21 ottobre

Stiamo dimostrando che in $d=2$ le soluzioni deboli di Leray sono uniche, sono in $C^0([0,+\infty), L^2)$ e soddisfano l'identità dell'energia.

Finora abbiamo dimostrato l'unicità a meno del claim

$$\partial_t u \in L^2((0,T), H^{-1}(\mathbb{R}^2, \mathbb{R}^2)) \quad \forall\, T>0$$

Abbiamo dimostrato che le soluzioni possono essere prese come funzioni test

$$|u(t)|^2_{L^2} = \int_0^t \left(-\nu |\nabla u|^2_{L^2} + \langle u, \partial_t u\rangle - \cancel{\langle \operatorname{div}(u\otimes u), u\rangle}\right) dt' + |u_0|^2_{L^2}.$$

$$\frac{d}{dt} |u(t)|^2_{L^2} = 2\langle u, \partial_t u\rangle$$

$$\int_0^t \langle u, \partial_t u\rangle\, dt' = \frac{1}{2}|u(t)|^2_{L^2} - \frac{1}{2}|u_0|^2_{L^2}$$

$$\boxed{\frac{1}{2}|u(t)|^2_{L^2} + \nu\int_0^t |\nabla u|^2_{L^2}\, dt' = |u_0|^2_{L^2}}$$

$$\langle \operatorname{div}(u\otimes u), u\rangle = 0$$

$$= \langle \partial_j(u^j u^k), u^k\rangle = -\langle u^j u^k, \partial_j u^k\rangle = -\frac{1}{2}\langle u^j, \partial_j(u^k u^k)\rangle = \frac{1}{2}\langle \underbrace{\partial_j u^j}_{0}, u^k u^k\rangle$$

Ora dimostriamo che ogni soluzione soddisfa

$$\partial_t u \in L^2((0,T), L^2) = L^2((0,T)\times\mathbb{R}^d, \mathbb{R}^d)$$

$\forall\ T > 0$.

Utilizzando una funzione test $\phi(x) \in C^\infty_{c\sigma}(\mathbb{R}^2, \mathbb{R}^2)$

$$\langle u(t), \phi\rangle - \langle u(0), \phi\rangle = \int_0^t \left(\nu\langle u, \Delta\phi\rangle - \langle \operatorname{div}(u\otimes u), \mathbb{P}\phi\rangle\right) dt'$$

Questa formula si può estendere a tutti i

$\phi \in H^1(\mathbb{R}^2, \mathbb{R}^2)$.

<u>Lemma</u> Siano $u, g \in L^1(I, X)$, I interv., e X B-space, e supponiamo che $\forall\, \phi \in X^*$

$$\langle u(t_2), \phi\rangle_{XX^*} - \langle u(t_1), \phi\rangle_{XX^*} = \int_{t_1}^{t_2} \langle g(t), \phi\rangle_{XX^*}\, dt$$

Allora $\underline{\partial_t u = g \quad \text{in} \quad \mathcal{D}'(I, X)} =: \mathcal{L}(\mathcal{D}(I,\mathbb{R}), X)$ $\square$

Applichiamo $X = H^{-1}(\mathbb{R}^2, \mathbb{R}^2)$, $X^* = H^1(\mathbb{R}^2, \mathbb{R}^2)$

$$\langle \overbrace{\nu\Delta u - \mathbb{P}\operatorname{div}(u\otimes u)}^{g}, \phi\rangle$$

$$\langle u(t), \phi\rangle - \langle u(0), \phi\rangle = \int_0^t \left(\nu\langle u, \Delta\phi\rangle - \langle \operatorname{div}(u\otimes u), \mathbb{P}\phi\rangle\right) dt'$$

$$\Delta u \in L^2((0,T), H^{-1})$$

$$\operatorname{div}(u\otimes u) \in L^2((0,T), H^{-1})$$

$$\partial_t u = \nu\Delta u - \mathbb{P}\operatorname{div}(u\otimes u) \in L^2((0,T), H^{-1})$$

$$u \in L^\infty((0,T), L^2), \quad \nabla u \in L^2((0,T), L^2),$$

$$u \in L^2((0,T), H^1) \Longrightarrow \boxed{\Delta u \in L^2((0,T), H^{-1})}$$

$$\| \operatorname{div}(u \otimes u) \|_{L^2((0,T), H^{-1})} \leq \| u^2 \|_{L^2((0,T), L^2)} =$$

$$= \| \, | u^2 |_{L^2_x} \|_{L^2(0,T)} =$$

$$= \| \, |u|^2_{L^4_x} \|_{L^2(0,T)} \leq C \| \, |u|_{L^2_x} |\nabla u|_{L^2_x} \|_{L^2(0,T)}$$

$$\leq C \, |u|_{L^\infty((0,T), L^2_x)} \, |\nabla u|_{L^2((0,T), L^2_x)} < \infty$$

$$\Longrightarrow \quad \partial_t u \in \underline{L^2((0,T)}, H^{-1}) \quad \forall T > 0$$

$$u_t - \nu \Delta u = Q_{NS}(u,u)$$

$$Q_{NS}(u,u) = -\mathbb{P}\,\mathrm{div}(u\otimes u)$$

$$Q_{NS}(u,v) \doteq -\frac{1}{2}\,\mathbb{P}\,\mathrm{div}(u\otimes v) - \frac{1}{2}\,\mathbb{P}\,\mathrm{div}(v\otimes u).$$

Lemma $d = 2, 3$

$$\forall\ (u, v, \varphi) \in C_c^\infty \times C_c^\infty \times C_c^\infty \longrightarrow \langle \mathrm{div}(u\otimes v), \varphi\rangle \in \mathbb{R}$$

Allora vale

$$(**)\quad \left|\langle \mathrm{div}(u\otimes v), \varphi\rangle\right| \leq C\, \|\nabla u\|_{L^2}^{\frac{d}{4}}\, \|\nabla v\|_{L^2}^{\frac{d}{4}}\, \|u\|_{L^2}^{1-\frac{d}{4}}\, \|v\|_{L^2}^{1-\frac{d}{4}}\, \|\nabla \varphi\|_{L^2}$$

$*$ si estende in una forma trilineare limitata

$$H^1 \times H^1 \times H^1 \longrightarrow \mathbb{R}$$

E inoltre, se $\mathrm{div}(u) = 0$

$$\Longrightarrow \langle \mathrm{div}(u\otimes v), v\rangle = 0$$

Dimostrazione teor di Leray

$$\begin{cases} \partial_t u + \mathbb{P}\,\mathrm{div}(u\otimes u) - \nu \Delta u = 0 \\ u_{|t=0} = u_0 \in H \end{cases}$$

$$(1) \quad \begin{cases} \partial_t u + \mathbb{P} \operatorname{div}(u \otimes u) - \nu \Delta u = 0 \\ u|_{t=0} = u_0 \in H \end{cases}$$

$$(2) \quad \begin{cases} \partial_t u_n + P_n \mathbb{P} \operatorname{div}(P_n u_n \otimes P_n u_n) - \nu P_n \Delta u_n = 0 \\ u_n|_{t=0} = P_n u_0 \in H \end{cases}$$

$$\widehat{P_n f} = \chi_{B(0,n)} \hat{f} \qquad P_n = \chi_{[0,1]}\left(\frac{-\Delta}{n^2}\right) = \chi_{[0,n^2]}(-\Delta)$$

Vedremo ora che (2) e' una ODE in L^2

$\dot{H}^N$

<u>Lemma</u> $u_n \in C^\infty([0,+\infty), H^N) \quad \forall N \in \mathbb{N} \cup \{0\}$.

$$P_n u_n = u_n, \qquad \mathbb{P} u_n = u_n.$$

$$\begin{cases} \partial_t u_n + P_n \mathbb{P}\, \mathrm{div}(P_n u_n \otimes P_n u_n) - \nu P_n \Delta u_n = 0 \\ u_n|_{t=0} = P_n u_0 \in H \end{cases} \qquad L^2$$

$$\partial_t u_n = F_n(u_n)$$

$F_n : L^2 \to L^2$ ed e' localmente Lipchitz.

$\forall\, M > 0 \quad \exists\, L(M)$ t.c.

$$|F(u) - F(v)|_{L^2} \le L(M)\, |u - v|_{L^2}, \quad \text{se } |u|_{L^2} \le M \text{ e } |v|_{L^2} \le M.$$

$$|F_n(v)|_{L^2} \le \nu |P_n \Delta v|_{L^2} + |P_n \mathbb{P}\, \mathrm{div}(P_n v \otimes P_n v)|_{L^2}$$

$$= \nu\, |\chi_{[0,n]}(|\xi|)\, |\xi|^2 \hat{v}|_{L^2} + |\chi_{[0,n]}(|\xi|)\, \xi\, \widehat{P_n v \otimes P_n v}|_{L^2}$$

$$\le \nu n^2 |v|_{L^2} + n\, |P_n v \otimes P_n v|_{L^2}$$

$$\le \nu n^2 |v|_{L^2} + n\, |P_n v|^2_{L^4} \qquad |f|_{L^4} \le |f|_{L^2}^{1-\frac{d}{4}} |\nabla f|_{L^2}^{\frac{d}{4}}$$

$$\le \nu n^2 |v|_{L^2} + Cn\, |P_n v|_{L^2}^{2-\frac{d}{2}} |\nabla P_n v|_{L^2}^{\frac{d}{2}}$$

$$\le \nu n^2 |v|_{L^2} + C n^{1+\frac{d}{2}} |v|^2_{L^2}$$

$$F_n \in C^\infty(L^2, L^2) \qquad H^N$$

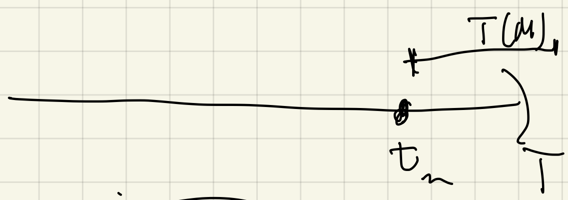
$$u_n \in C^1([0, T_n), L^2) \qquad L^2$$

se $T_n < +\infty \implies \lim_{t \to T_n^-} |u_n(t)|_{L^2} = +\infty$

Questo è legato al fatto che $\forall$ dato iniziale u_0 $\exists$ un tempo minimo di esistenza $T(\|u_0\|_{L^2})>0$

Se io avessi $T<\infty$ con una successione $\|u(t_n)\|_{L^2} < M$ con $t_n \to T^-$

$T(M) > 0$



$$P_n u_n = u_n = P_n u_n + (1-P_n) u_n$$

$$\mathbb{P} u_n = u_n \qquad (1-\mathbb{P}) u_n$$

$$\begin{cases} \partial_t u_n - \nu P_n \Delta u_n + \mathbb{P} P_n \operatorname{div}(P_n u_n \otimes P_n u_n) = 0 & (1-P_n) \\ u_n|_{t=0} = P_n u_0 & (1-P_n) \end{cases}$$

$$\partial_t (1-P_n) u_n - \nu \cancel{(1-P_n) P_n} \Delta u_n + \cancel{\mathbb{P}(1-P_n) P_n \operatorname{div}(P_n u_n \otimes P_n u_n)} = 0$$

$$\partial_t (1-P_n) u_n = 0$$

$$(1-P_n) u_n|_{t=0} = \underbrace{(1-P_n) P_n}_{0} u_0$$

$$\partial_t u_n - \nu \Delta u_n + \mathbb{P} P_n \operatorname{div}(u_n \otimes u_n) = 0 \,. \quad \langle \cdot , u_n \rangle$$

$$\frac{1}{2}\partial_t |u_n|_{L^2}^2 + \nu |\nabla u_n|_{L^2}^2 + \langle \cancel{\mathbb{P} P_n} \operatorname{div}(u_n \otimes u_n), u_n \rangle = 0$$

$$0$$

$$\boxed{|u_n(t)|_{L^2}^2 + 2\nu \int_0^t |\nabla u_n(t')|_{L^2}^2 \, dt' = |P_n u_0|_{L^2}^2 \le |u_0|_{L^2}^2}$$

$$\Longrightarrow \quad u_n \in C^1(\underline{\underline{[0,+\infty)}}, \underset{\substack{\cap \\ H^N}}{L^2})$$

$\{u_n\}$ converge du quelche norte?

Prop $\exists\, u \in L^\infty(\mathbb{R}_+, L^2)$, $\nabla u \in L^2(\mathbb{R}_+, L^2)$ con $\operatorname{div} u = 0$ ed esiste una sottosucc. di $\{u_n\}$ t.c. $\forall\, T > 0$ e $\forall\, K \subset\subset \mathbb{R}^d$ si ha

$$u_n \to u \quad \text{in} \quad L^2((0,T)\times K, \mathbb{R}^d)$$

Inoltre

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^d} (u_n - u)\cdot \phi \, dt\, dx = 0 \quad \forall\, \phi \in L^2((0,T)\times\mathbb{R}^d)$$

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^d} \nabla(u_n - u) : \Psi \, dt\, dt = 0 \quad \forall\, \Psi \in L^2((0,T)\times\mathbb{R}^d)$$

Inoltre $\forall\, \varphi \in C^0([0,\infty), H^1(\mathbb{R}^d, \mathbb{R}^d))$

$$\langle u_n(t), \varphi(t)\rangle_{L^2_x} \longrightarrow \underline{\underline{\langle u(t), \varphi(t)\rangle_{L^2_x}}} \quad \text{in } L^\infty([0,T]) \quad \forall\, T \geq 0$$

Fissiamo $T > 0$ e $K \subset\subset \mathbb{R}$ arbitrari

Lemma 1 $\{u_n\}$ è rel. compatto $L^2([0,T] \times K, \mathbb{R}^d)$

Lemma 2 $\forall \varepsilon > 0$ esiste un ricoprimento finito $\{u_n\}$ in $L^2([0,T] \times K, \mathbb{R}^d)$, ricoprimento formato da palle di raggio ε.

Riduzione: è possibile ~~sostituire~~ sostituire $\{u_n\}$ con $\underline{\{P_{n_0} u_n\}}$ con n_0 ~~opportuno~~ arbitrario

$$|u_n - P_{n_0} u_n|_{L^2([0,T] \times K)} \le |(1-P_{n_0}) u_n|_{L^2([0,T] \times \mathbb{R}^d)}$$

$$= |\chi_{B^c(0,n_0)} \hat{u}_n|_{L^2([0,T] \times \mathbb{R}^d)} \le |\cancel{\chi_{B^c(0,n_0)}} \frac{\xi}{n_0} \hat{u}_n|_{L^2([0,T] \times \mathbb{R}^d)}$$

$$= \frac{1}{n_0} |\nabla u_n|_{L^2([0,T], L^2_x)} \le \frac{1}{n_0} \frac{1}{\sqrt{2}} |\cancel{P_n} u_0|_{L^2_x}$$

Per $n_0 >> 1$ $\quad |u_n - P_{n_0} u_n|_{L^2((0,T) \times K)} < \frac{\varepsilon}{2} \ \forall n.$

Pertanto, se $\left\{ B_{L^2}(f_j, \frac{\varepsilon}{2}) \right\}_{j=1,\dots,k}$ è un ricoprimento di $\boxed{\{P_{n_0} u_n\}}$ in $L^2((0,T) \times K)$ allora

$\left\{ B_{L^2}(f_j, \varepsilon) \right\}_{j=1,\dots,k}$ è un ricoprimento $\{u_n\}$.