$f, g : E \subseteq X \to \mathbb{R}^{m}$ $\quad (X, d)$ sp. metrico

$x_0 \in X$ punto di accumulazione per E

$\exists \lim\limits_{x \to x_0} f(x) = \alpha \qquad \lim\limits_{x \to x_0} g(x) = \beta \qquad a, b \in \mathbb{R}$

$(af + bg)(x) = a \cdot f(x) + b g(x)$

$\Rightarrow \exists \lim\limits_{x \to x_0} (af + bg)(x) = a\alpha + b\beta.$

$f, g : E \subseteq X \to \mathbb{R} \qquad \lim\limits_{x \to x_0} (f \cdot g)(x) = \alpha \cdot \beta$

1) Teorema della permanenza del segno $\quad M = 1$

2) Teorema della limitatezza locale

1) $f : E \subseteq X \to \mathbb{R}$ $\quad \lim\limits_{x \to x_0} f(x) = \alpha \overset{<}{>} 0$

allora esiste $U \in \mathcal{N}_{x_0}$ tale che $\forall x \in U \cap E$ $\; x \neq x_0$ $\; f(x) \overset{<}{>} 0$

2) $f : E \subseteq X_1 \to X_2$ $\quad \exists \lim\limits_{x \to x_0} f(x) = \alpha$ allora esiste

$U \in \mathcal{N}_{x_0}$ tale che $f|_{U \cap E}$ è limitata ; cioè esiste

una palla $B(y_0, r) \subset X_2$ tale che $f(x) \in B(y_0, r)$ $\; \forall x \in U \cap E$

Altri limiti

$$\lim_{x \to x_0} f(x) = \alpha \qquad \boxed{f : X \to Y}$$

$$\lim_{x \to x_0} f(x) = \begin{matrix} +\infty \\ -\infty \end{matrix}$$

? ha senso se $Y = \mathbb{R}$

$+\infty$ $\forall M \in \mathbb{R}$ $\exists$ $\delta > 0$ tale che $\forall x \in E$ $x \neq x_0$

$d_1(x, x_0) < \delta \Rightarrow f(x) > M$

$<$

$$\lim_{x \to \begin{smallmatrix} +\infty \\ -\infty \end{smallmatrix}} f(x) = \alpha$$

? ha senso se $X = \mathbb{R}$

In generale $\quad f : X_1 \to X_2$

$$\cancel{\lim_{x \to +\infty} f(x)} \qquad \lim_{\|x\| \to +\infty} f(x) = \overset{\alpha}{0}$$

$$\lim_{x \to x_0} \|f(x)\| = +\infty \qquad \lim_{x \to 0} \left|\frac{1}{x}\right| = +\infty$$

Esempi importanti di campi vettoriali

$f : \mathbb{R}^N \to \mathbb{R}^K$ sono le <u>applicazioni lineari</u>

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Un'applicazione lineare $f : \mathbb{R}^N \to \mathbb{R}^K$ si può rappresentare mediante una matrice M K righe N colonne

(fissate le basi)

$\mathbb{R}^N$ base $e_1, e_2, \ldots, e_N$ $\quad \mathbb{R}^K$

$$M = \begin{pmatrix} \vdots & \vdots & & \vdots \\ f(e_1) & f(e_2) & \cdots & f(e_N) \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

$f(e_1) = (y_1, y_2, \ldots, y_K)^T$

$$f(v) = M \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$v = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$

$\boxed{\text{Se } K = 1}$ forma lineare

$M = \left(f(e_1) \; \cdots \; f(e_n) \right)$ matrice riga

$$f(x_1, x_2, \dots, x_n) = \left(f(e_1), \dots, f(e_n) \right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} =$$

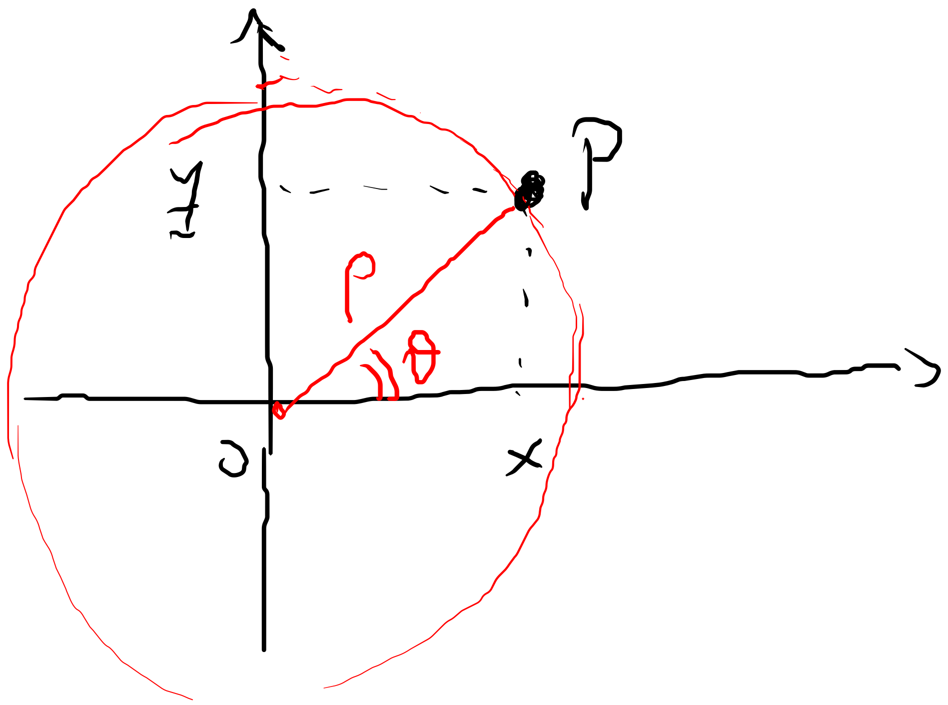
$$= \Big\langle \underbrace{\left(f(e_1), \dots, f(e_n) \right)^T}_{\text{vettore di } \mathbb{R}^n}, (x_1, \dots, x_n)^T \Big\rangle$$

OSSERVAZIONE Teorema di Riesz

Ogni forma lineare $f: \mathbb{R}^n \to \mathbb{R}$ ammette una rappresentazione del tipo $f(v) = \langle w, v \rangle \quad \forall v \in \mathbb{R}^n$ dove $w \in \mathbb{R}^n$ è fissato

Coordinate polari

$\mathbb{R}^2$



$$\boxed{P = (x,y)^T = [\rho, \vartheta]}$$

$$\begin{cases} x = \rho \cos\vartheta \\ y = \rho \,\mathrm{sen}\,\vartheta \end{cases}$$

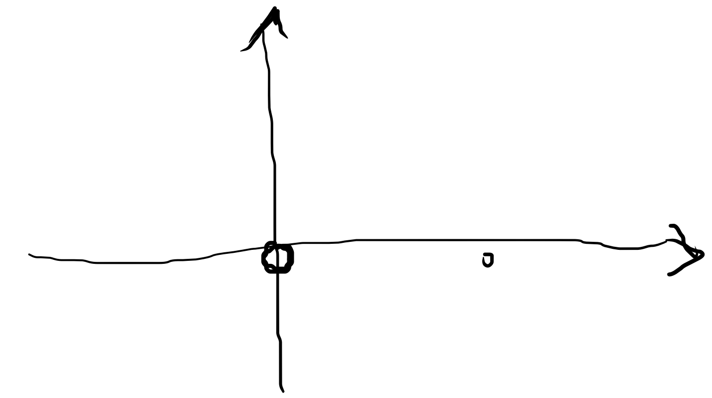
La coppia $[\rho, \vartheta]$ individua il punto $P \ (x,y)^T$

⚠ ϑ non è univocamente determinato. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$

$$\rho = 1 \qquad \vartheta = \frac{\pi}{4}, \ \frac{9}{4}\pi, \ -\frac{7}{4}\pi \ \ldots$$

Scegliamo di prendere $\theta \in [0, 2\pi[$

$\rho \geq 0$



Il punto $(0,0)^T$ $\quad \rho = 0 \quad \theta = ?$

Corrispondenza biunivoca tra i punti di $\mathbb{R}^2 \setminus \{(0,0)^T\}$ e

$]0,+\infty[\times [0,2\pi[$

$]0,+\infty[\times]0, 2\pi[$

$$\underbrace{\varphi : \mathbb{R}^2 \setminus \cancel{\{(0,0)^T\}} \longrightarrow]0,+\infty[\times [0,2\pi[}_{\text{biiettiva}}$$

$\mathbb{R}^2 \setminus [0,+\infty[$

(cambio di variabili)

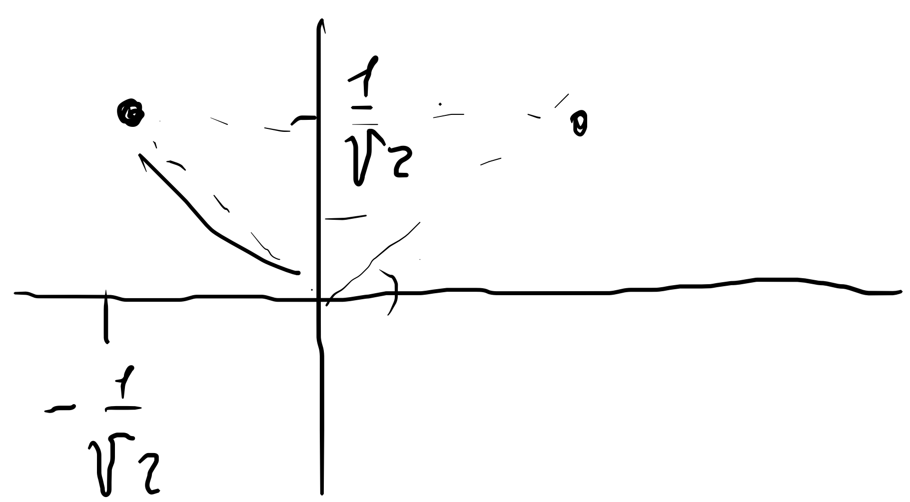
$$\boxed{\begin{aligned} x &= \rho \cos\vartheta \\ y &= \rho \,\mathrm{sen}\,\vartheta \end{aligned}}$$

$$\rho = \sqrt{x^2+y^2}$$

$$\cancel{\vartheta = \mathrm{arctg}\left(y/x\right)}$$

$$\frac{y}{x} = \frac{\cancel{\rho}\,\mathrm{sen}\,\vartheta}{\cancel{\rho}\cos\vartheta} = \mathrm{tg}\,\vartheta$$

$$(x,y)^T = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T \qquad \rho = 1$$



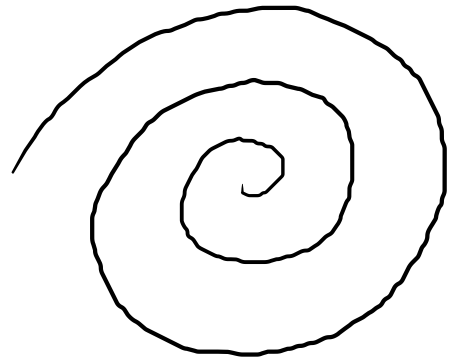
$$\mathrm{arctg}\left(\frac{1/\sqrt{2}}{-\frac{1}{\sqrt{2}}}\right) = \mathrm{arctg}(-1) = -\frac{\pi}{4} \neq \vartheta \;!$$

$$\vartheta = \frac{3}{4}\pi \;!$$

Es: l'equazione del cerchio $x^2+y^2=4$

in coordinate polari diventa $\boxed{\rho = 2}$

spirale



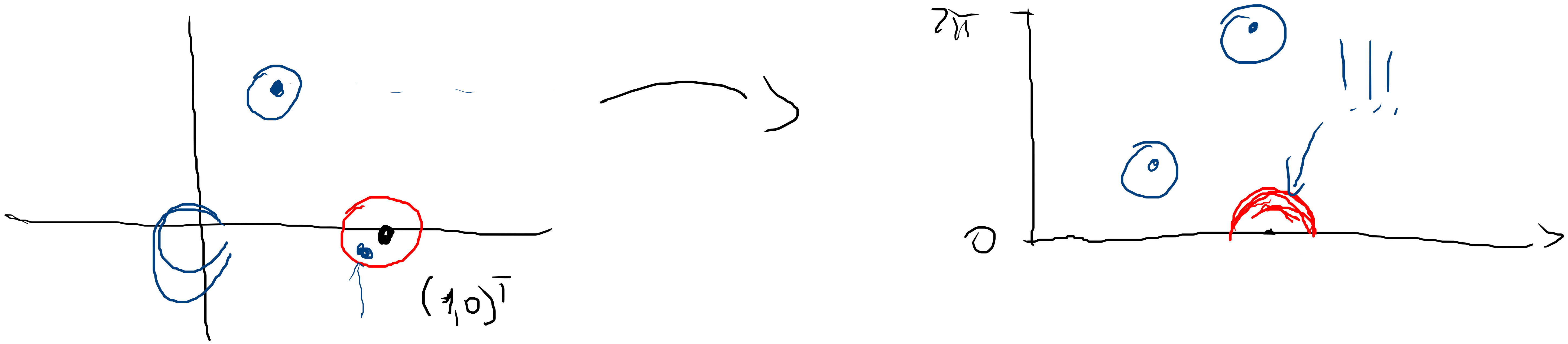
$\rho = 2\theta$

φ NON è continua

omomorfismo

$f(x+y) = f(x)+f(y)$

omeomorfismo = continua invertibile con inversa continua



$$\varphi(x,y) = (\rho, \vartheta) \quad \text{dove} \quad x = \rho\cos\vartheta \quad y = \rho \,\text{sen}\,\vartheta$$

$\vartheta \in [0, 2\pi[$

$\rho > 0$

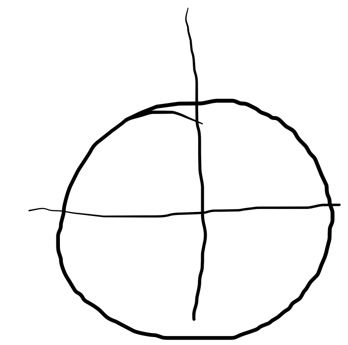
$$\varphi(1,0) = (1,0)$$

$\forall$ W intorno di $\varphi(1,0)$ $\exists$ U intorno di $(1,0)^T$ tale che $\forall (x,y)^T \in U$ si ha $\varphi(x,y) \in W$

$x = \rho \cos\vartheta$

$y = \rho \operatorname{sen}\vartheta$

$\rho = 2$



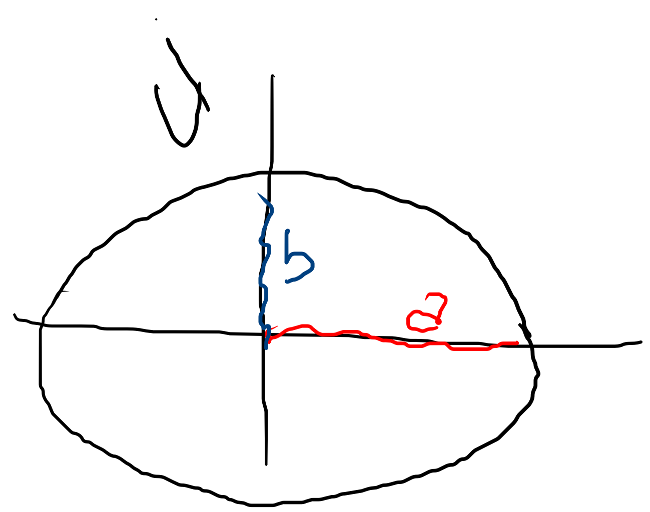
Eq. in forme canonico di un ellisse con semiassi a e b

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a = b = r$ cerchio

$u = ax \quad v = by$

$u^2 + v^2 = 1$



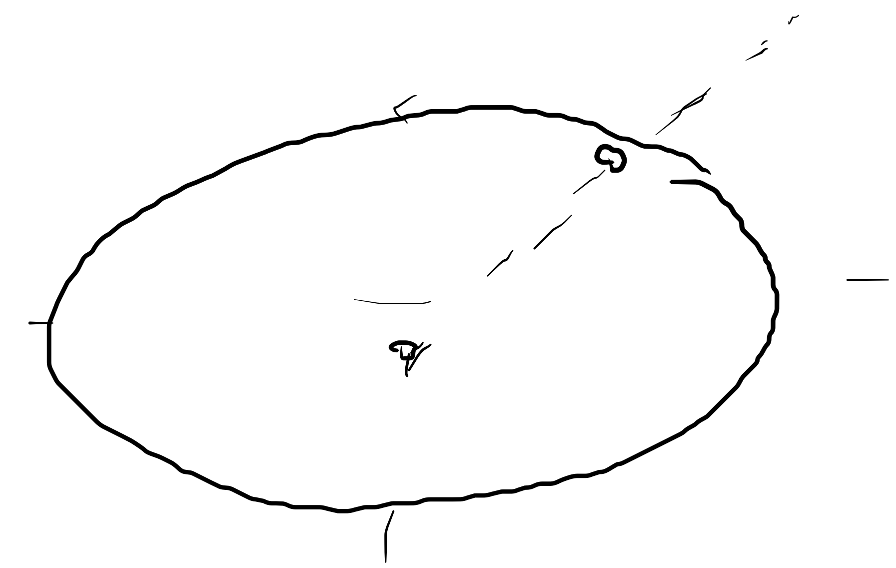
$x = r\cos\vartheta$

$y = r \operatorname{sen}\vartheta$

$x^2 + y^2 = (r^2\cos^2\vartheta) + (r^2\operatorname{sen}^2\vartheta) = r^2$

$$\frac{(ax)^2}{a^2} + \frac{(by)^2}{b^2} = x^2 + y^2$$

$$\begin{cases} x = \rho a \cos\vartheta \\ y = \rho b \operatorname{sen}\vartheta \end{cases}$$



Coordinate ellittiche

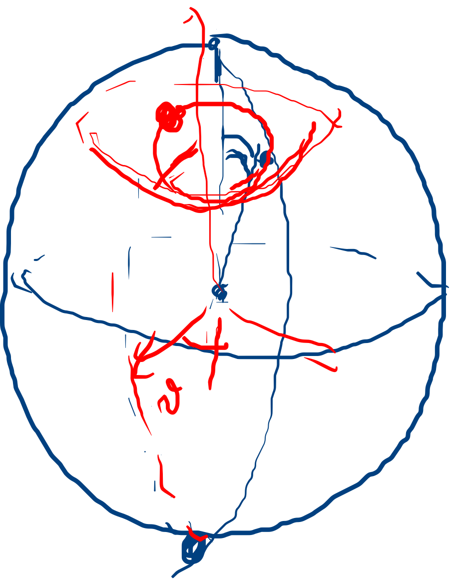
Coordinate sferiche

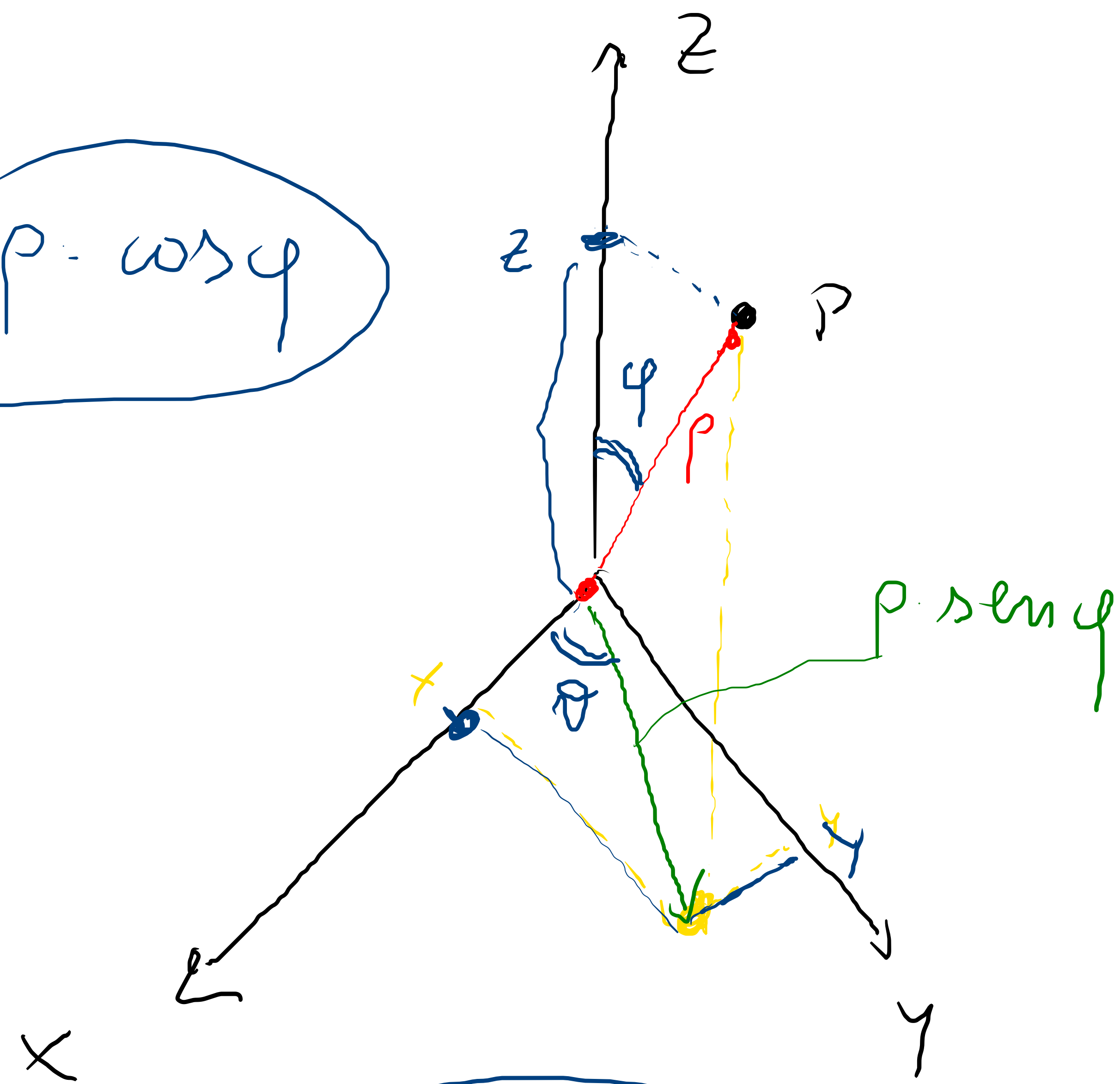
$P \simeq (x,y,z)^T \rightsquigarrow [\rho, \ , \ ]$

ρ

$\rho = \sqrt{x^2+y^2+z^2}$

$z = \rho \cdot \cos\varphi$





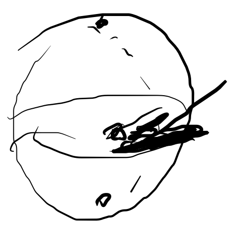
$x = \rho \operatorname{sen}\varphi \cos\vartheta$

$y = \rho \operatorname{sen}\varphi \operatorname{sen}\vartheta$

$\rho \geq 0 \qquad \varphi \in [0, \pi]$

$\theta \in [0, 2\pi]$

$\sigma : \mathbb{R}^3 - \{(0,0,0)^T\} \to \,]0,+\infty[\times [0,\pi] \times [0,2\pi[$

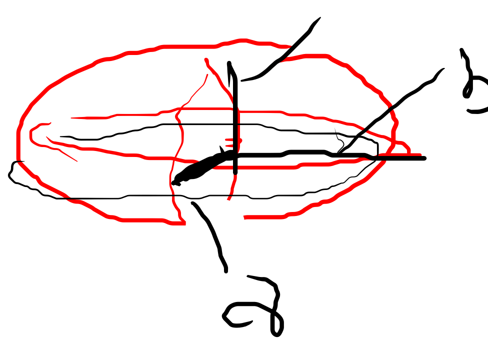


$\sigma(x,y,z) = [\rho, \varphi, \vartheta]$ tali che

$$\begin{cases} x = a\rho \operatorname{sen}\varphi \cos\vartheta \\ y = b\rho \operatorname{sen}\varphi \operatorname{sen}\vartheta \\ z = c\rho \cos\varphi \end{cases}$$

$a, b, c > 0$

ellissoidali



l'equazione $x^2+y^2+z^2 = r^2$ equivale a $\rho = r$

Coniche

parte lineare

$$a_{11}x^2 + 2a_{12}xy + 2a_{13}x + a_{22}y^2 + 2a_{23}y + a_{33} = 0$$

parte quadratica

grado 0

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

B

$$A \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$\det A \neq 0$

$\det B$:
- > 0 ellisse $x^2 + y^2 = 1$
- $= 0$ parabola $x^2 + y = 0$
- < 0 iperbole $x^2 - y^2 = 1$

$$x^2 - 15y^2 + 2xy + 2x - 4y + 1 = 0$$

$$\begin{vmatrix} 1 & 1 \\ 1 & -15 \end{vmatrix} = -16 < 0 \qquad \text{iperbole}$$

Quadriche $\quad a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2 +$

$$+ 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44}$$

B

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} = A$$

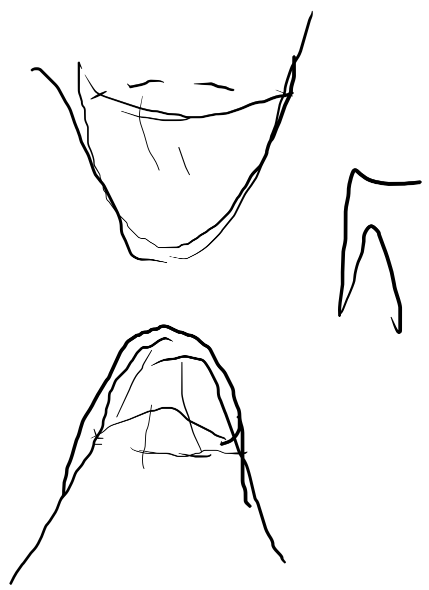
$\det A \neq 0 \qquad \det B \neq 0$ / autovalori concordi

discordi

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ ellissoide

iperboloide iperbolico a 1 foglio

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \quad +1$$ iperboloide ellittico a 2 fogli

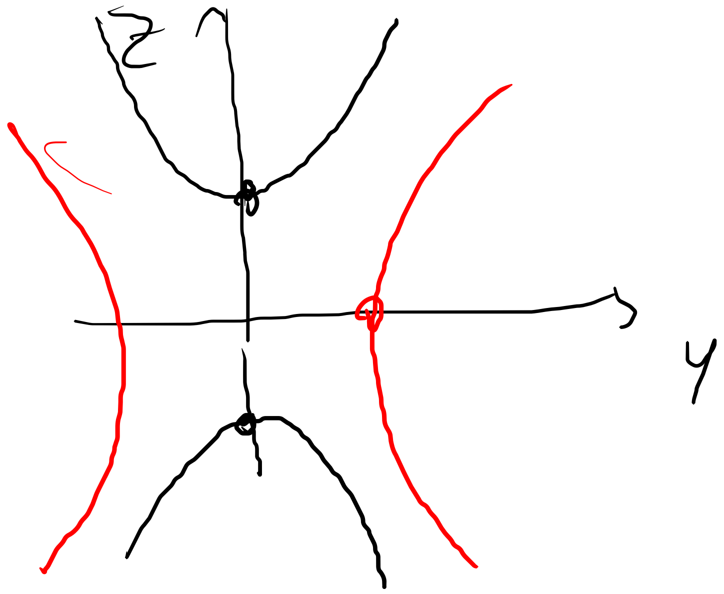


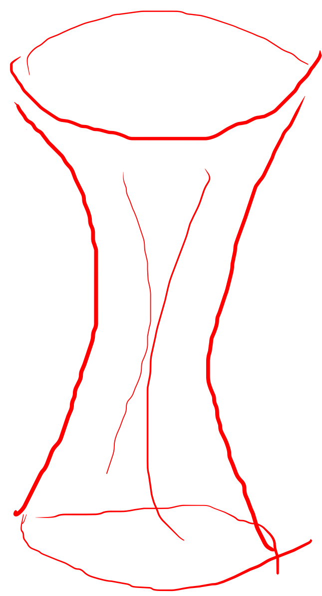
Proiezioni sui piani

$x=0 \qquad y^2-z^2=-1 \quad +1$

$\cancel{z^2-y^2=1}$

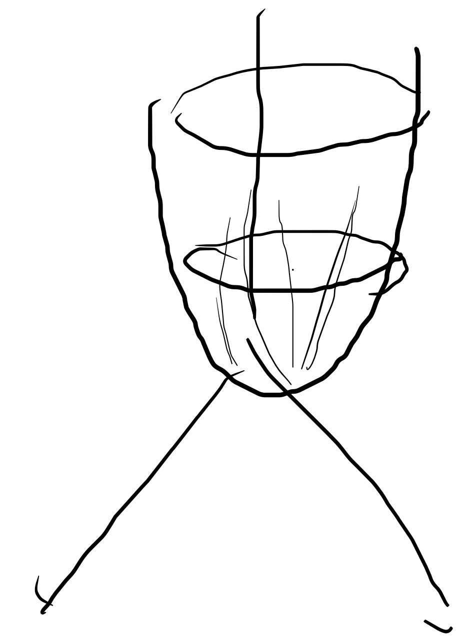
$y=0 \quad z^2=1$

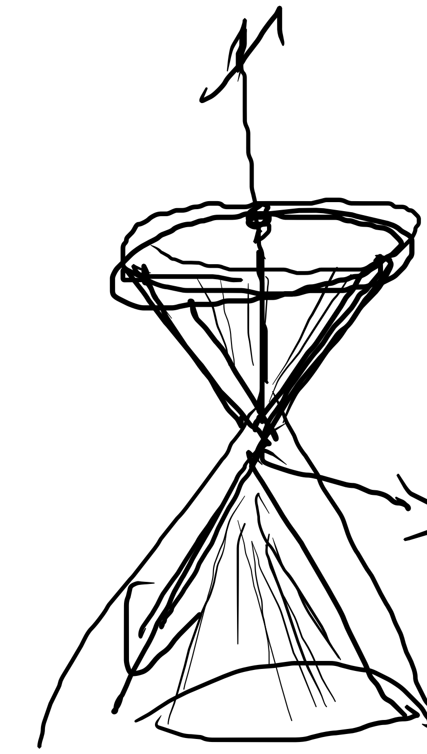




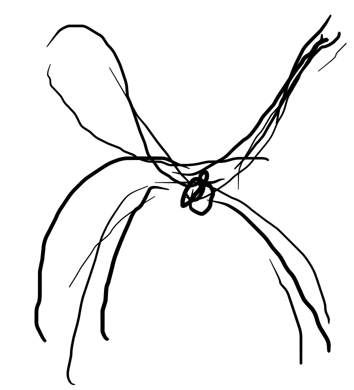
$x^2+y^2-z=0$ $\quad$ $z=x^2+y^2$

paraboloide ellittico $\quad$ $f(x,y)=x^2+y^2$





$x^2-y^2-z=0$ $\quad$ $z=x^2-y^2$

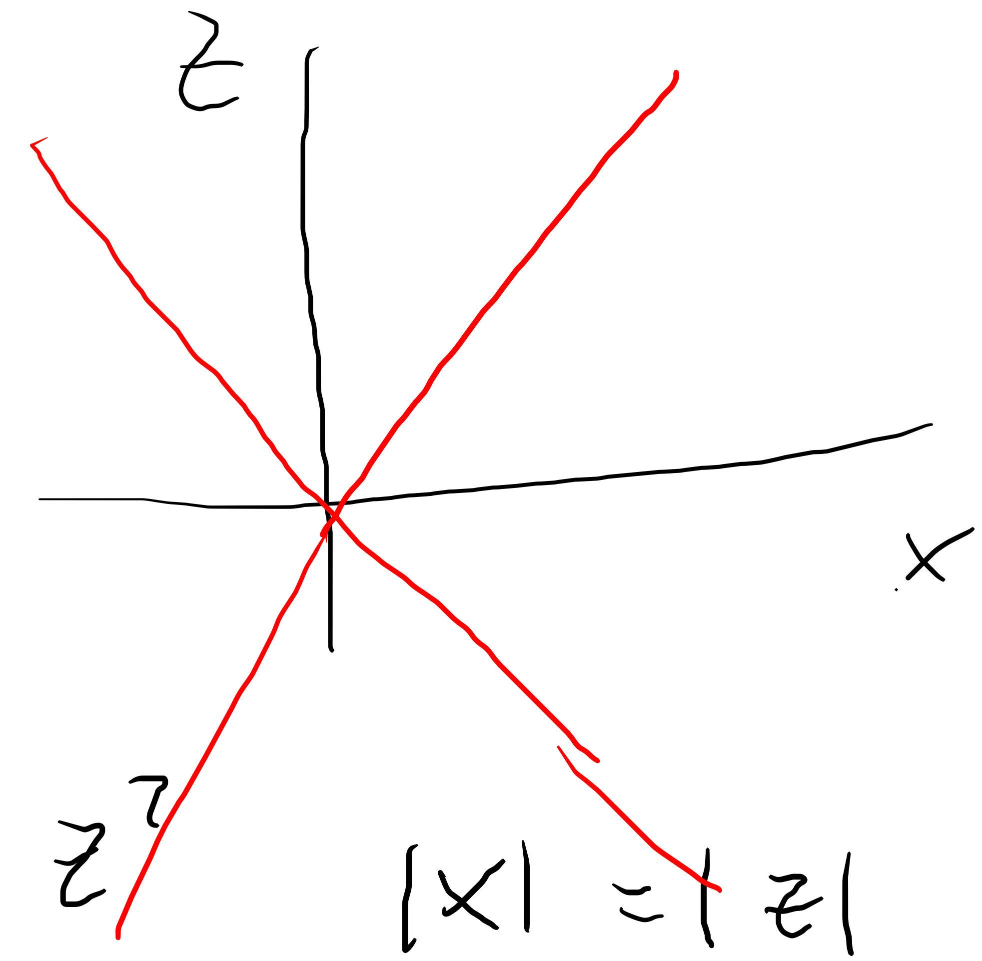


paraboloide iperbolico

$\boxed{x^2+y^2-z^2=0} \rightsquigarrow z^2=x^2+y^2$

$z>0$ $\quad$ $z=\sqrt{x^2+y^2} = \|(x,y)\|$

$x^2+y^2-z^2=0$ $\qquad y=0$



$x^2=z^2$ $\qquad |x|=|z|$

$x>0 \quad z>0 \quad z=x$

Derivate per campi scalari $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ $\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$ $\rightsquigarrow$ $\boxed{\lim_{h \to 0} \frac{f(x_0+h\bar{v}) - f(x_0)}{h}}$

la "pendenza" dipende dalla direzione

$\{ \bar{v} \quad \|\bar{v}\| = 1$ versore

$\varphi(h) = f(x_0 + h\bar{v})$

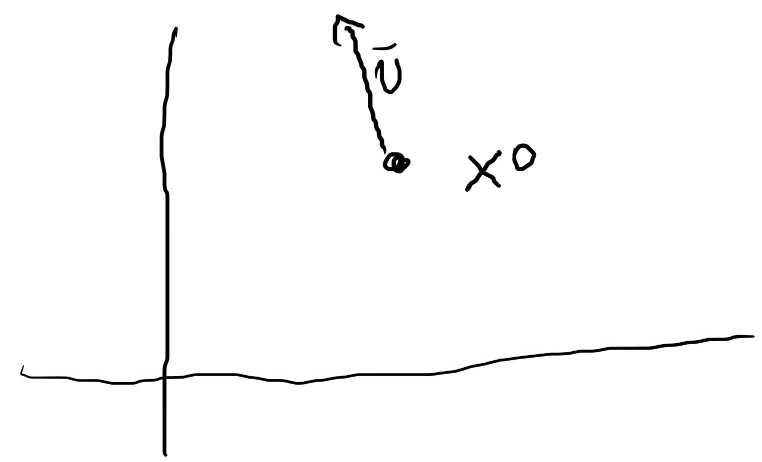
$\varphi: \mathbb{R} \to \mathbb{R}$

"pendenza nella direzione di v"

$\varphi'(0)$

$f(x_0 + h\bar{v})$

mi sposto nella direzione $\bar{v}$



Sia $x^0 \in A$ $\quad f: A \subseteq \mathbb{R}^n \to \mathbb{R}^M$ esista $B(x^0, \varepsilon) \subset A$; $v \in \mathbb{R}^n$ $\|v\|=1$

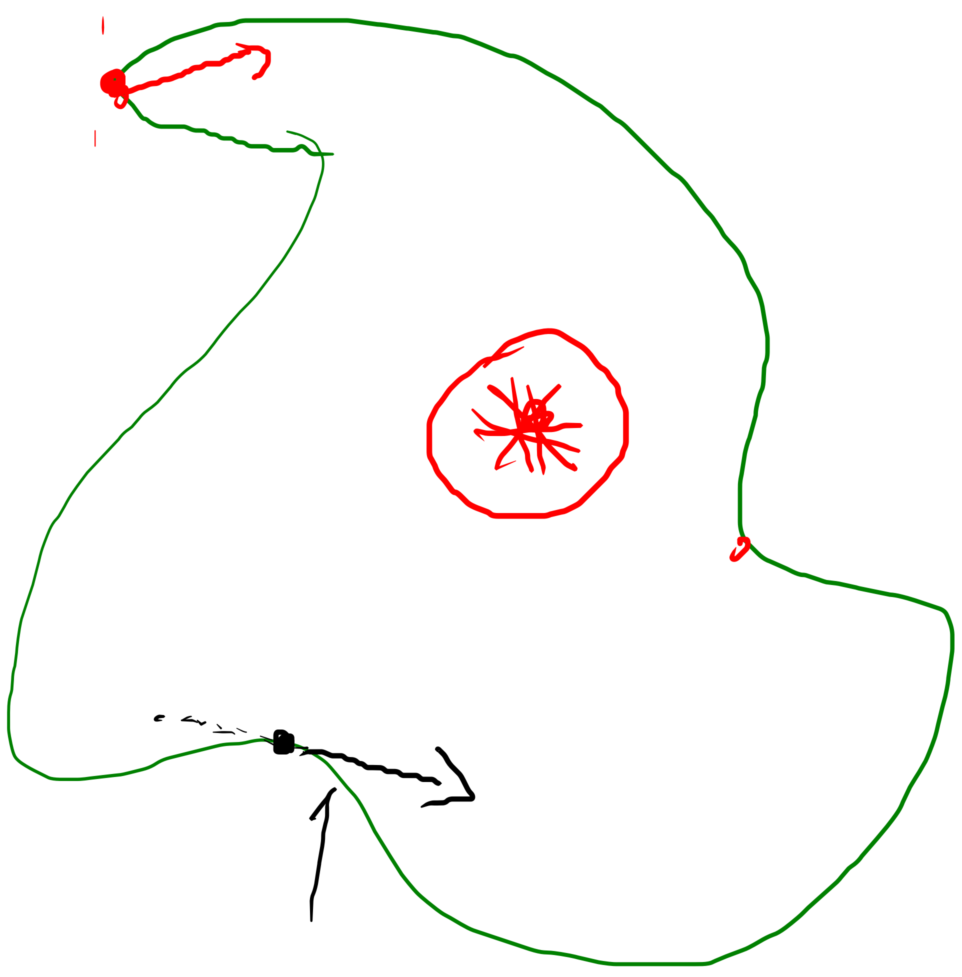
considero la funzione $\varphi :]-\varepsilon, \varepsilon[\to \mathbb{R}$ $\quad \varphi(t) = f(x^0 + t v)$.

Se esiste $\varphi'(0)$ questo si dice derivata direzionale di f in x^0 nella direzione v. Scriveremo

$$\boxed{\frac{\partial f}{\partial v}(x^0)} = \lim_{t \to 0} \frac{f(x^0 + t v) - f(x^0)}{t} = \varphi'(0)$$

Se $M > 1$ $\quad \varphi'(0) \in \mathbb{R}^M$

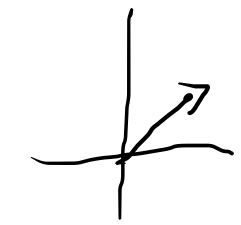
Se $M=2$ $\quad f = (f_1, f_2)^T \quad \dfrac{\partial f}{\partial v}(x^0) = \left(\dfrac{\partial f_1}{\partial v}(x^0), \dfrac{\partial f_2}{\partial v}(x^0) \right)^T$



$x^0 + t\bar{v}$ ho senso qualche volta considerare

$\frac{\partial f}{\partial v}(x^0)$ anche se x^0 non è interno al dominio

Esempio $f(x,y) = xy + 2x$ $\quad x^0 = (1,0)^T$ $\quad \bar{v} = \frac{1}{\sqrt{2}}(1,1)^T$



$$\boxed{\frac{\partial f}{\partial v}(1,0)} = \lim_{t \to 0} \frac{f\left((1,0)^T + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T\right) - f(1,0)}{t} =$$

$$\cancel{+2} + \frac{2t}{\sqrt{2}}$$

$$= \lim_{t \to 0} \frac{f\left(1 + \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right) - f(1,0)}{t} = \lim_{t \to 0} \frac{\left[\left(1 + \frac{t}{\sqrt{2}}\right)\frac{t}{\sqrt{2}} + 2\left(1 + \frac{t}{\sqrt{2}}\right)\right] - \cancel{2}}{t} =$$

$$= \lim_{t \to 0} \frac{\cancel{t}}{\cancel{t}}\left(\left(1 + \frac{t}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} \quad //$$

$\mathbb{R}^n$ direzioni speciali

$v = e_i$

Sia $\quad v = e_i = (0, \dots, \underset{\underset{i}{\uparrow}}{1}, 0, \dots, 0)^T \qquad x^0 = (x_1^0, x_2^0, \dots, x_i^0, \dots, x_N^0)^T$

$f : \mathbb{R}^N \to \mathbb{R}$

$$\frac{\partial f}{\partial e_i}(x^0) = \underbrace{\lim_{t \to 0} \frac{f(x_1^0, x_2^0, \dots, x_i^0 + t, \dots, x_N^0) - f(x_1^0, x_2^0, \dots, x_i^0, \dots, x_N^0)}{t}}$$

Consideriamo la funzione $\underset{\substack{\uparrow \\ x \in \mathbb{R}}}{g(x)} := \underbrace{f(x_1^0, x_2^0, \dots, \underset{\underset{i}{\uparrow}}{x}, x_{i+1}^0, \dots, x_N^0)}$

$g : \mathbb{R} \to \mathbb{R}$

$$\underline{g'(x_i^0)} = \lim_{t \to 0} \frac{g(x_i^0 + t) - g(x_i^0)}{t} = \frac{\partial f}{\partial e_i}(x^0)$$

$\dfrac{\partial f}{\partial e_i}(x^0)$ si dice derivata parziale di f rispetto alla variabile x_i

e si scrive $\dfrac{\partial f}{\partial x_i}(x^0)$

$\dfrac{\partial f}{\partial x_i}(x^0) = \dfrac{\partial f}{\partial e_i}(x^0) = g'(x^0)$ dove g è la funzione f in cui

ho "congelato" tutte le variabili diverse da x_i

$f(x,y) = \underline{xy+2x}$ $\qquad \dfrac{\partial f}{\partial x}(1,0) = 2 \qquad \dfrac{\partial f}{\partial y}(1,0) = 1$

$\dfrac{\partial f}{\partial x}(x,y) = y+2$

$\dfrac{\partial f}{\partial y}(x,y) = x$

$$f(x,y,z) = \frac{\operatorname{sen}(xy^2)}{z}$$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{1}{z}\cos(xy^2)\cdot y^2 \qquad \frac{\partial f}{\partial y}(x,y,z) = \frac{1}{z}\cos(xy^2)\,2xy$$

$$\frac{\partial f}{\partial z}(x,y,z) = -\frac{\operatorname{sen}(xy^2)}{z^2}$$

$$f(x,y,z) = \left(\underline{x^2+2y+z},\ -x+\log y,\ x+z^2-1\right)^T \qquad f: A \subseteq \mathbb{R}^3 \to \mathbb{R}^3$$

$$\frac{\partial f}{\partial x}(x,y,z) = \left(2x,\ -1,\ 1\right)^T \qquad \frac{\partial f}{\partial y}(x,y,z) = \left(2,\ \frac{1}{y},\ 0\right)^T \qquad \frac{\partial f}{\partial z}(x,y,z) = \left(1,\ 0,\ 2z\right)^T$$

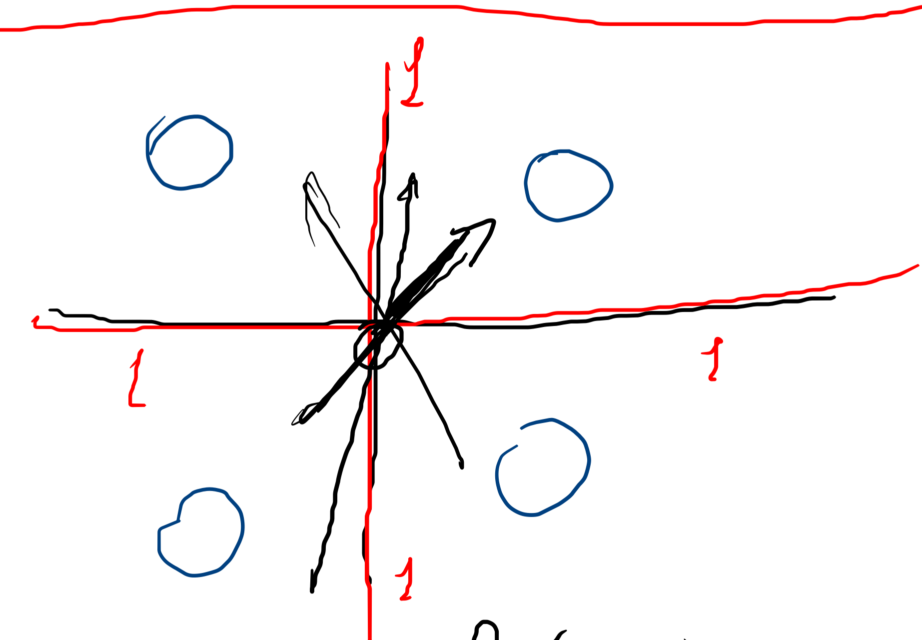
funzioni derivabili ? ... in una direzione

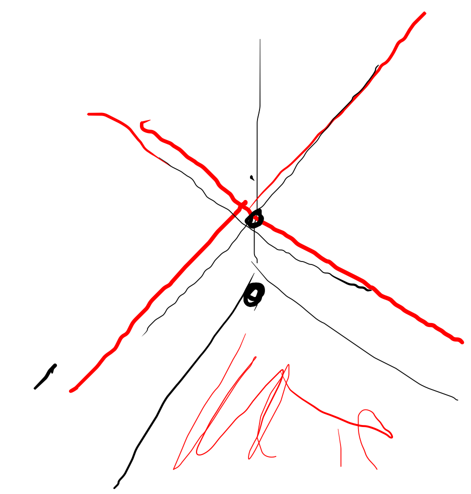
... rispetto ad una variabile ?

... in tutte le direzioni

... rispetto a tutte le variabili

es: $f(x,y) = \begin{cases} 1 & \text{se } xy = 0 \\ 0 & \text{se } xy \neq 0 \end{cases}$





$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$$

$$\frac{\partial f}{\partial v}(0,0) = ? \quad \lim_{t \to 0} \frac{f(tv) - f(0,0)}{t} = \lim_{t \to 0} \frac{0 - 1}{t} \nexists$$

$v = \frac{1}{\sqrt{2}}(1,1)^T$

⚠ Una funzione può avere tutte le derivate parziali ma non avere altre derivate direzionali

Una funzione può avere tutte le derivate direzionali ma NON essere continua (Es. 4)

differenziabile

Funzioni differenziabili

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m \qquad x^0 \in A$

f si dice differenziabile in x^0 se esiste un'applicazione lineare $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ tale che

$(0, 0 \dots, 0)^T \in \mathbb{R}^m$

$$\lim_{x \to x^0} \frac{f(x) - f(x_0) - L(x - x_0)}{\|x - x^0\|} = 0$$

L ?

Se f è differenziabile $L = df(x_0)$ si dice differenziale di f in x^0

NB $df(x_0)$ è un'applicazione lineare !

$$df(x_0)(v) = a v_1 + b v_2$$

$v = (v_1, v_2)^T$

$\cancel{dx}$

$\cancel{f(x) = x}$ ~~identità~~

$\cancel{df(x) = dx}$

Come calcolare il differenziale ?

$df(x_0)$ è un applicazione lineare $\quad df(x_0): \mathbb{R}^N \to \mathbb{R}^M$

Possiamo rappresentare il differenziale con una matrice M righe N colonne

$$\begin{pmatrix} df(x_0)(e_1) & df(x_0)(e_2) & \cdots & df(x_0)(e_N) \end{pmatrix} = Jf(x_0)$$

matrice Jacobiana di f in x_0

Teorema Proprietà di una funzione differenziabile

$f: A \subseteq \mathbb{R}^N \to \mathbb{R}^M$ $x^0 \in A$ f differenziabile in x^0. Allora

1) f è continua in x^0.

2) f è derivabile in x^0 lungo ogni direzione e si ha

$$\frac{\partial f}{\partial v}(x^0) = df(x^0)(v)$$

$$Jf(x^0) = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_N} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

In particolare $\boxed{\frac{\partial f}{\partial x_i}(x^0) = df(x^0)(e_i)}$

$$f(x,y) = \left(x^2+y^3,\ \operatorname{sen}(x)+y,\ y\right)^T \qquad f: \mathbb{R}^2 \to \mathbb{R}^3$$

$$Jf(x,y) = \begin{pmatrix} 2x & 3y^2 \\ \cos(x) & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{array}$$

$$v = \frac{1}{\sqrt{2}}(1,1)^T$$

$$\frac{\partial f}{\partial v}(0,1) = df(0,1)(v) = \begin{pmatrix} 0 & 3 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$$