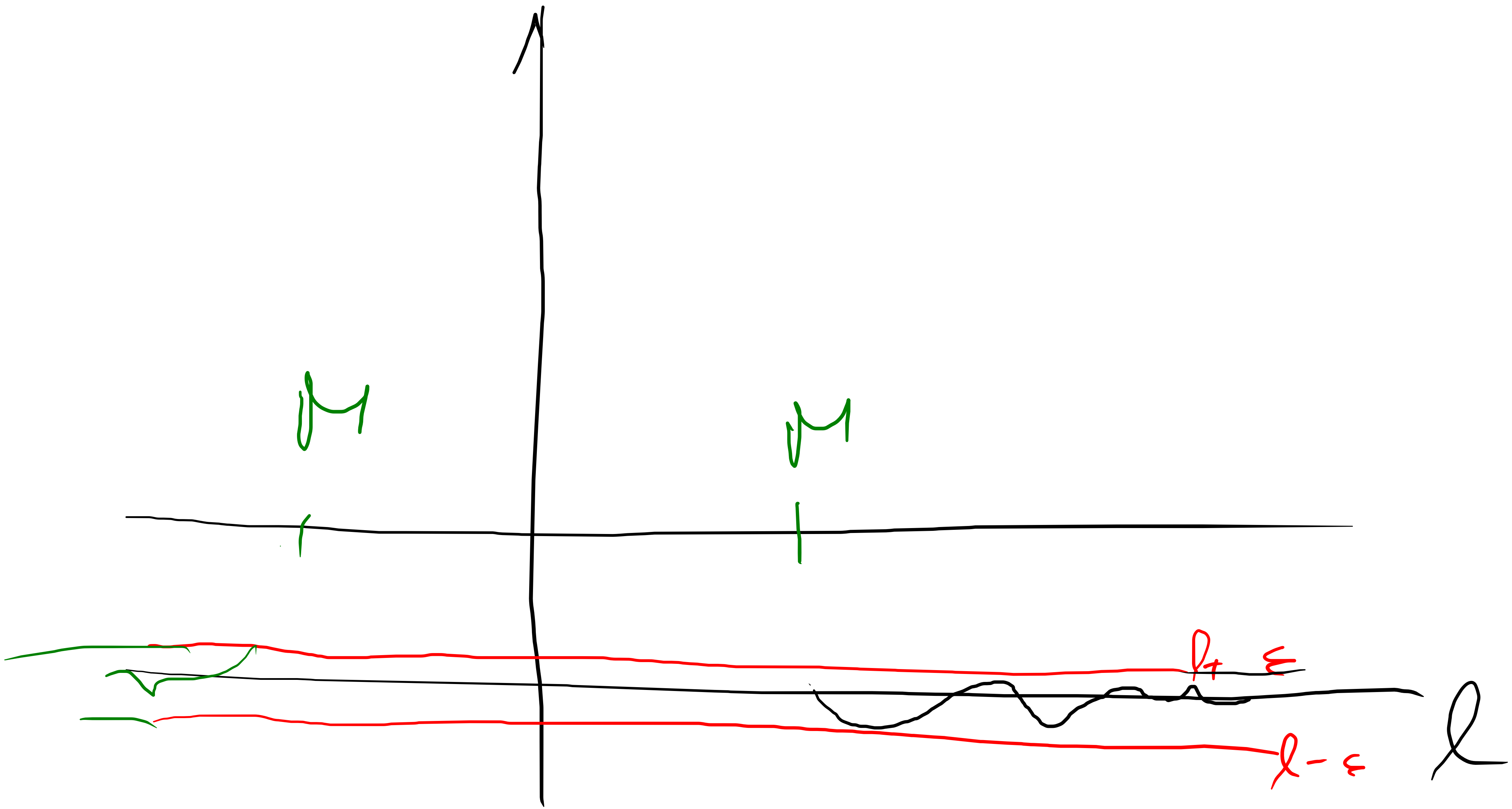
per ogni

$\exists M > 0$ $(M < 0)$ tale che

$(x < M)$

se $x > M \wedge x \in \mathrm{Dom} f \Rightarrow |f(x) - l| < \varepsilon$

$x \in I_{+\infty}$ $\qquad f(x) \in I_l$



M
M
l+ε
l
l−ε

$$\lim_{x \to x_0} f(x) = +\infty$$

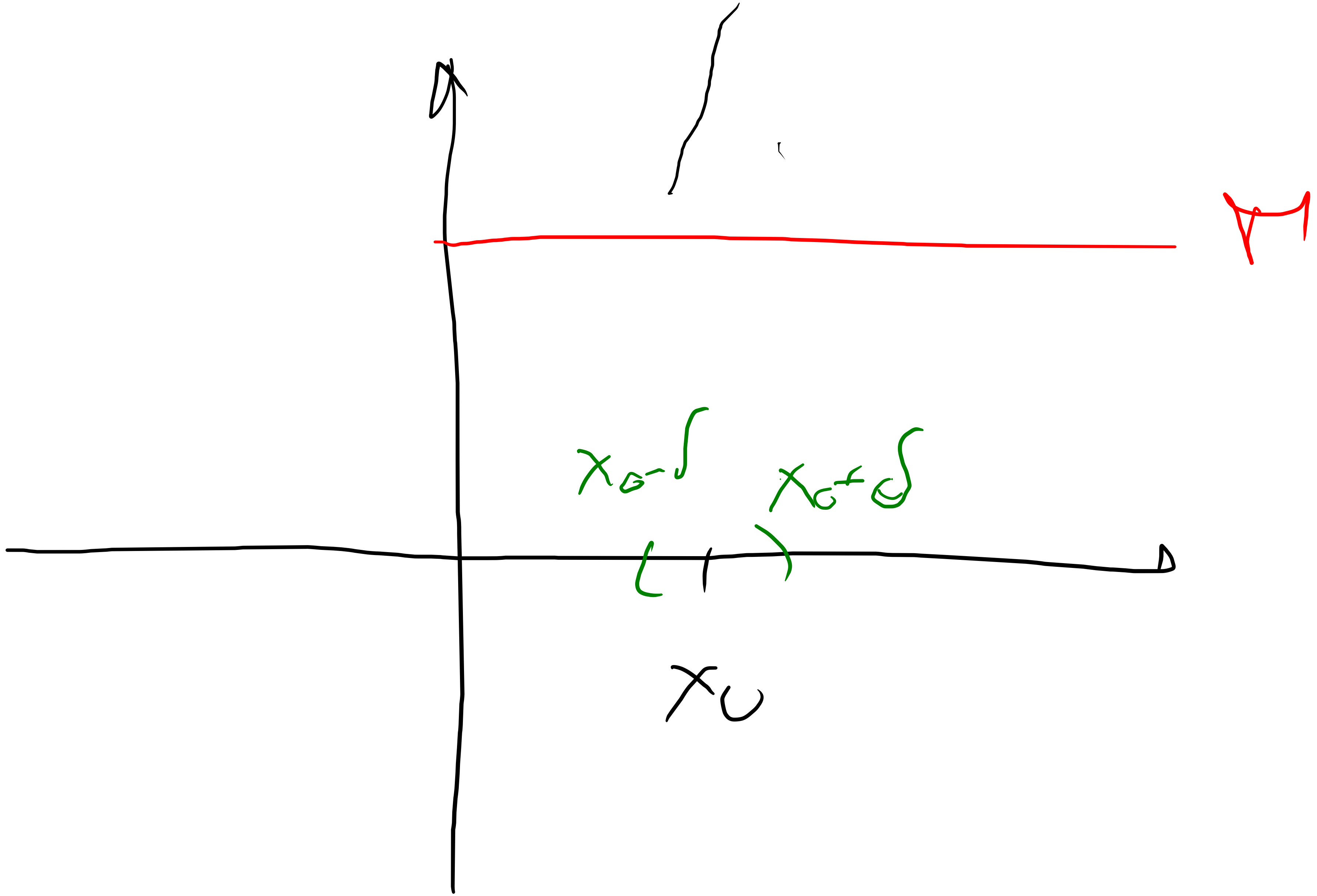
$x_0 \in \mathbb{R}$

$\forall M > 0$

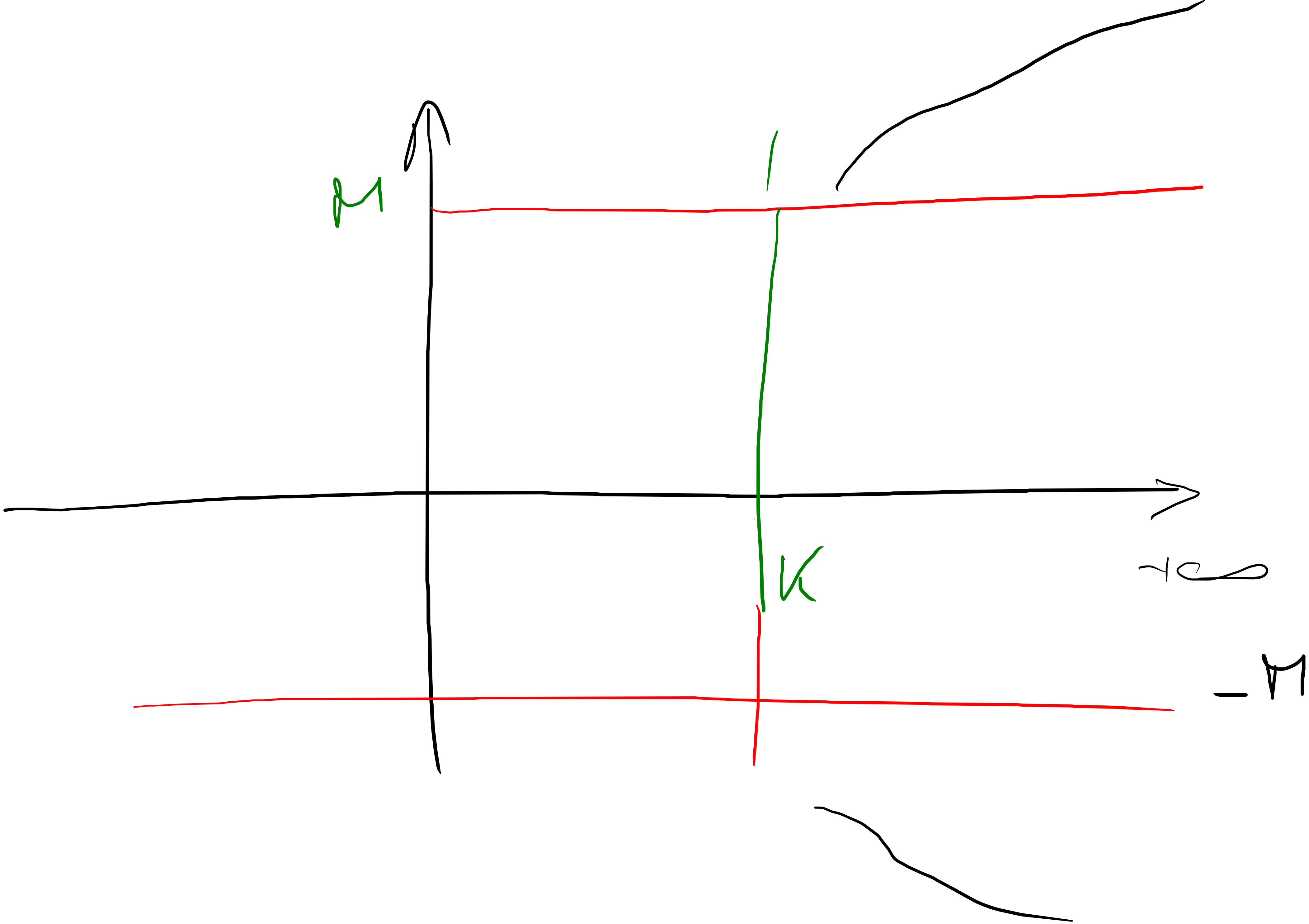
$\exists \delta > 0 \ : \ |x - x_0| < \delta$

$x \in \mathrm{Dom} f$

$f(x) > M$



M
$x_0-\delta$
$x_0+\delta$
x_0

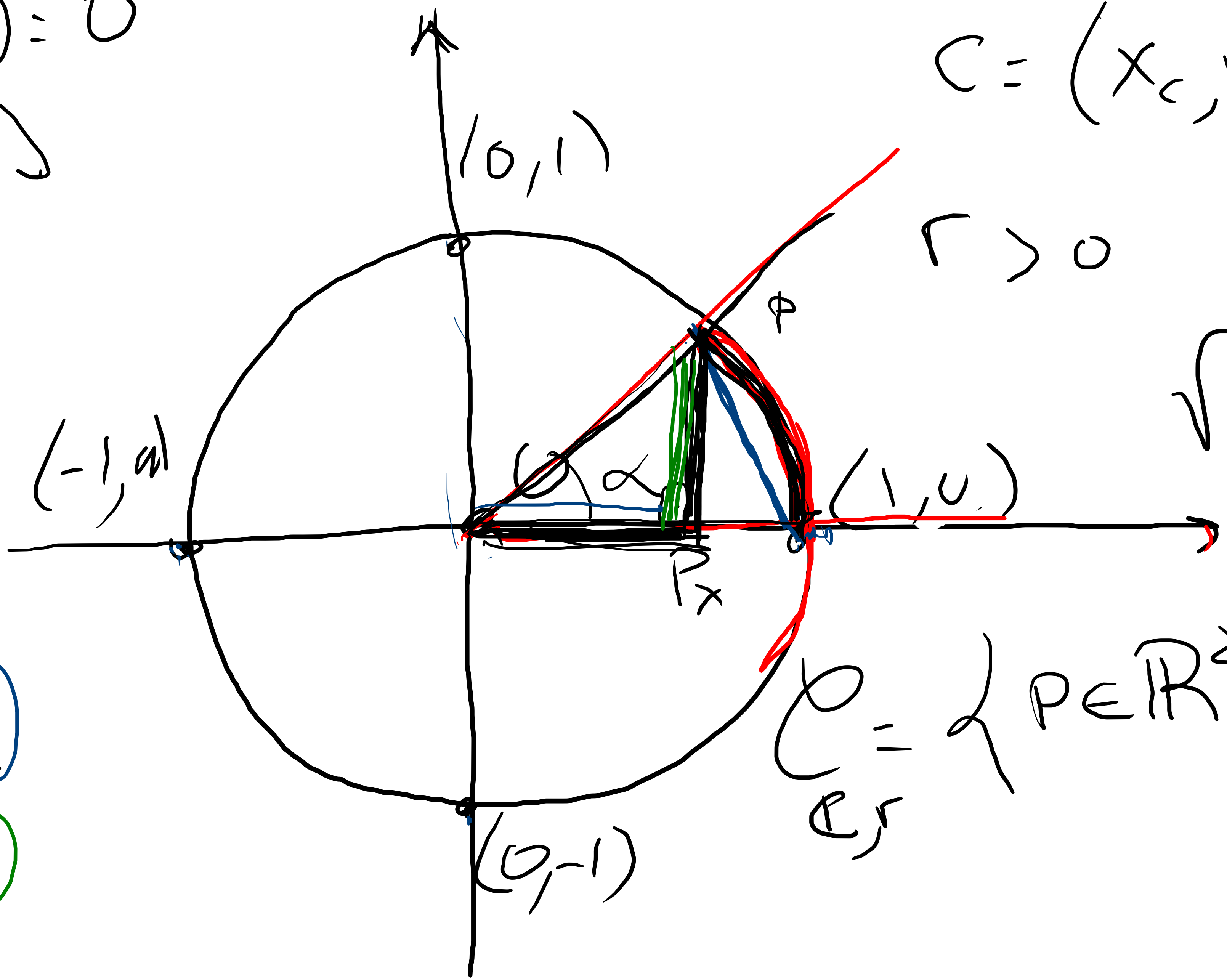

M
K
+∞
−M

Non è detto che una funzione abbia LIMITE per $x \to x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$

$\operatorname{sen} x$ non ha limite per $x \to +\infty$
$(\cos x)$ o $x \to -\infty$
benchè sia definita $\forall x \in \mathbb{R}$

$C = (0,0) = O$

$r = 1$



$C = (x_c, y_c)$

$r > 0$

$P = (x,y)$

$\sqrt{x^2+y^2} = 1$

$x^2+y^2=1$

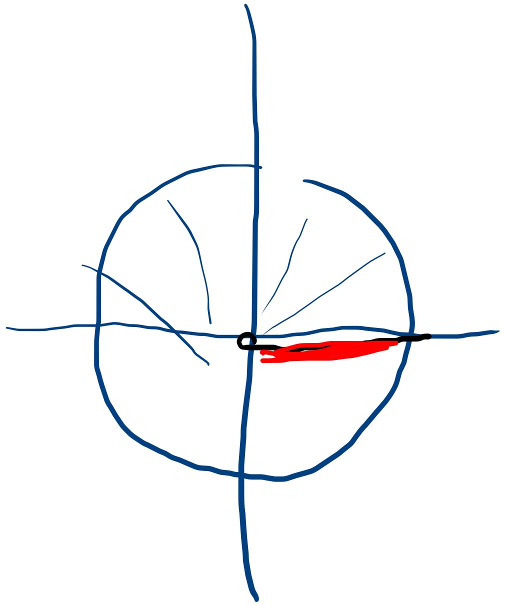
$\mathcal{C}_{C,r} = \{ P \in \mathbb{R}^2 : d(P,C) = r \}$

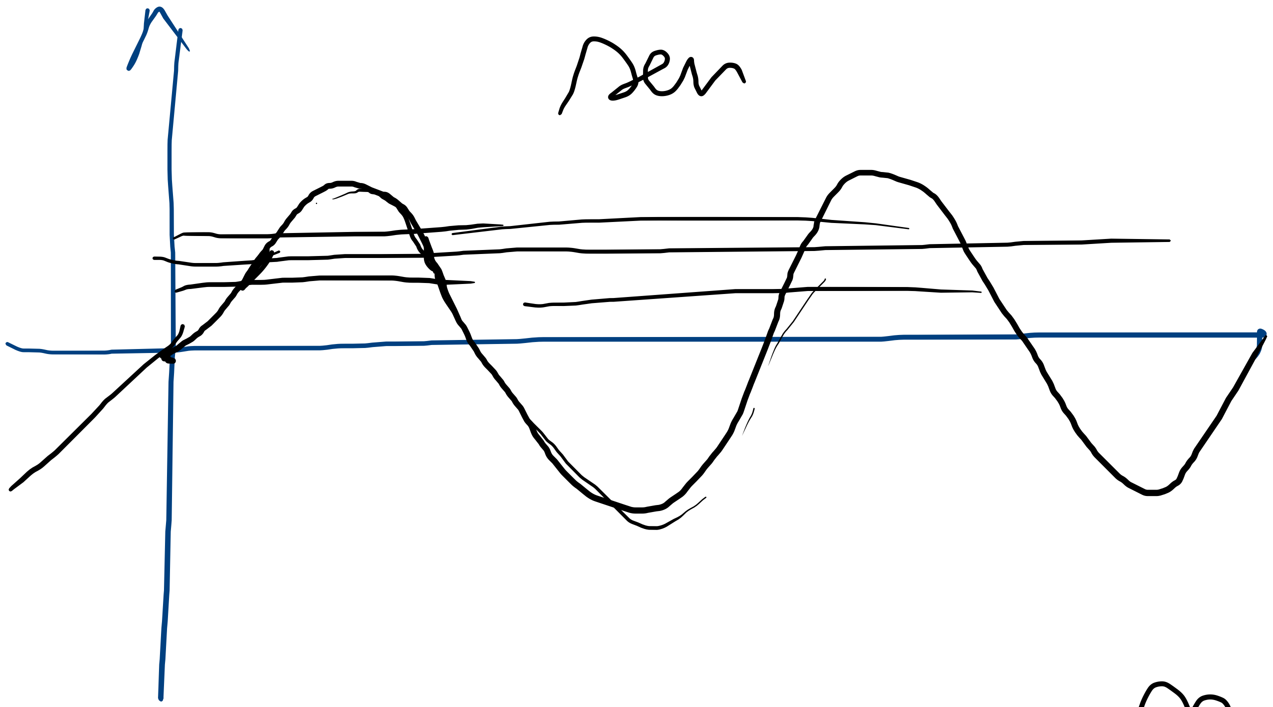
cos

$\alpha \longmapsto \overline{OP_x}$

sen

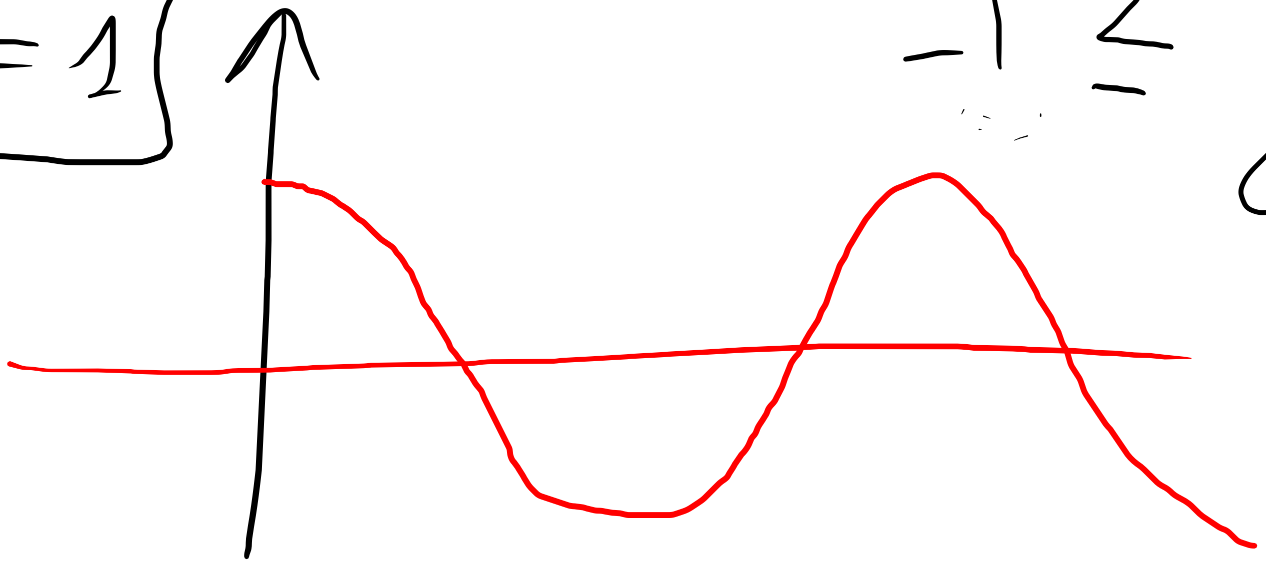
$\alpha \longmapsto \overline{PP_x}$



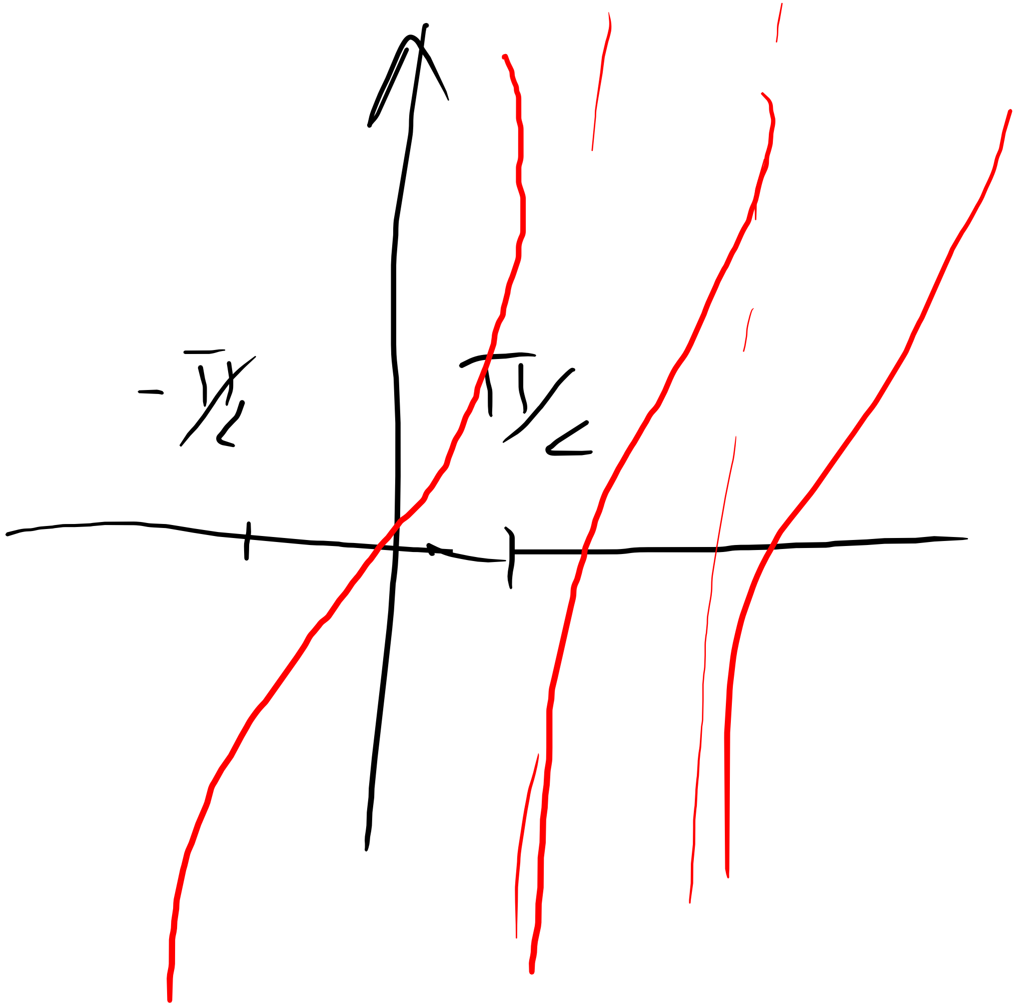


$$\boxed{\cos^2 x + \operatorname{sen}^2 x = 1}$$

$$-1 \leq \begin{matrix} \operatorname{sen} x \\ \cos x \end{matrix} \leq 1$$



$$\boxed{\operatorname{tg} x =: \frac{\operatorname{sen} x}{\cos x}}$$

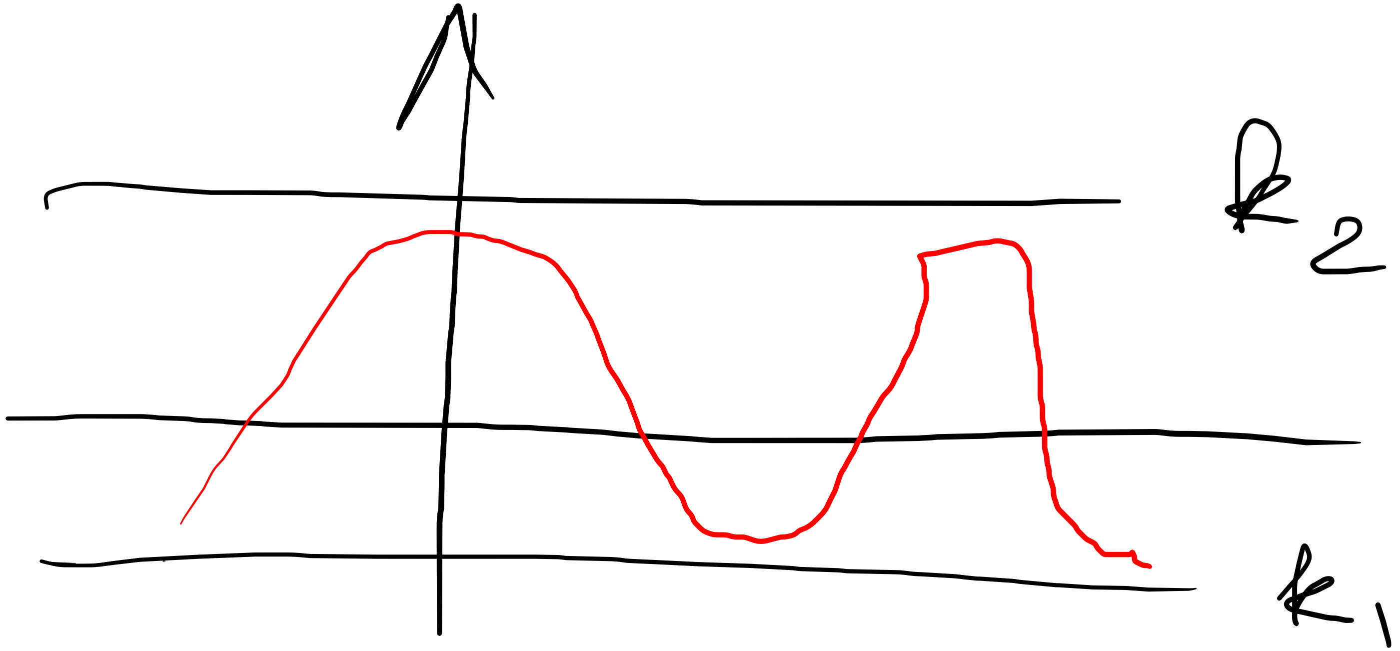


NON è limitata
ed è definita
in $\mathbb{R} \setminus \{ k\pi/2 \quad k \in \mathbb{Z} \}$

$$\lim_{x \to \pi/2^+} \operatorname{tg} x = -\infty$$

$$\lim_{x \to \pi/2^-} \operatorname{tg} x = +\infty$$

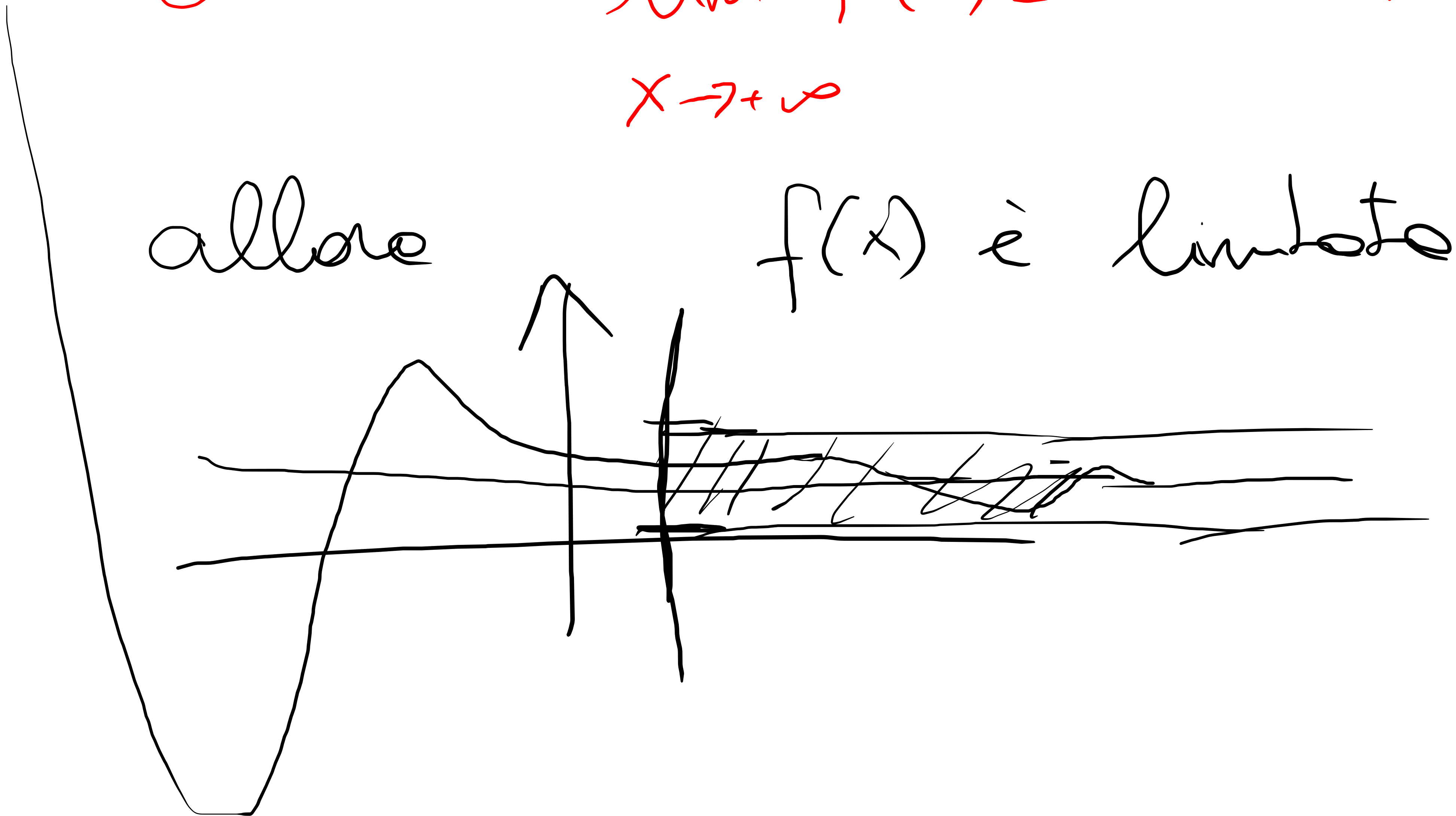
Def Una funzione reale si dice **LIMITATA** in un intervallo (a,b) se $\exists\, k_1, k_2 \in \mathbb{R}$: $\forall\, x \in (a,b)$ $\quad k_1 < f(x) < k_2$



<u>Se</u> f ammette limite ^FINITO^ per $x \to +\infty$

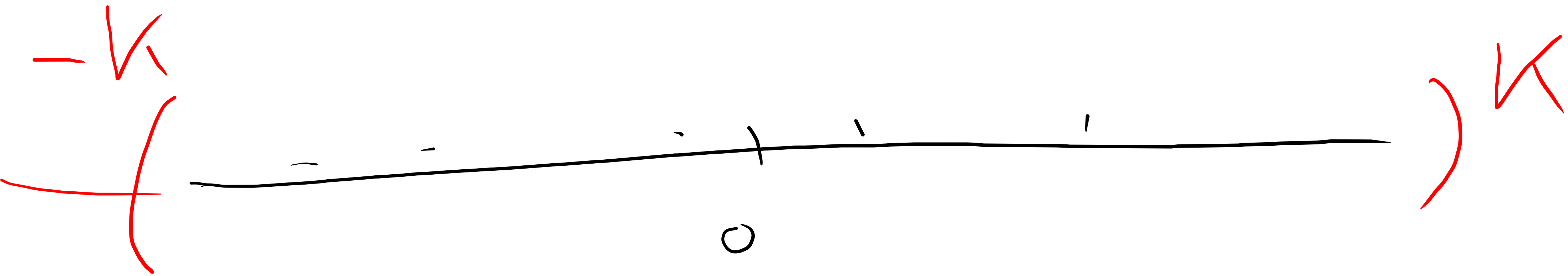
Cioè se $\lim\limits_{x \to +\infty} f(x) = l \qquad l \in \mathbb{R}$

allora $f(x)$ è limitata se $x > M$.



Diremo che un insieme A di numeri reali è limitato se $\exists\ K>0$

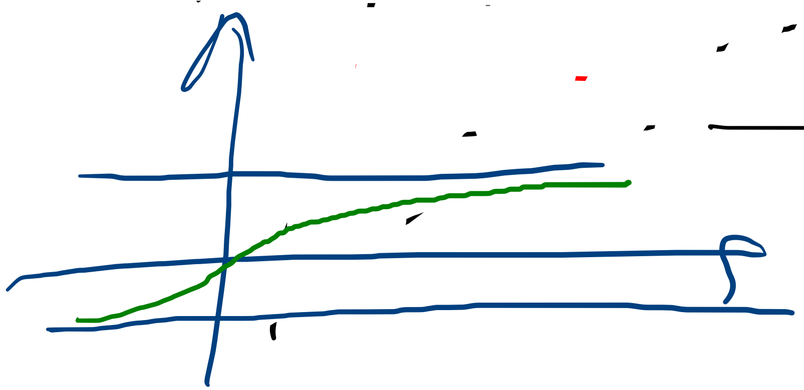
$$\forall\ x\in A \quad |x|<K \iff -K<x<K$$



Prop Una funzione monotona in $\mathbb{R}$ (o crescente o decrescente) ha limite per $x \to +\infty$, $x \to -\infty$

① f illimitata (f crescente)

② f limitata



Prop

Se $\exists$ il limite di una funzione

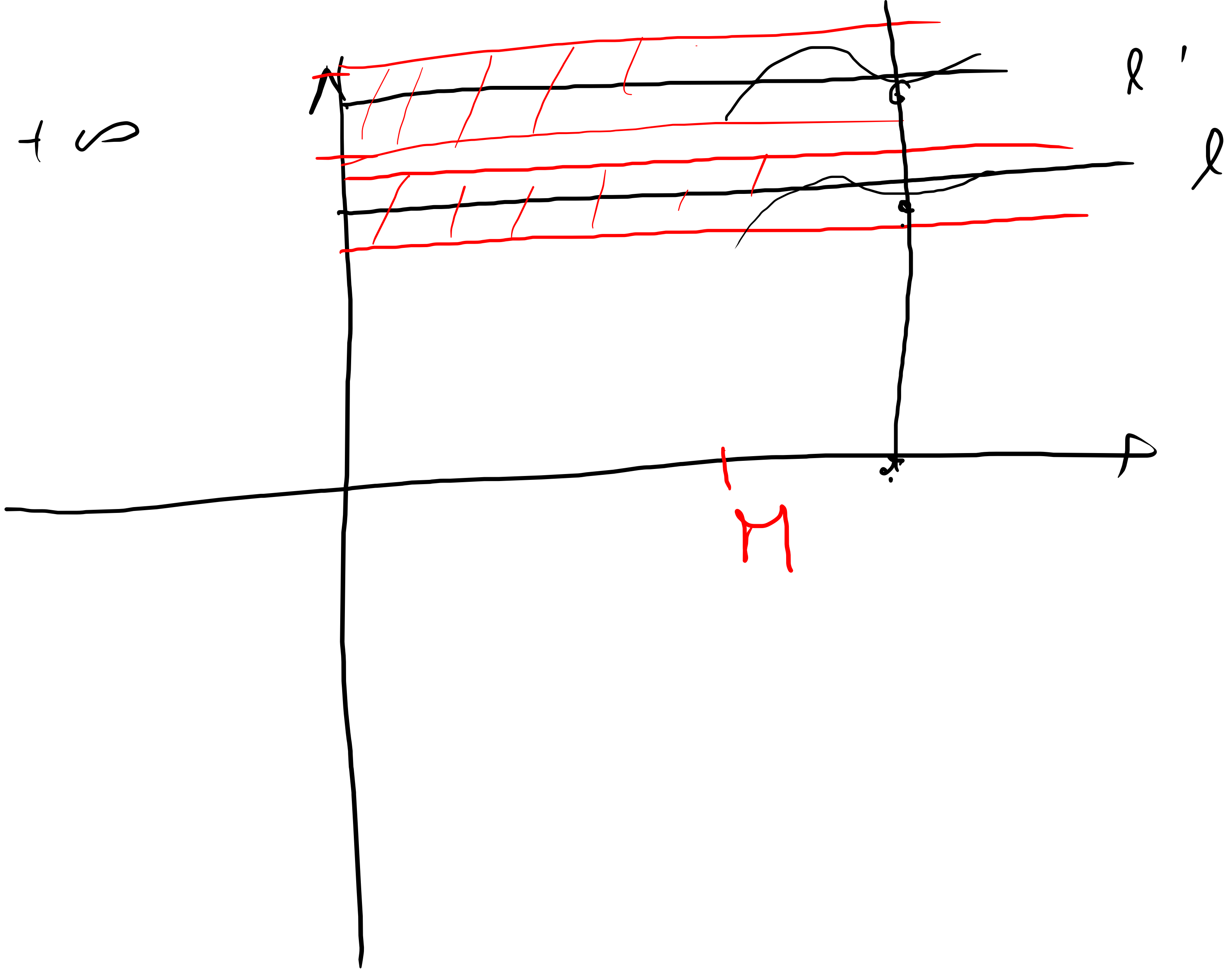
esso è UNICO, cioè se $\lim_{x \to x_0} f(x) = l$

$\Rightarrow$ l è unico

cioè non può essere che $\lim_{x \to x_0} f(x) = l$

$\lim_{x \to x_0} f(x) = l' \neq l$

$x_0 = +\infty$



$$\varepsilon < \frac{(\ell' - \ell)}{3}$$

$$I_{\ell,\varepsilon} \cap I_{\ell',\varepsilon} = \emptyset$$

$x \in \mathbb{R}$

f g f, g : funzioni reali

$x \longrightarrow f(x)$ $x \longrightarrow g(x)$

$x \xmapsto{h} f(x)+g(x)$ $h = f+g$

$x \xmapsto{c} f(x) \cdot g(x)$ $c = f \cdot g$

Prop (Compatibilità delle operazioni con i limiti)

Siano f, g reali di variabile reale

tali che $\lim_{x \to x_0} f(x) = l$ $\quad x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$

$\lim_{x \to x_0} g(x) = l'$ $\quad l', l \in \mathbb{R} \cup \{-\infty, +\infty\}$

① $\lim_{x \to x_0} \left(f(x)+g(x)\right) = l+l'$ salvo il caso in cui siamo $l = +\infty$ $l' = -\infty$ oppure $l = -\infty$ $l' = +\infty$

e con la convenzione che se $l = l' = +\infty$ $(-\infty)$ $\Rightarrow$ $l + l' = +\infty$ $(-\infty)$

② $\lim_{x \to x_0} f(x) \cdot g(x) = l \cdot l'$

SALVO i casi in cui $l = \pm\infty$ $l' = 0$

$l = 0$ $l' = \pm\infty$

con le convenzione se $l = +\infty$ $l' = +\infty$

$l \cdot l' = +\infty$

se $l = +\infty$ $l' = -\infty$

$l \cdot l' = -\infty$

se $l = -\infty$ $l' = -\infty$

$\boxed{l \cdot l' = +\infty}$

se $l = -\infty$ $l' = +\infty$

$l \cdot l' = -\infty$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{l}{l'}$$

se $l' \neq 0$ e $l \in \mathbb{R}$

SALVO i casi $l = \pm\infty$ $l' = \pm\infty$

$l = 0$ $l' = 0$

---

con la convenzione $l = +\infty$ $l' = 0$ $\frac{l}{l'} = {+\infty \atop -\infty}$ a seconda del segno

$l \in \mathbb{R}$ $l' = \pm\infty$ $\boxed{\frac{l}{l'} = 0}$

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \qquad \frac{x}{x^2+2} = \frac{x}{\cancel{x} \cdot \left(x + \frac{1}{x}\right)} =$$

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0 \qquad = \frac{1}{x + \frac{1}{x}} \to 0$$

$$\lim_{x \to +\infty} \frac{2-x}{x^2+1} = \lim_{x \to +\infty} \left(\frac{2}{x^2+1} - \frac{x}{x^2+1} \right) = 0$$

Def

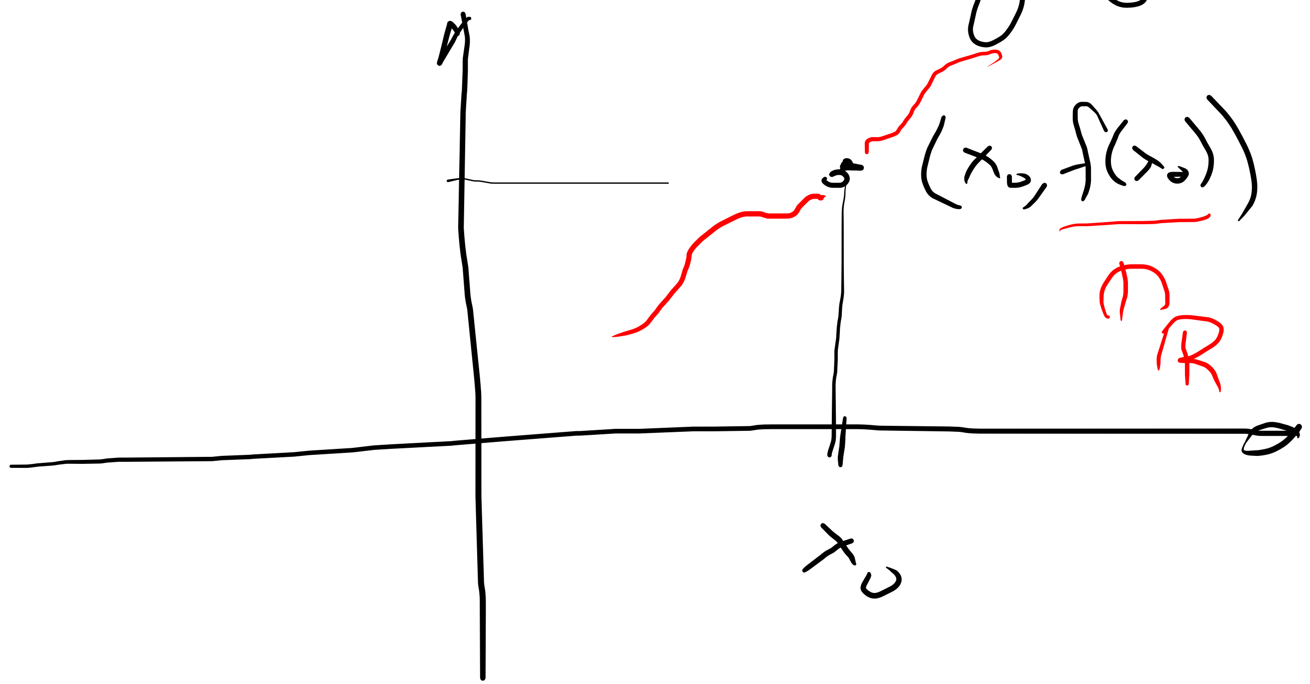
A aperto

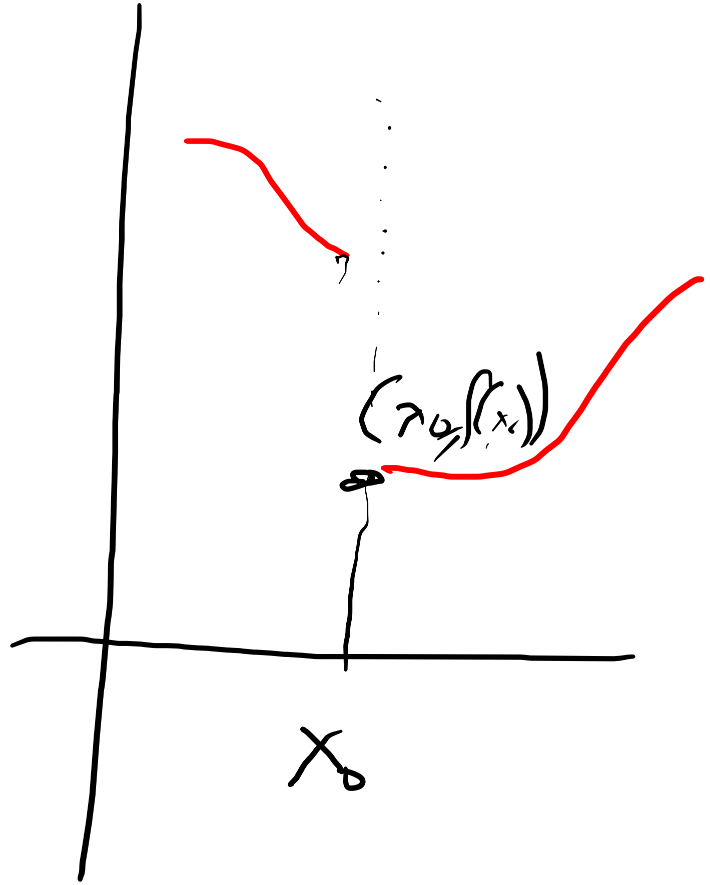
Diremo che la funzione $f: A \subseteq \mathbb{R} \to \mathbb{R}$

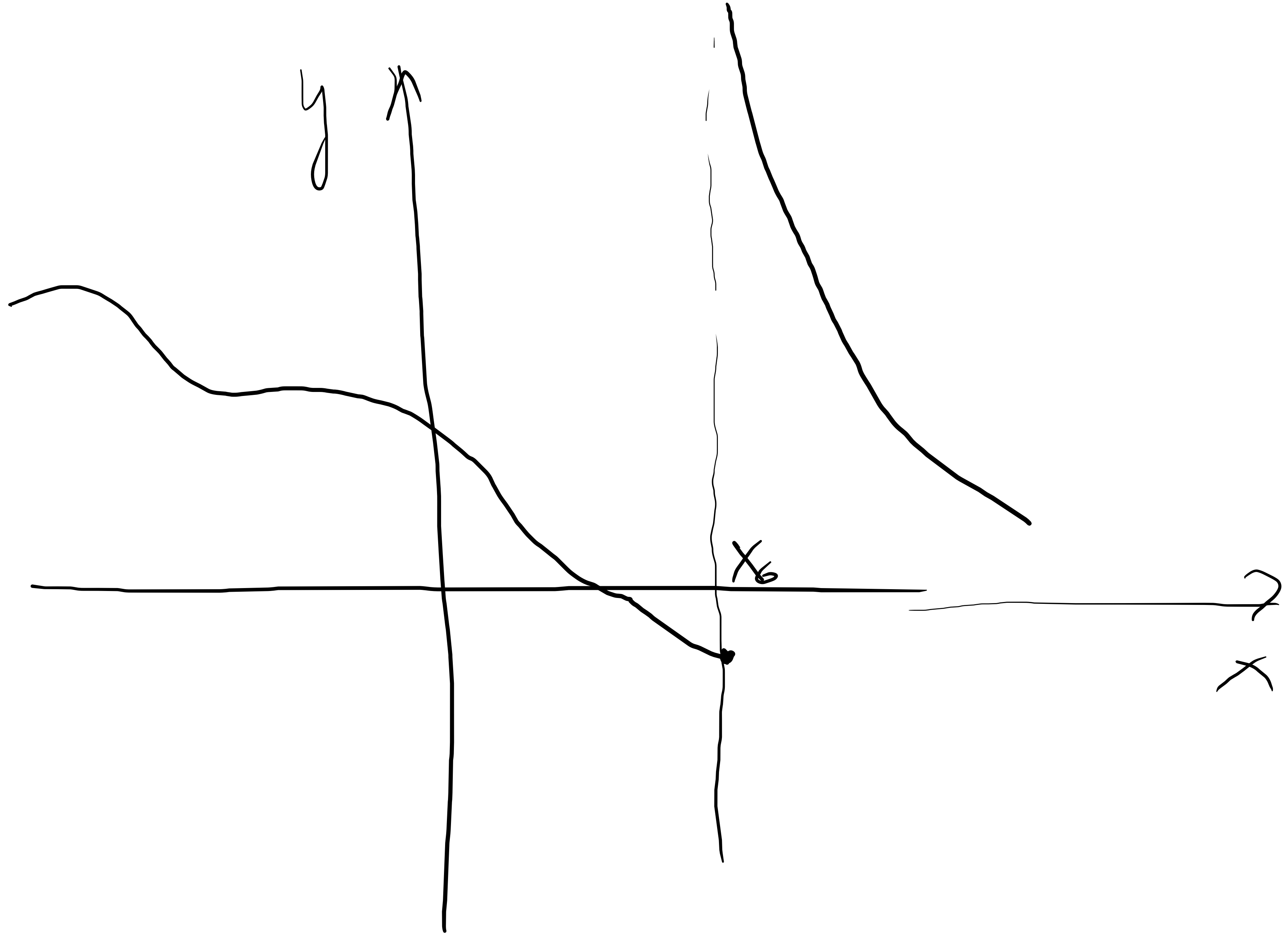
è CONTINUA in $x_0 \in A$ se

$$\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$$

Graficamente, una funzione CONTINUA non può presentare dei SALTI o interruzioni nel grafico






y
x_0
x

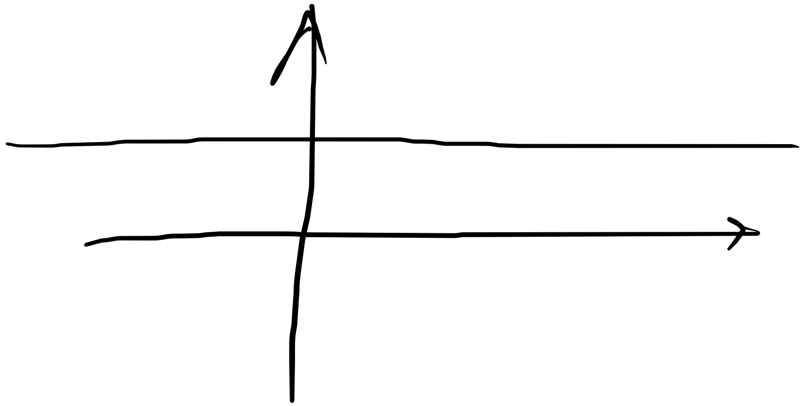
Se utilizziamo la proposizione che mostra la compatibilità delle operazioni con i limiti, riusciamo a concludere che se f è continua in x_0 e g è continua in x_0, allora $f \pm g$ e $f \cdot g$ sono continue in x_0.

Inoltre se $g(x_0) \neq 0$, anche f/g è continua in x_0

Prop Se f è continua in x_0 e g è continua in $f(x_0)$ (e se ha senso considerare la funzione composta $g \circ f$) allora $g \circ f$ è continua in x_0

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[rr, bend right, "g \circ f"'] & f(x) \arrow[r, mapsto, "g"] & g(f(x)) \end{tikzcd}$$

$f(x) = x$ è continua



$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x = x_0 = f(x_0) \checkmark$$

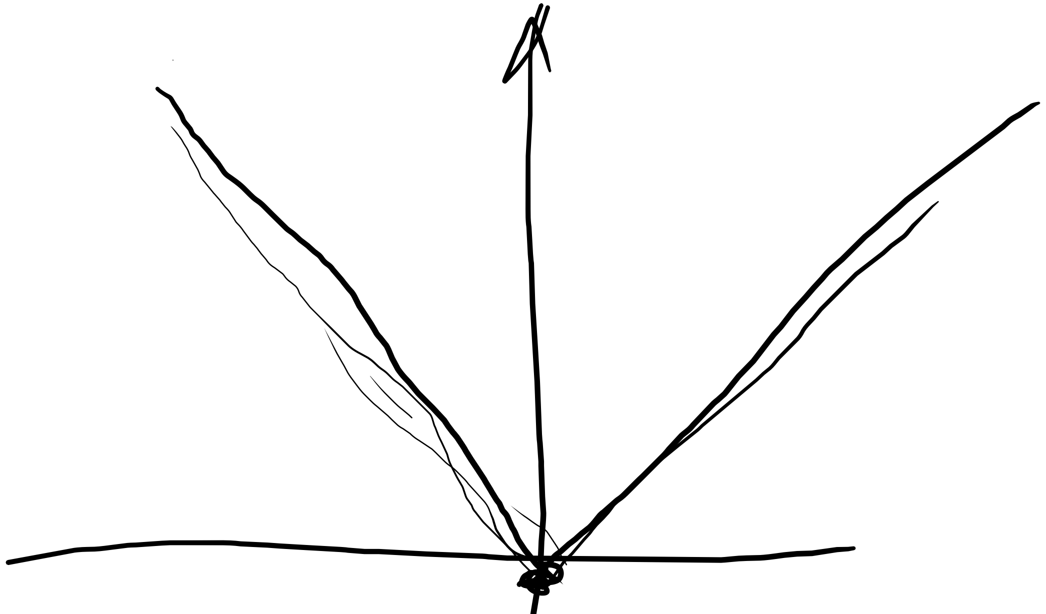
Quindi x^2, x^3, ..., x^4 sono continue

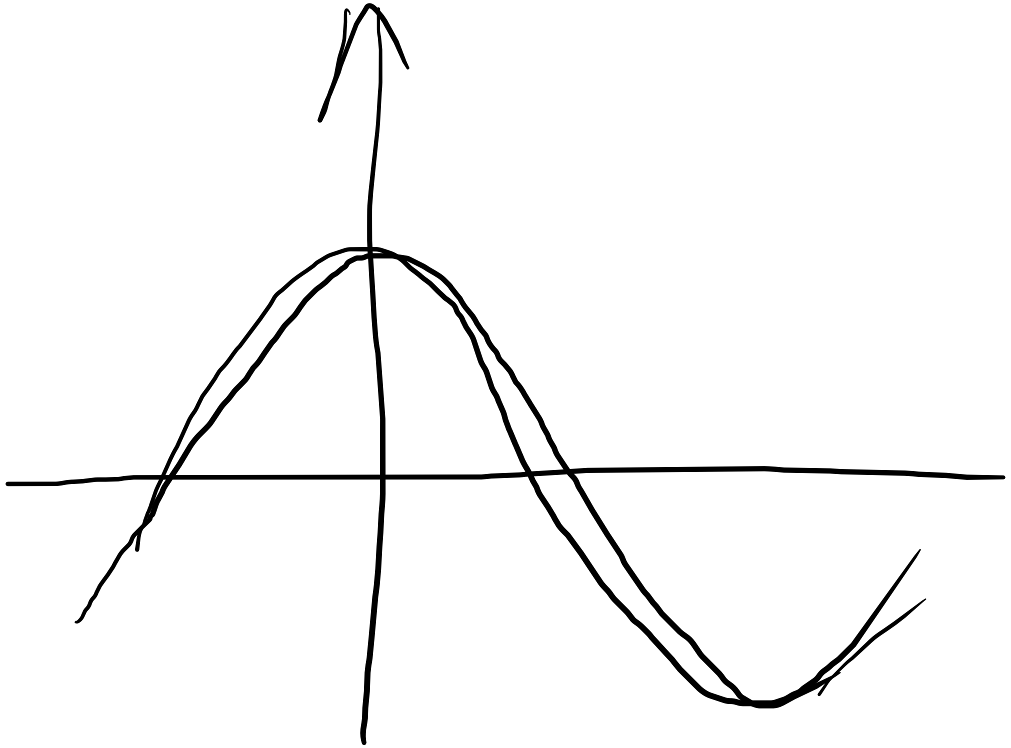
$x \mapsto 2x^2 + 3x^5 - x^7$ è continua

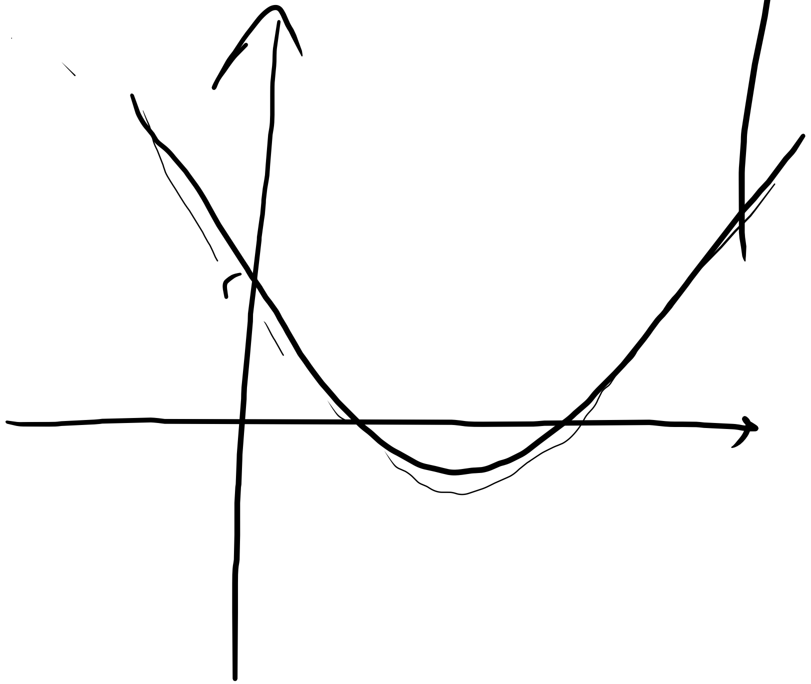
Se $p(x)$ è un polinomio $x \mapsto p(x)$ è continua in x_0 $\forall x_0 \in \mathbb{R}$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\sqrt{x^2} = |x|$$







$y = 2x^2 - 3x + 1$

sen e cos sono funzioni continue

(pertanto tg è continua in $\mathbb{R} \setminus \{k\frac{\pi}{2}\}$)

$$tg(x) = \frac{\operatorname{sen} x}{\cos x}$$

$x \longmapsto a^x$ $\qquad a \in \mathbb{R} \quad a > 0 \quad (a \neq 1)$

esponenziale di base a

$x \longmapsto x^k$ $\qquad k \in \mathbb{R}$

In particolare $a > 1$

$x \longmapsto a^x$ è <u>monotono crescente</u>

$a > 1 \qquad a = 1+r \qquad r > 0$

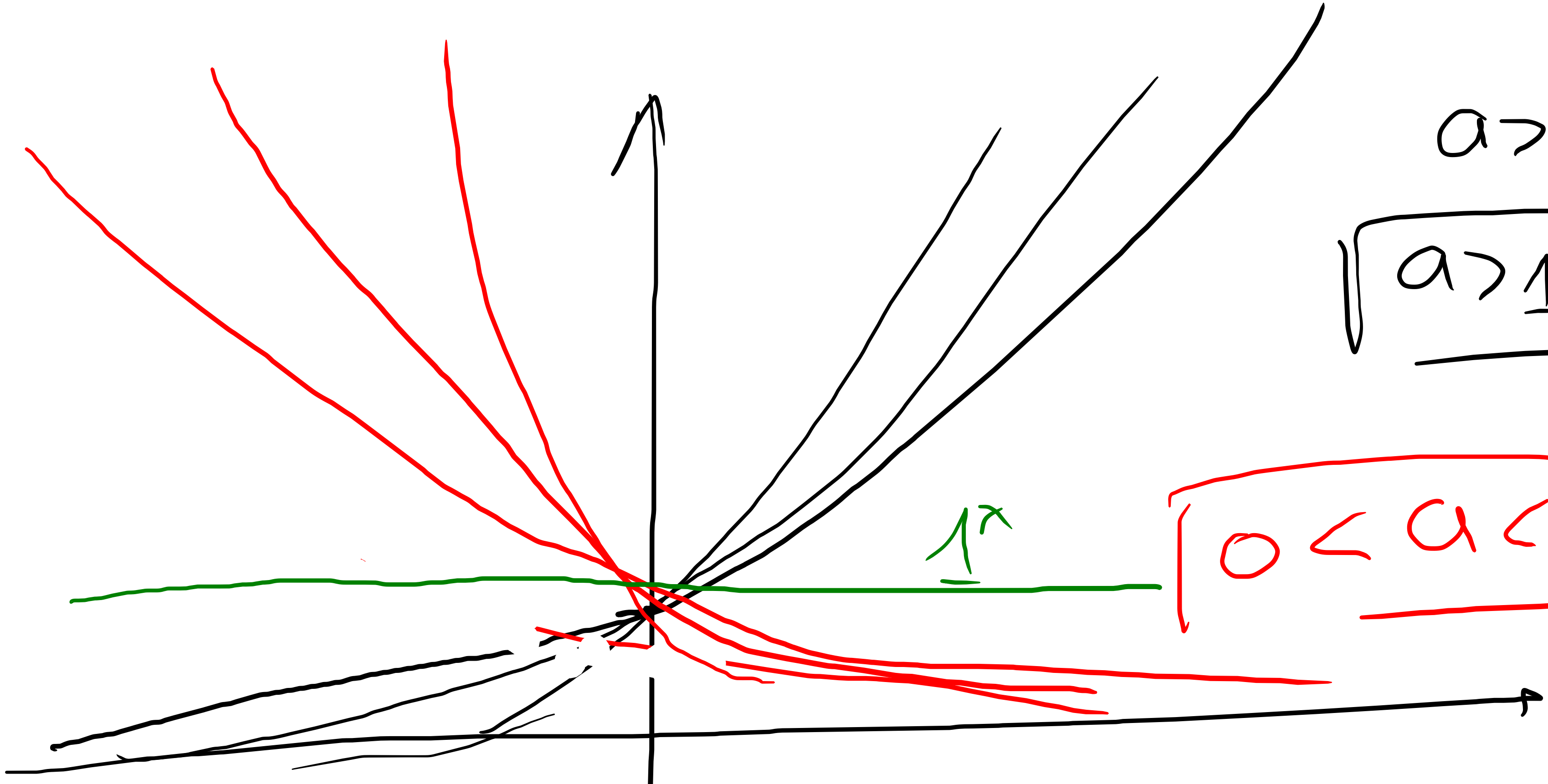
$(1+r)^{x_1} > (1+r)^{x_2} \qquad x_1 > x_2$

$$\boxed{a^0 = 1}$$

$a > 0 \quad a^x > 0$

$\boxed{a > 1}$

$a^0 = 1$



$\boxed{0 < a < 1}$

$\boxed{a \neq 0}$

$a > 0$

Ogni funzione ESPONENZIALE

$x \mapsto a^x \quad 0 < a \quad (a \neq 1)$

è MONOTONA in particolare

crescente se $a > 1$ decrescente se $0 < a < 1$.

$\exists$ una funzione inversa di a^x detta LOGARITMO in base a

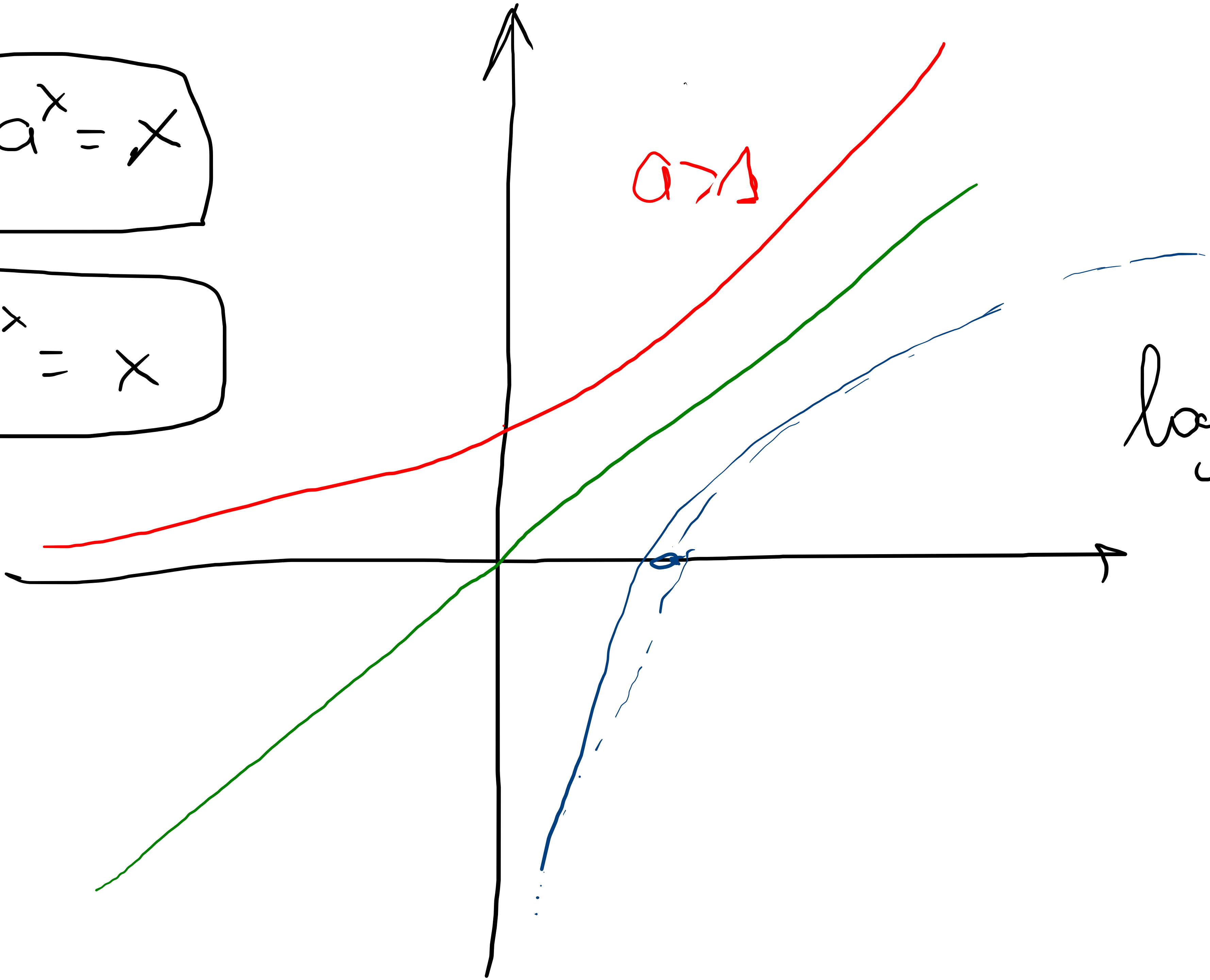
a^x $\qquad x = 1/2 \qquad a^{1/2} = \sqrt{a}$

$0 < a \qquad \boxed{a \neq 1}$

$x \mapsto \log_a x$ è l'inverso di $x \mapsto a^x$

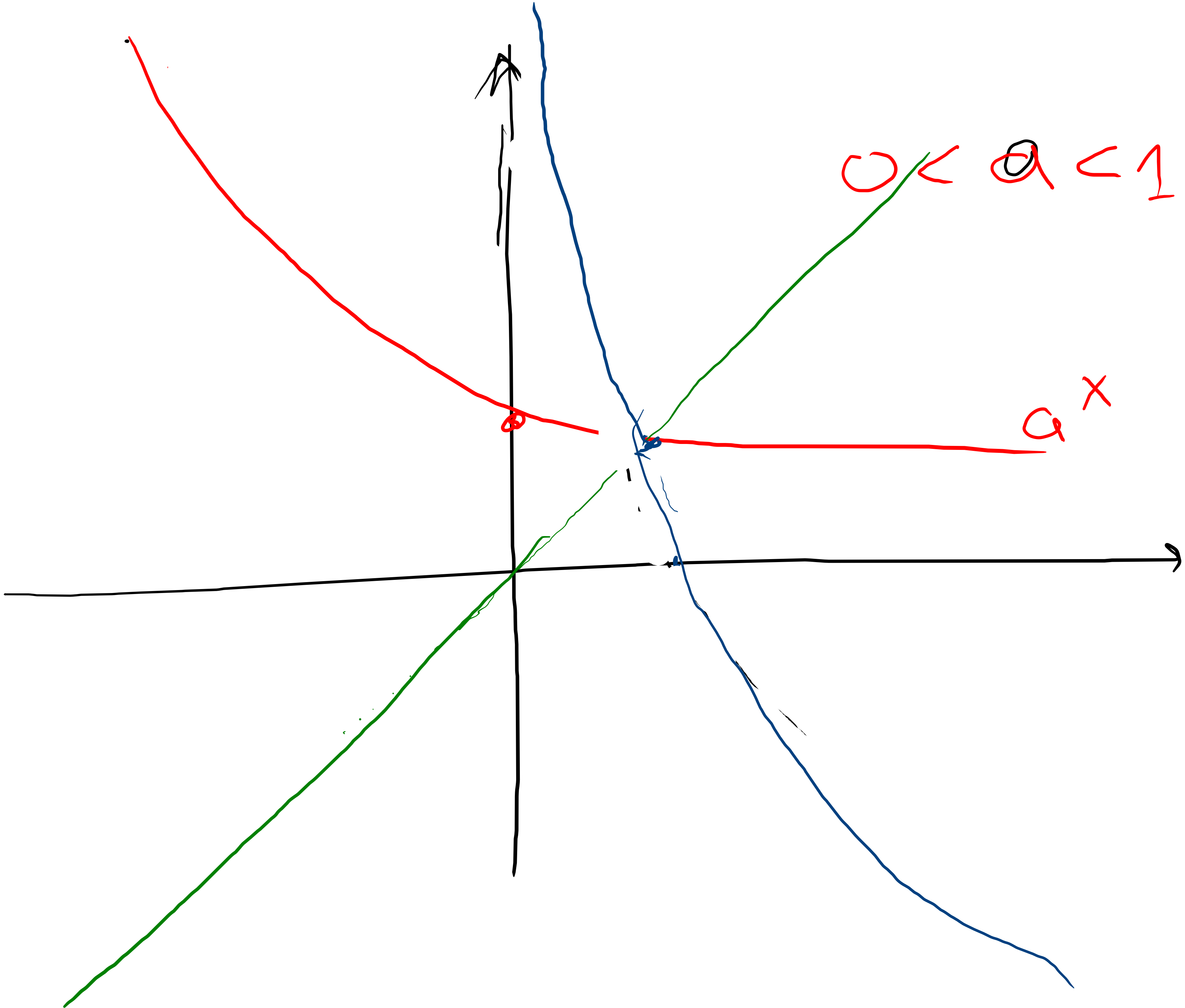
$$\boxed{\log_a a^x = x}$$

$$\boxed{a^{\log_a x} = x}$$



$$\boxed{a^0 = 1}$$

$$\log_a(1) = 0$$



$\log_a(1) = 0$

Fra tutte le basi a $\quad 0<a \quad a\neq 1$

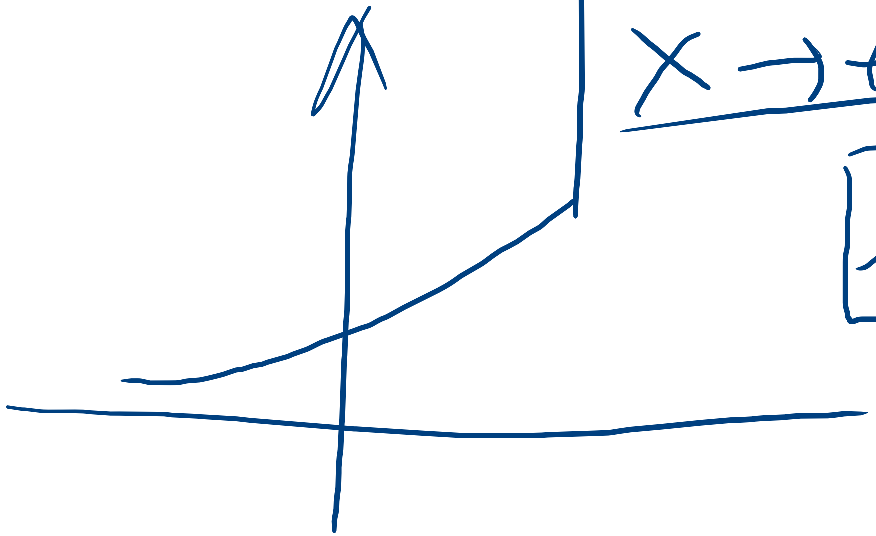
si predilige la base $a = e =$

$$= \boxed{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x} = e$$

numero di Nepero

$$\boxed{\log_e = \ln}$$

$$\boxed{2<e<3}$$



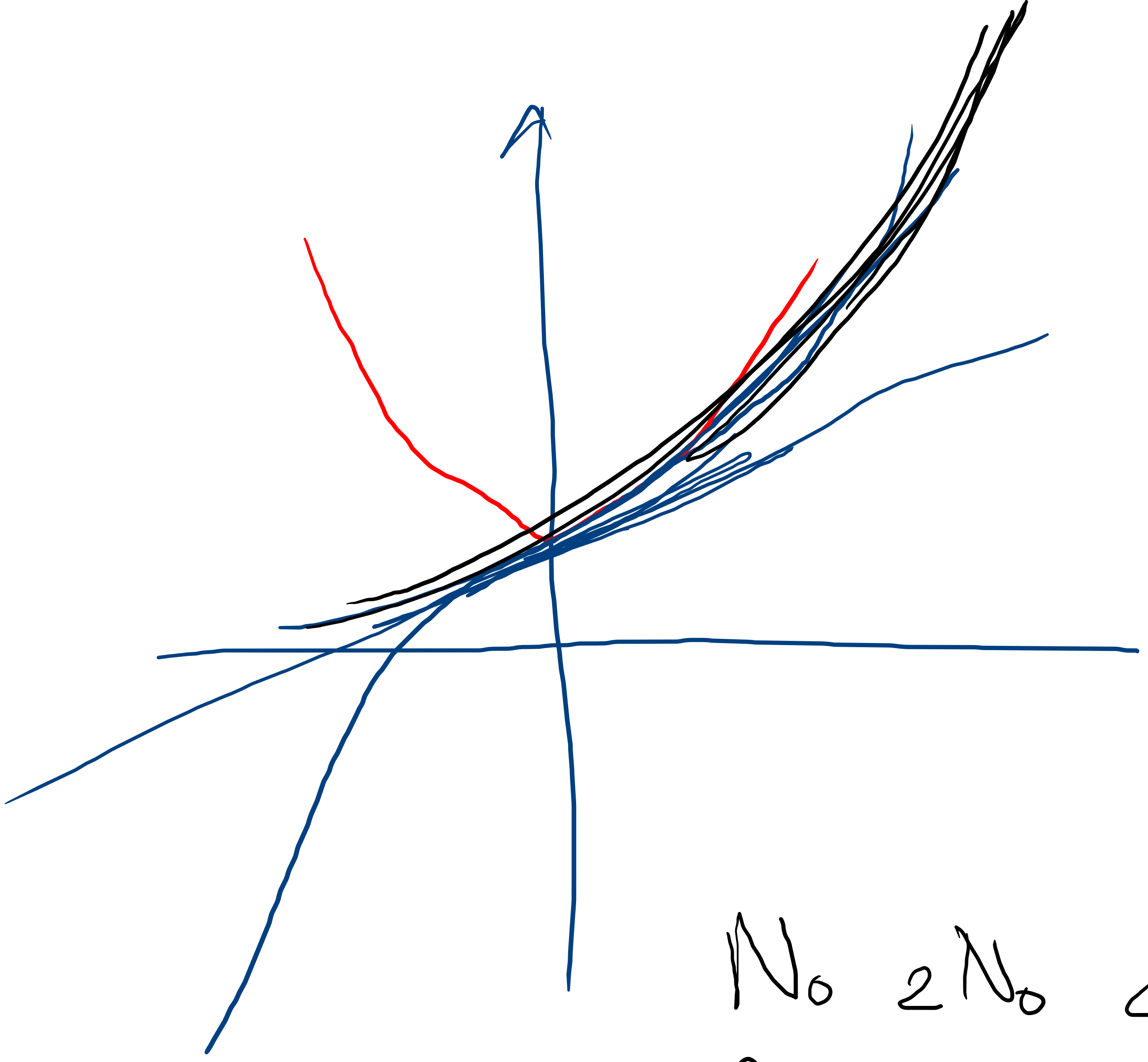
$\lim_{x \to +\infty} \log_a x = +\infty$ meno velocemente di x^α

$1 < a$

$\alpha \in \mathbb{R}^+$

$\log_a x \ll x^\alpha \ll a^x$

$1 < a$



$$\boxed{R_t}$$

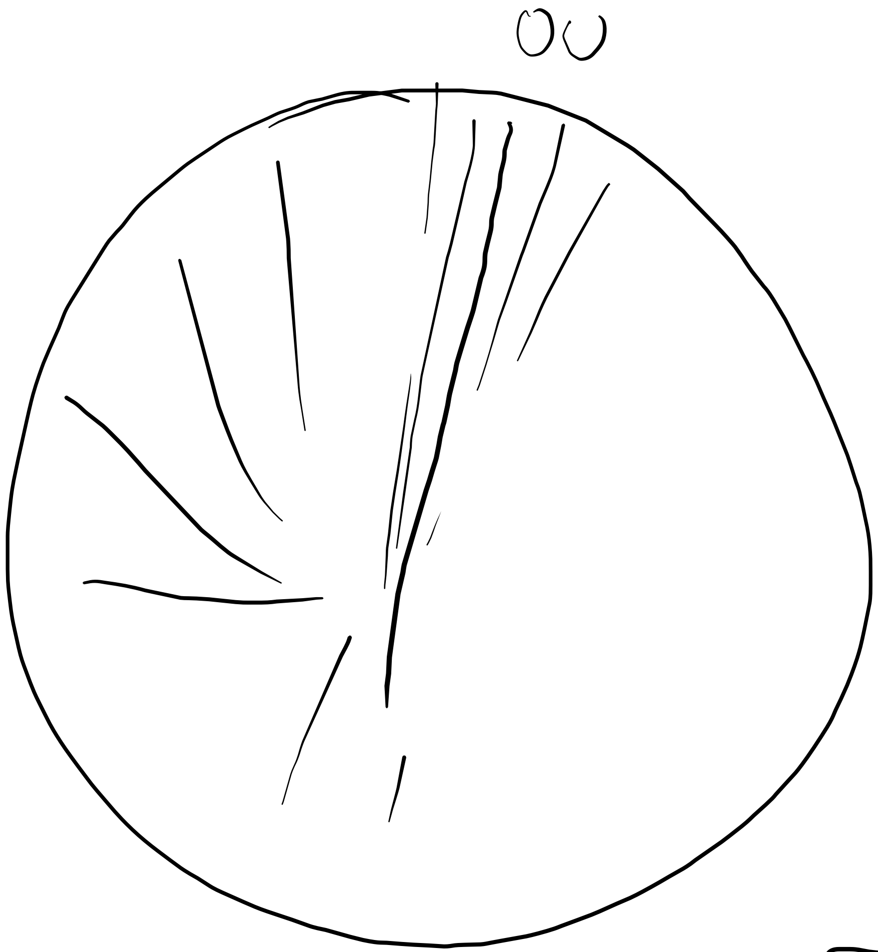
$$C_{t_0} = N_0$$

$$C_{t_1} = 2N_0$$

$N_0 \quad 2N_0 \quad 4N_0 \ldots \; 8N_0 \ldots \quad \boxed{2^n \cdot N_0}$

$\uparrow \quad \uparrow \quad \uparrow$

2 bore



P D.

R N

$1 \to 2$

$\mathbb{N}_0 \quad (1+2+4+8+\dots \dots + 2^n + \dots)$

$\boxed{q \neq 1}$

$$\underset{q^0}{1} + q^1 + q^2 + \dots + q^n = \frac{1-q^{n+1}}{1-q}$$

$2^3 = 8$

$2^4 = 16$

$\vdots$

Se $q = 1$

$$\boxed{1+1 \dots +1 = n}$$

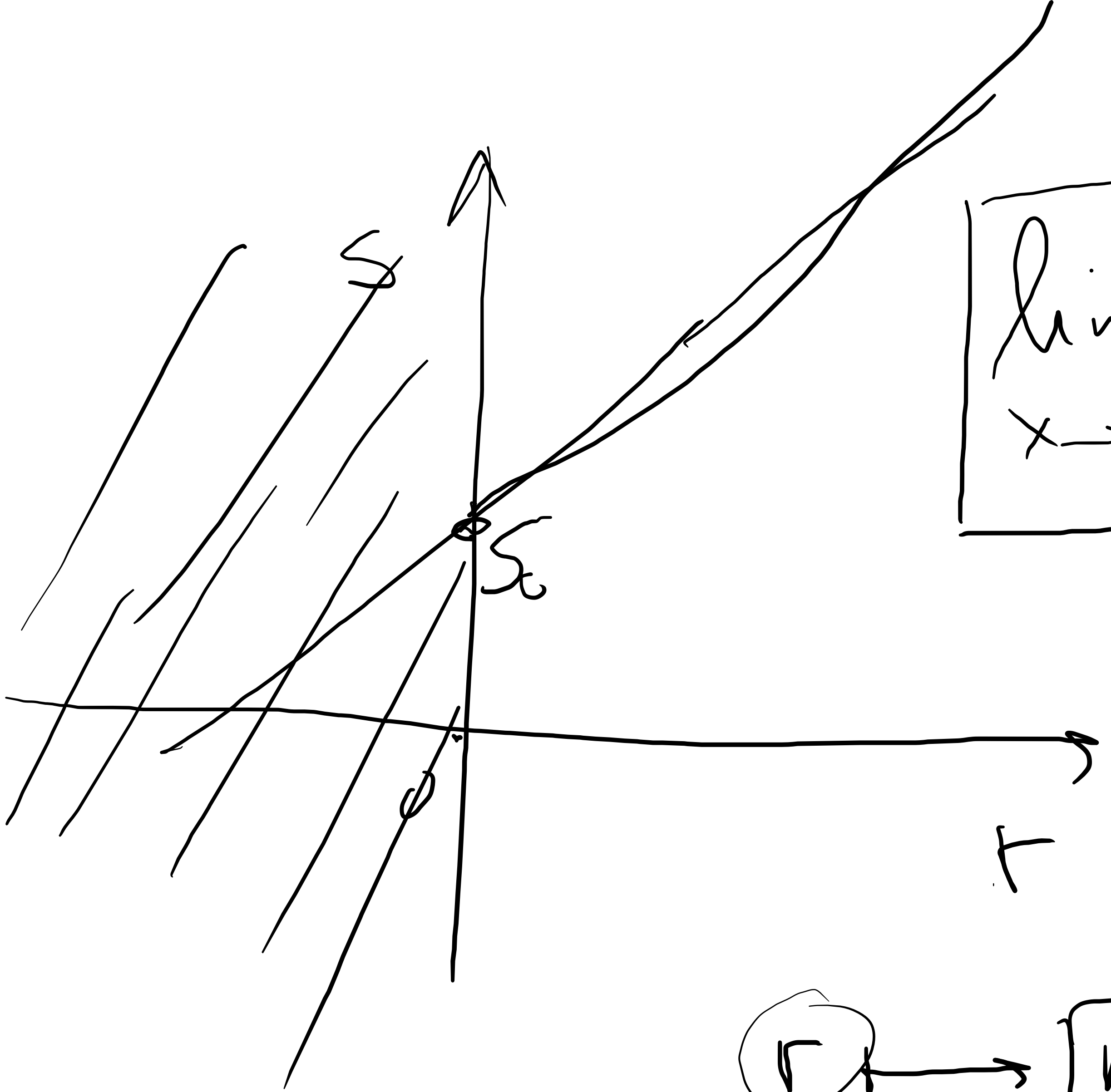
$$\frac{1-2^{65}}{1-2} = 2^{65} - 1$$

10^{15} $\quad$ 10^{11} Kg

$$> 2^{60} = (2^4)^{15}$$

Prop Le funzioni esponenziali e i logaritmi sono funzioni CONTINUE.

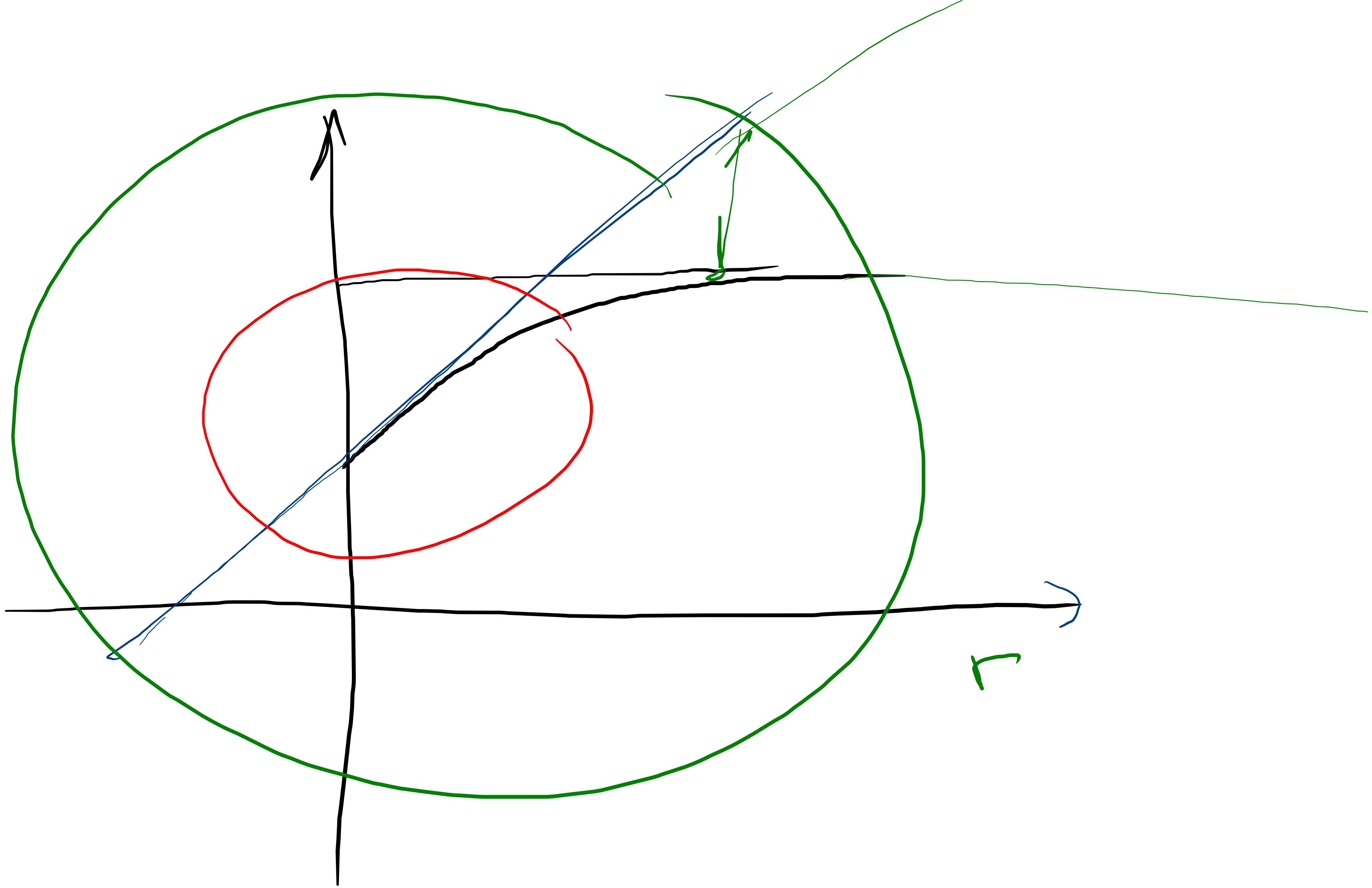
Oss Anche $|x|$ è continua

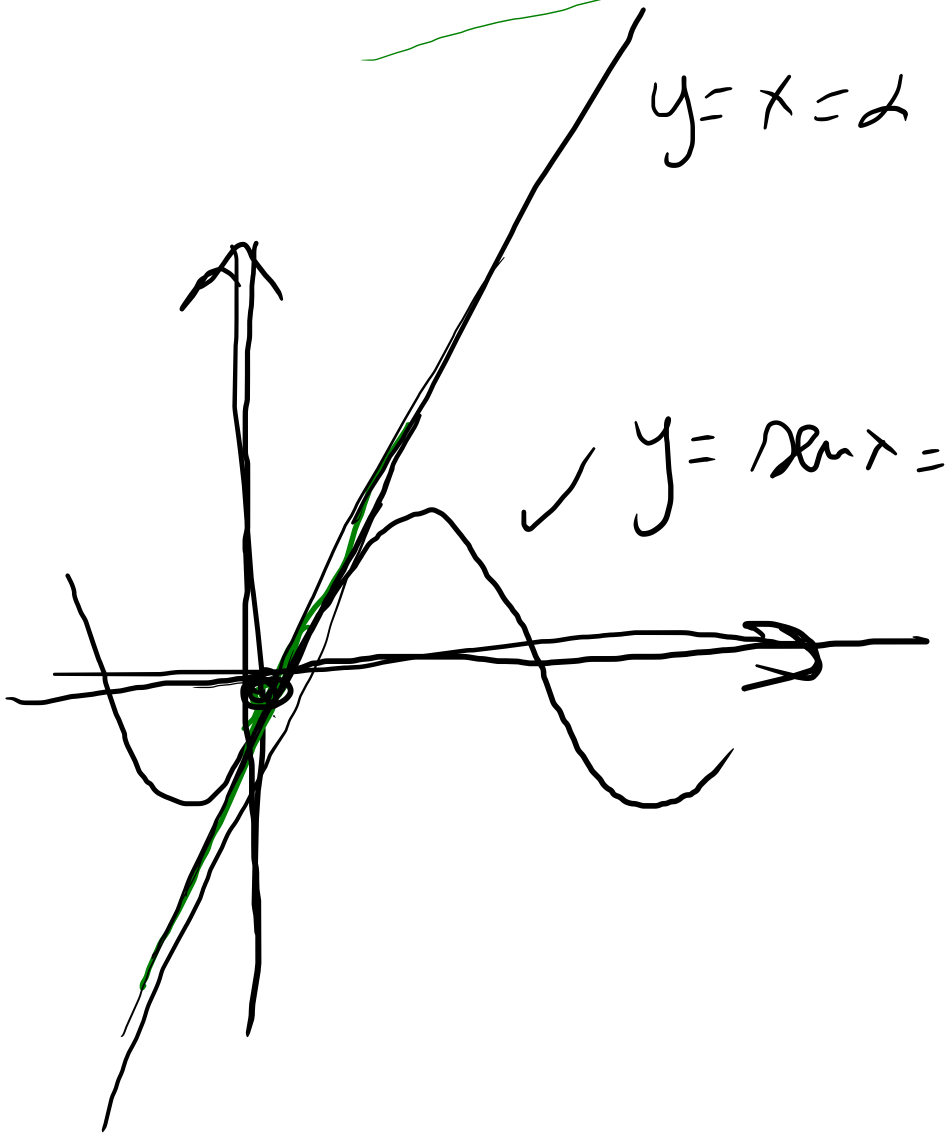


$$\boxed{\lim_{x \to +\infty} (\underline{mx + q}) = +\infty}$$

$$\boxed{m > 0}$$

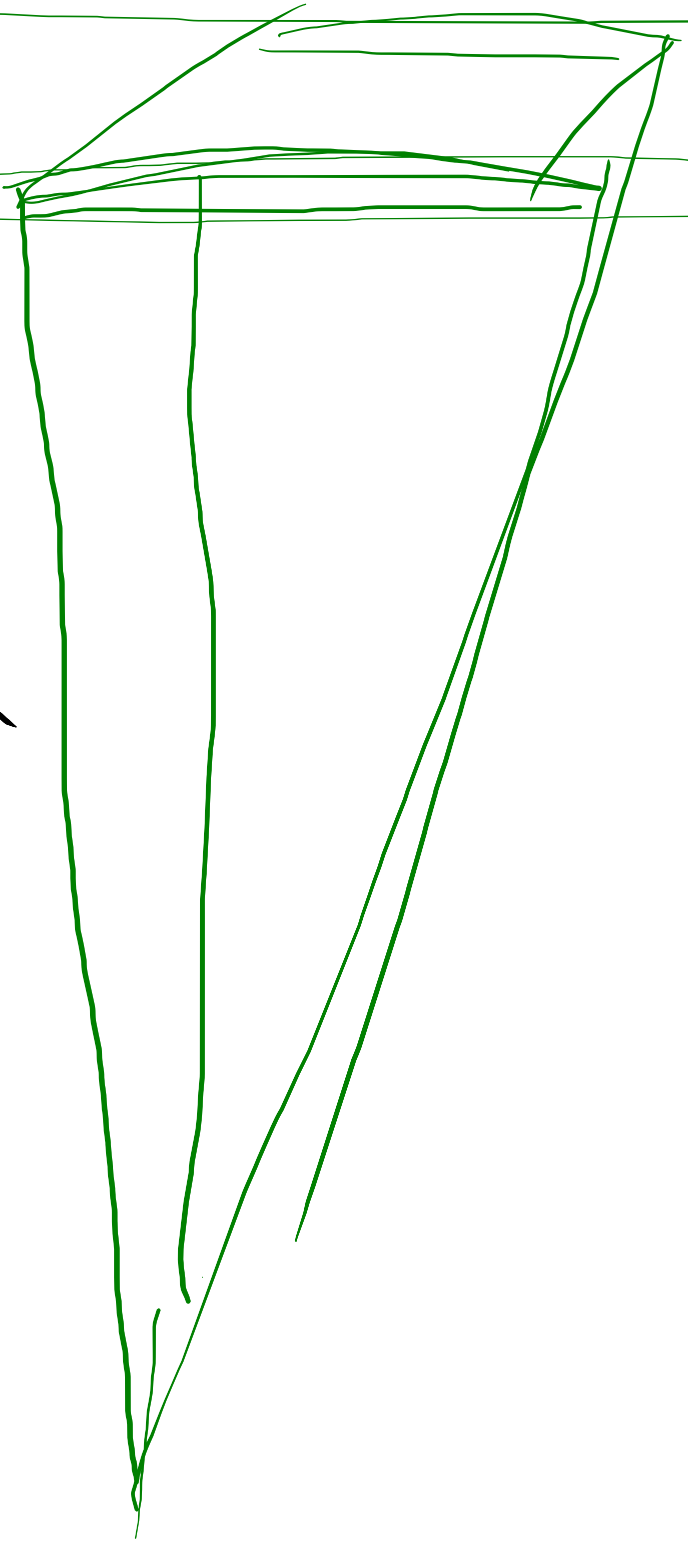
$$r \longmapsto \boxed{\underline{m}r + s_0}$$

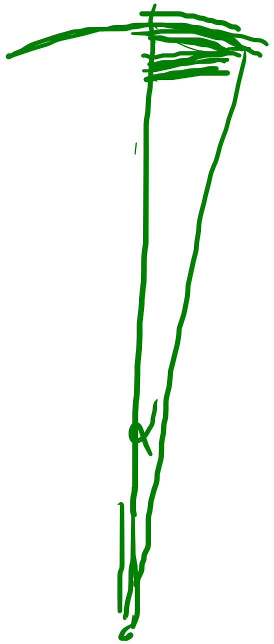

r



$y = x = \alpha$

$\checkmark$ $y = \text{sen}\, x = \text{sen}\, \alpha$





$\boxed{\alpha \simeq \text{sen}\, \alpha}$

α piccolo