Image Processing for Physicists

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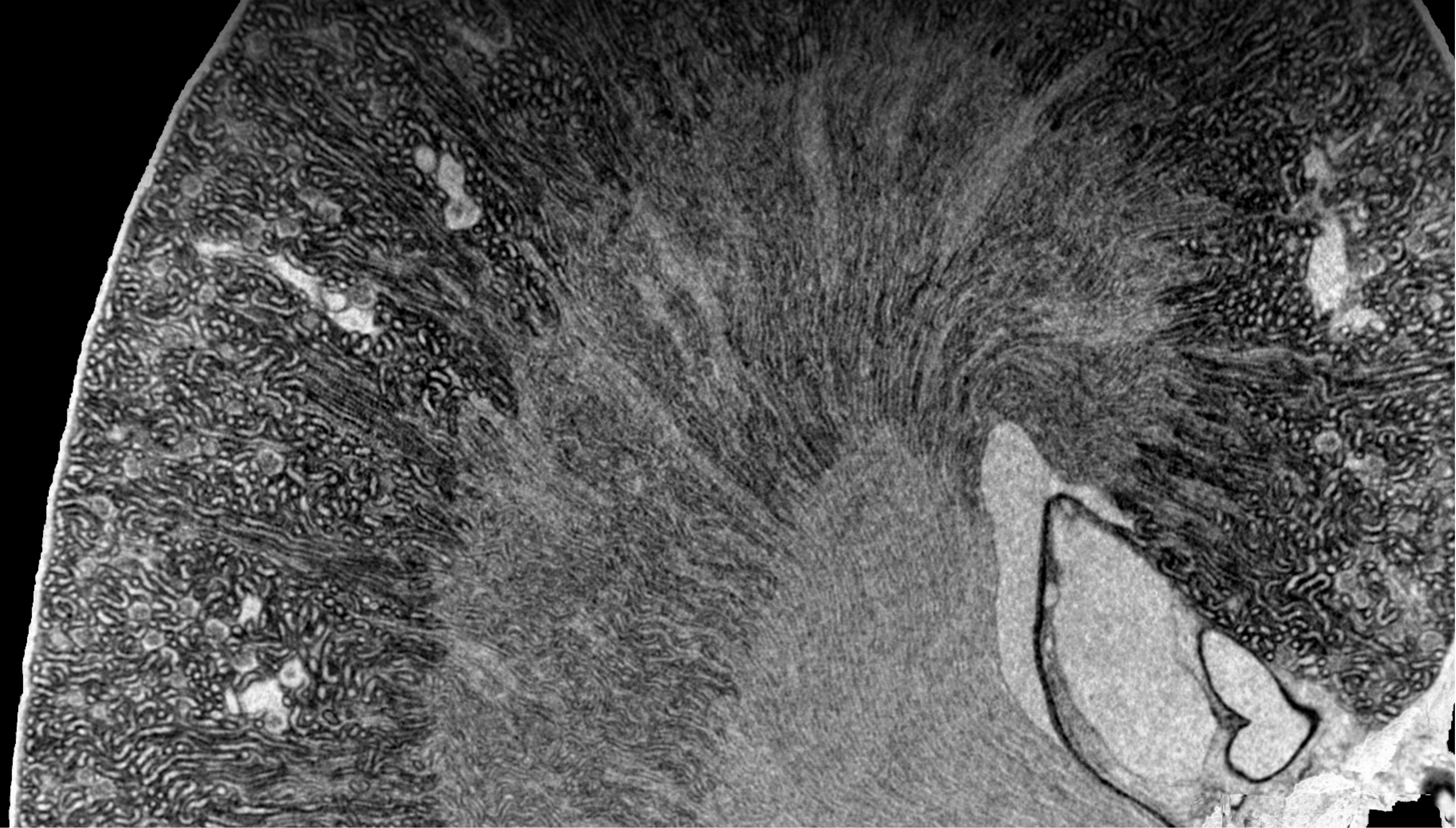


Overview

- The Fourier transform (FT)
  - introduction, properties
  - Fourier series, convolution, Dirac comb
  - Discrete Fourier transform (DFT), sampling, aliasing
- Linear filters
  - smoothing, sharpening, edge detection

Literature

- Rafael C. Gonzalez, “Digital Image Processing”, Prentice Hall International; (2008)
- E. Oran Brigham, “Fast Fourier Transform and Its Application”, Prentice Hall International; (1988)
- J.D. Gaskill, “Linear Systems, Fourier Transforms, and Optics”, John Wiley and Sons, (1978)

The Fourier transform

- First introduced by Joseph Fourier

  (1768-1830) to describe heat transfer
- today extremely important
- widely used in many fields
- fast computational implementation (FFT)
- original motivation: representation by easier-to-handle functions
- basis functions: oscillations (sine and cosine)
- describe signal by its frequency spectrum

What's a spatial frequency?

Analogy with time domain:

Temporal frequency : $\frac{\#\text{ cycles}}{\text{unit of time}}$.

For images:

spatial frequency : $\frac{\#\text{cycles}}{\text{unit length}}$

e.g. printer resolution "300 dpi" dots per <u>inches</u>

What's a spatial frequency?

High spatial frequencies:

- "fast" changes in image content, small details, edges, ...

Low spatial frequencies:

- "slow" changes in image content, large areas, plane regions, ...

Single frequencies are not localized in an image!

Definitions

convention most often used in imaging

- Continuous Fourier transform

$$\mathcal{F}\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\, e^{\overbrace{-2\pi i}^{} ux}\, dx$$

$$\mathcal{F}^{-1}\{F(u)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{2\pi i u x}\, du$$

(physics: $e^{-iq\cdot x}$

$u = \frac{q}{2\pi}$)

- Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k\, e^{2\pi i k\, x/p}$$

← period of $f(x)$

$$c_k = \frac{1}{p}\int_{-p/2}^{p/2} f(x)\, e^{-2\pi i k\, x/p}\, dx$$

- Discrete Fourier transform

$$F_k = \sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{kn}{N}}$$

f_n: sample points of a function

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k\, e^{2\pi i \frac{nk}{N}}$$

Properties

- linearity

$$a\,f(x) + b\,g(x) \xrightarrow{\mathcal{F}} a\,F(u) + b\,G(u)$$

- scaling

$$f(a.x) \xrightarrow{\mathcal{F}} \frac{1}{|a|} F\left(\frac{u}{a}\right)$$

- shifting/modulation

$$f(x - x_0) \xrightarrow{\mathcal{F}} F(u)\, e^{2\pi i u x_0}$$

"phase ramp"

- Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(u)|^2\, du$$

- 0-frequency term

$$F(u=0) = \int_{-\infty}^{\infty} f(x)\, dx$$

"DC term"
(constant term that doesn't oscillate)

Dirac distribution

- "sifting" property

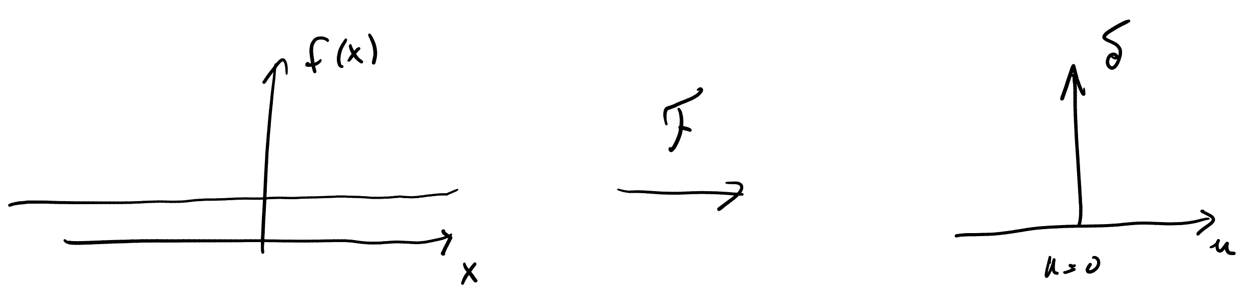
$$\int_{-\infty}^{\infty} f(x)\,\delta(x - x_0)\,dx = f(x_0)$$

- normalization

$$\int_{\infty}^{\infty} \delta(x)\,dx = 1$$

- relation to Fourier transforms

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{2\pi i u x}\,dx = \delta(u)$$



Convolution

- definition

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds$$

- commutativity, associativity, distributivity

$$f * g = g * f \qquad f * (g + h) = f * g + f * h$$

$$(f * g) * h = f * (g * h)$$

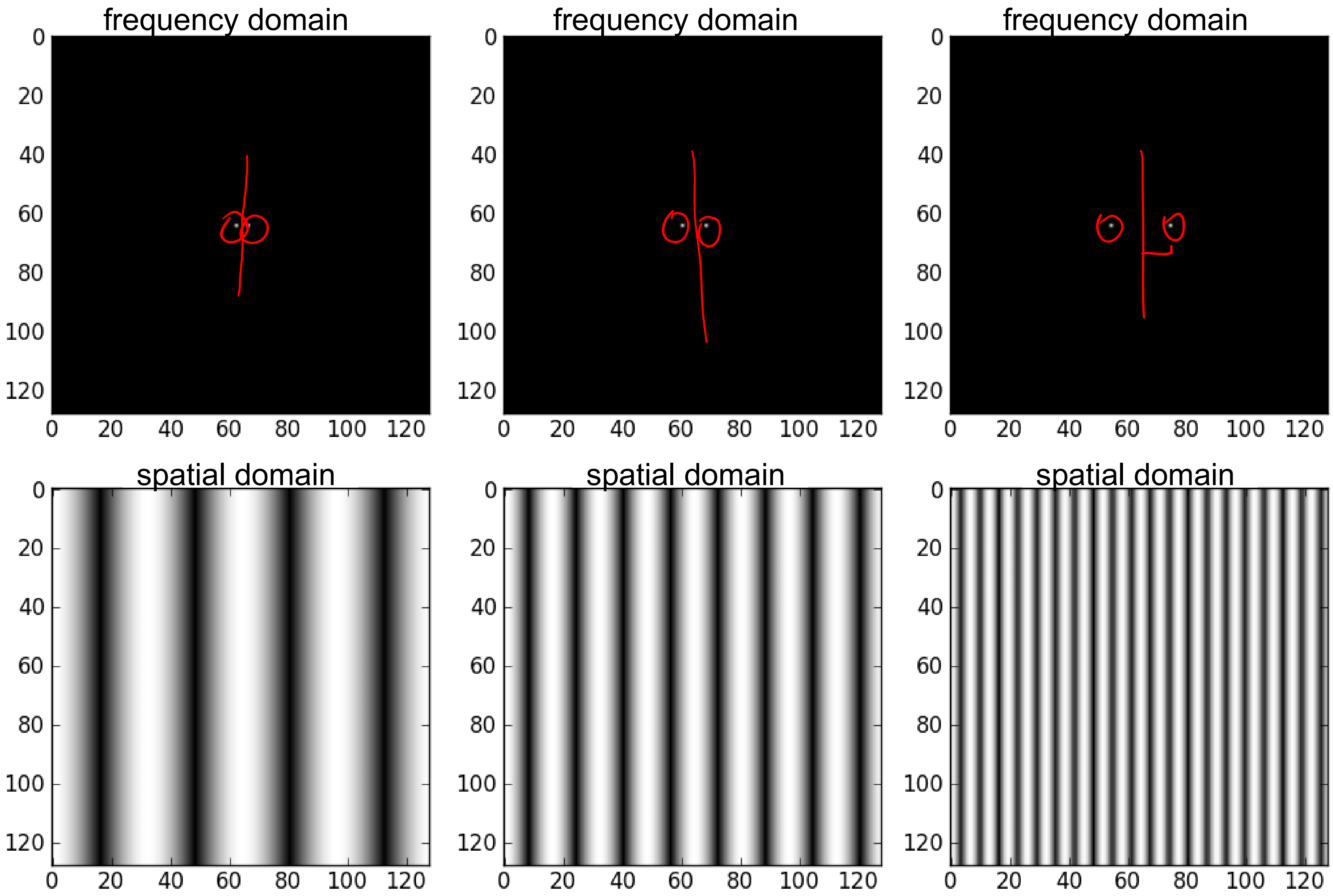
- Dirac distribution: indentity/translation

$$f(x) * \delta(x - x_0) = f(x - x_0)$$

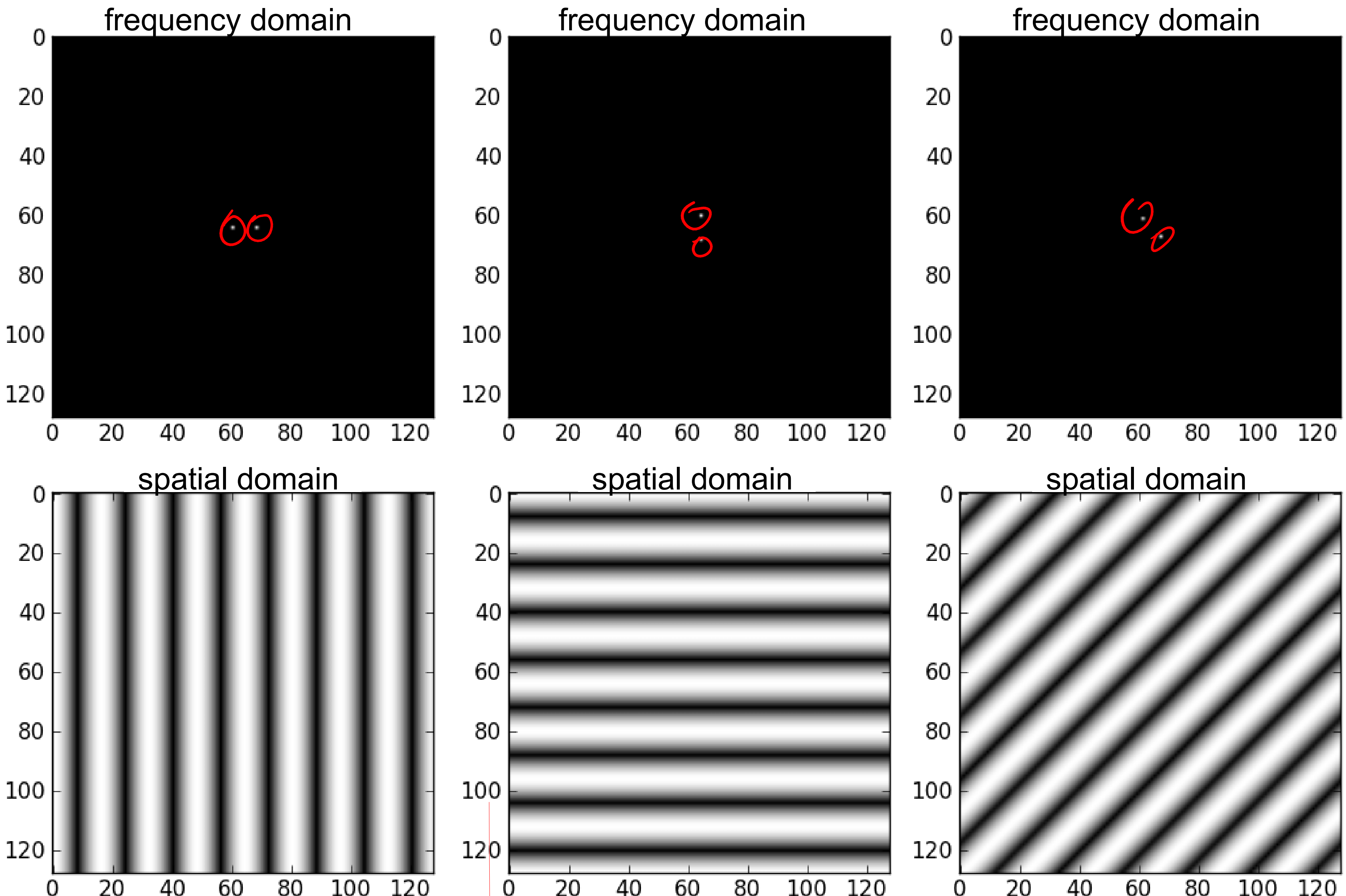
- relation to Fourier transforms

$$\mathcal{F}\{f * g\} = F(u) \cdot G(u)$$

Basic functions and their spectra



Basic functions and their spectra



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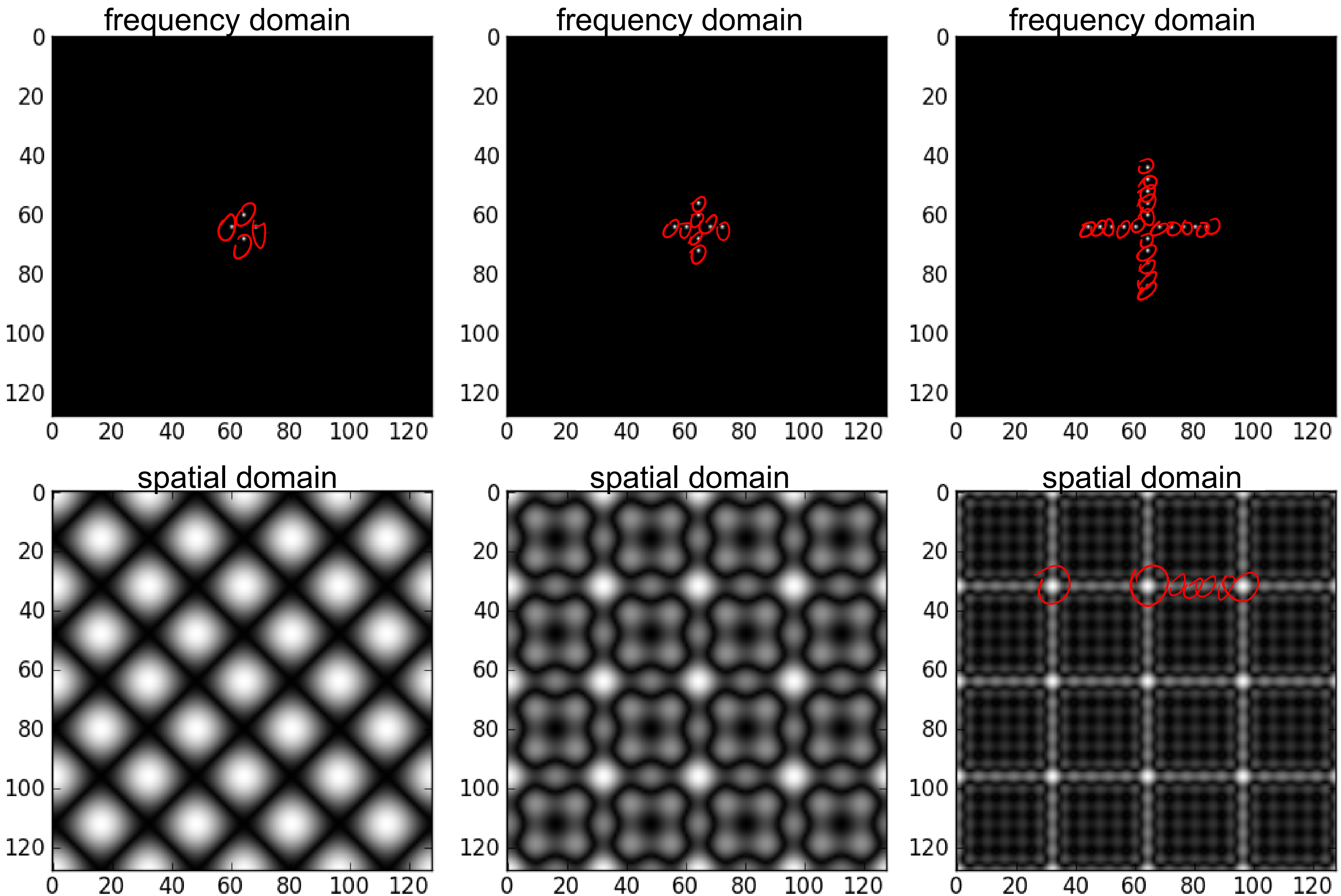
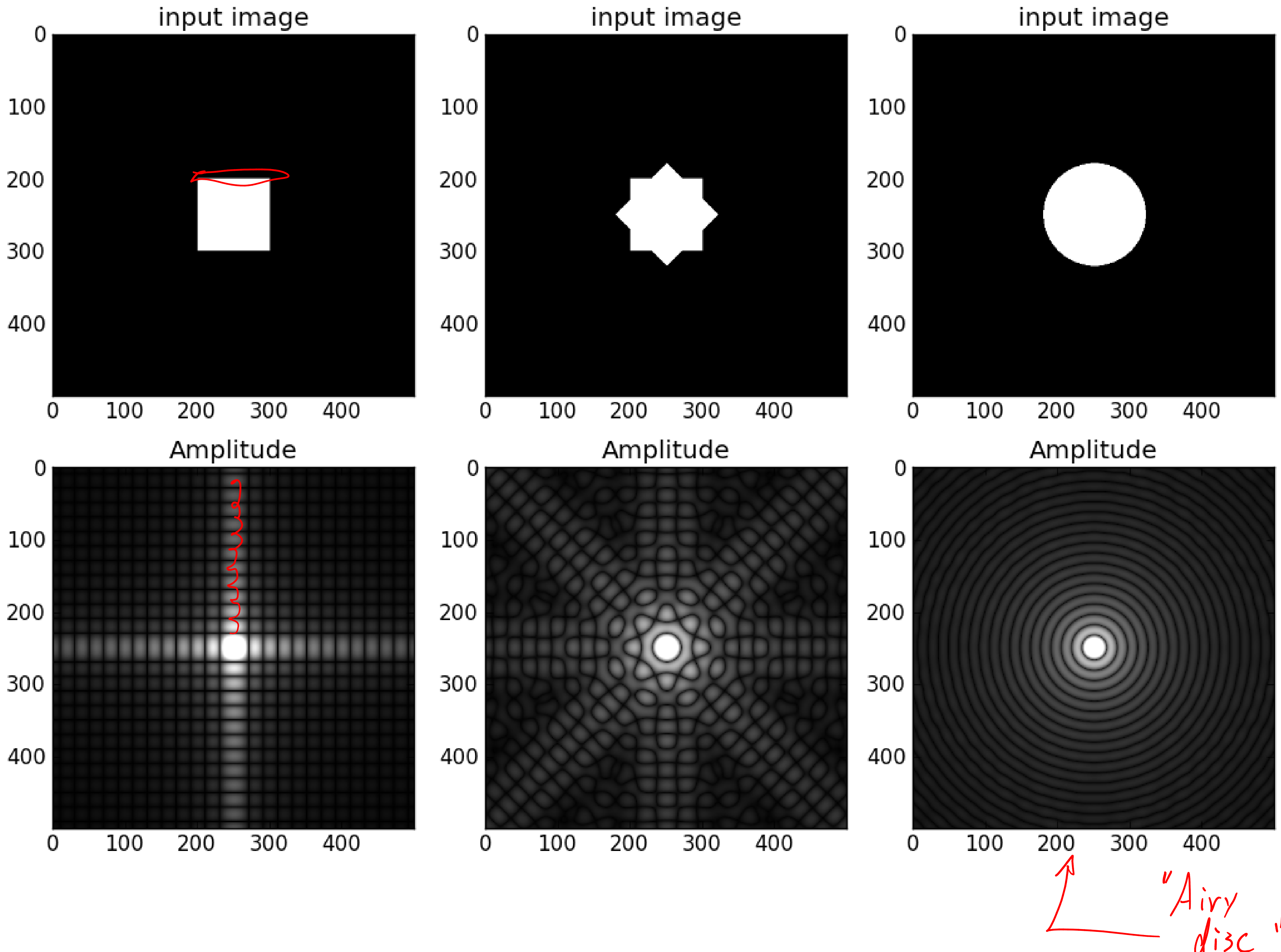
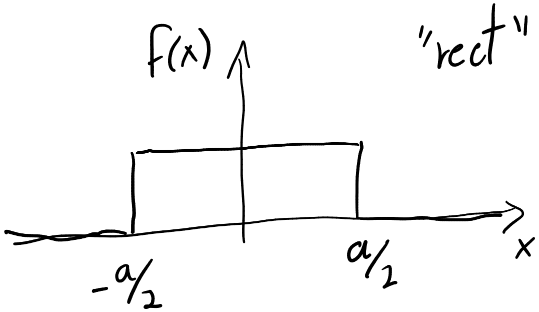
Basic functions and their spectra

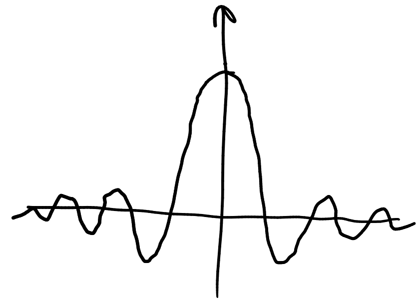


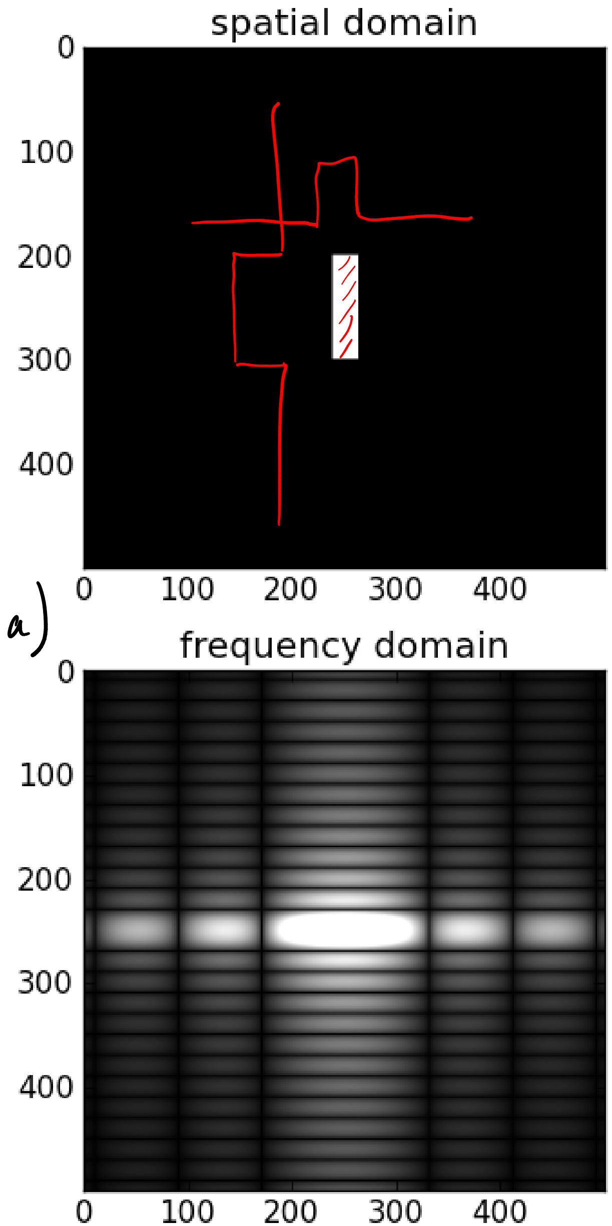
Basic functions and their spectra



Basic functions and their spectra







$$\mathcal{F}\{f(x)\} = \int_{-a/2}^{a/2} e^{-2\pi i u x}\, dx$$

$$= \frac{1}{-2\pi i u} e^{-2\pi i u x} \Big|_{-a/2}^{a/2}$$

$\sin(\pi u a)$

$$= \frac{1}{\pi u} \left(\frac{e^{\pi i u a} - e^{-\pi i u a}}{2i} \right)$$

$$= \frac{\sin(\pi u a)}{\pi u} \qquad \text{"sinc"} \qquad \frac{\sin x}{x}$$

$$= a \,\mathrm{sinc}(ua)$$

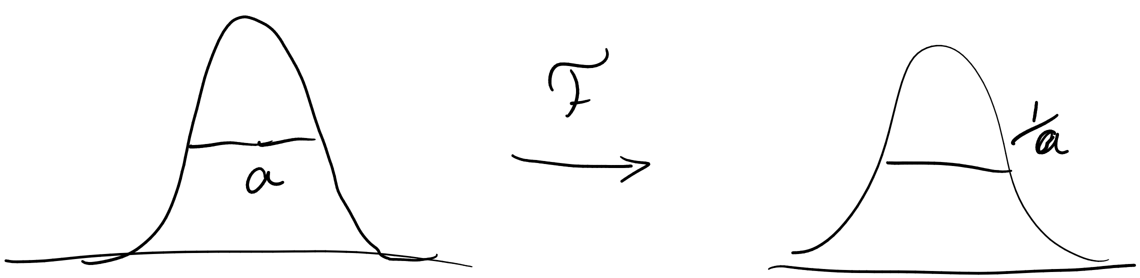
Basic functions and their spectra

Gaussian $e^{-x^2/2a^2}$ ← $\Delta x \sim a/2$

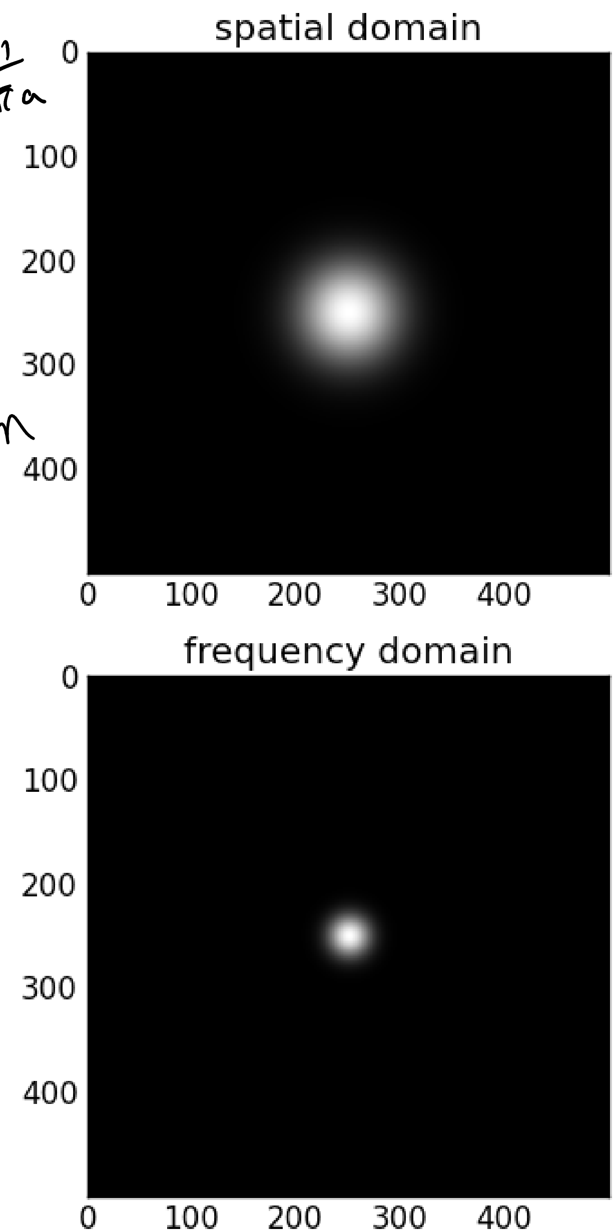
$\Delta u \sim \frac{1}{2\pi a}$

$$\mathcal{F}\{e^{-x^2/a^2}\} \propto e^{-\pi^2 u^2 a^2/2}$$

also a gaussian



"Δx" "Δu"



Additional properties

- uncertainty principle

$$\Delta x \, \Delta u \geq \frac{1}{4\pi}$$

- power spectrum

$$P(u) = |F(u)|^2$$

- derivatives

$$\mathcal{F}\left\{ \frac{\partial}{\partial x} f(x) \right\} = 2\pi i u \, F(u)$$

$$\frac{\partial^n}{\partial x^n} f \xrightarrow{\mathcal{F}} (2\pi i u)^n F(u)$$

- "Friedel" (crystallography terminology) symetry:

complex conjugate

if $f(x) \in \mathbb{R}$ then $F(u) = F^*(-u)$

Periodic signals

$f(x)$: Periodic function with period p

$$f(x) = \int_{-\infty}^{\infty} F(u)\, e^{-2\pi i u x}\, dx \qquad \text{(Fourier transform)}$$

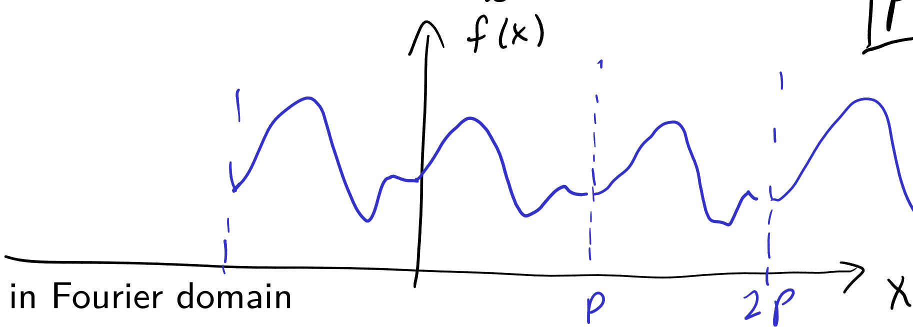
but also

$$f(x) = \sum_{k=-\infty}^{\infty} c_k\, e^{-2\pi i k x / p} \qquad \text{(Fourier series)}$$

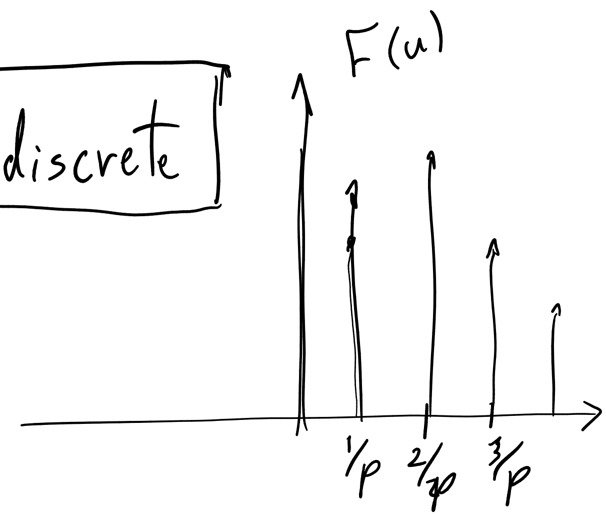
$$= \int_{-\infty}^{\infty} e^{-2\pi i u x} \sum_{k=-\infty}^{\infty} c_k\, \delta(u - k/p)\, dx$$

$$\Rightarrow F(u) = \sum_{k=-\infty}^{\infty} c_k\, \delta(u - k/p)$$

$\boxed{\text{periodic} \longrightarrow \text{discrete}}$



$\xrightarrow{\mathcal{F}}$



The Dirac comb

A periodic series of Delta functions

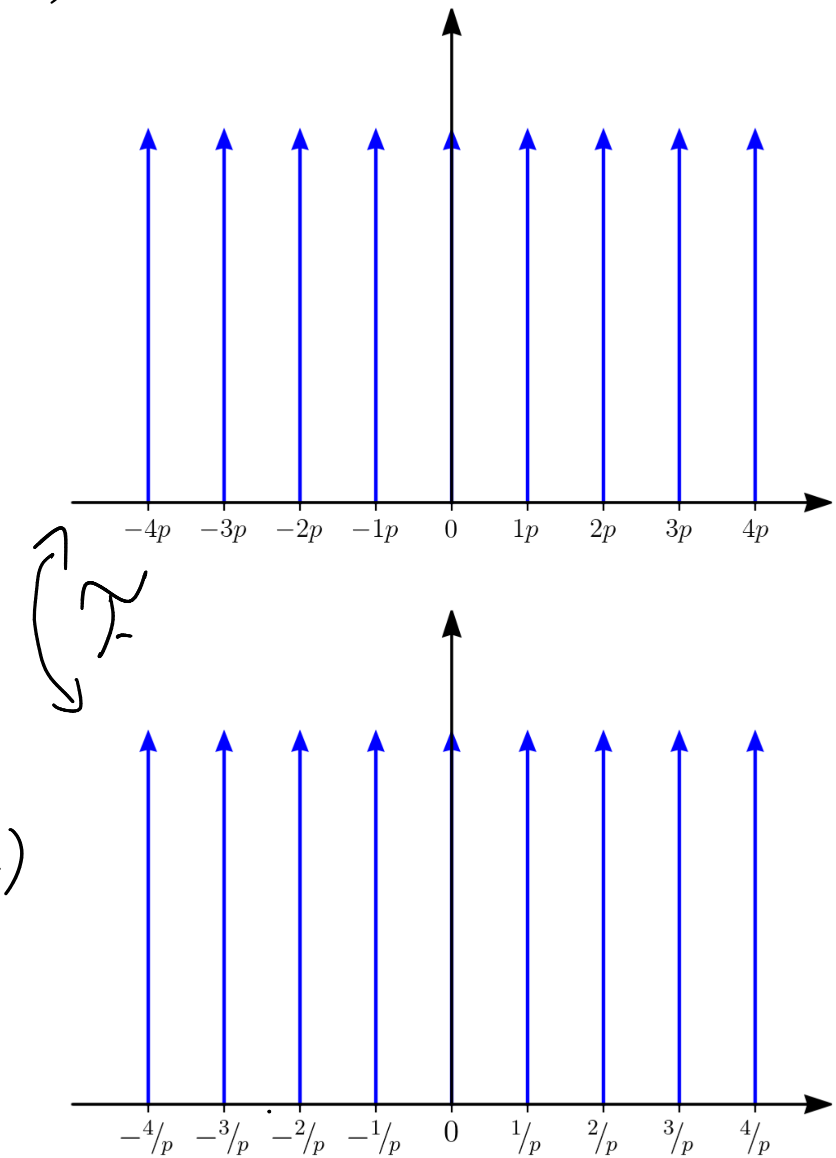
$$\Delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x-np)$$

$$\Delta_{\frac{1}{p}}(u) = \sum_{n=-\infty}^{\infty} \delta\left(u-\tfrac{k}{p}\right)$$

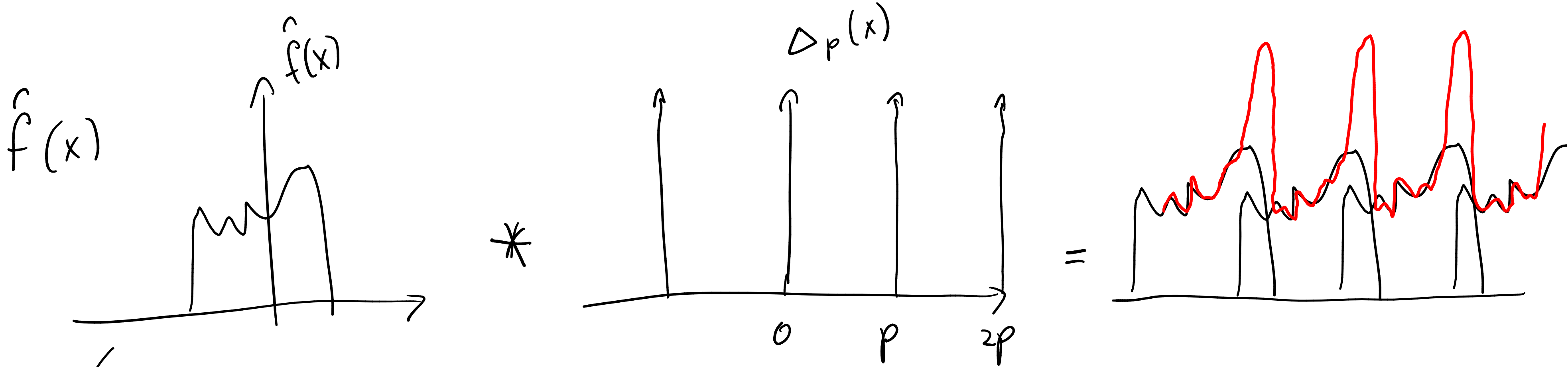
$$\mathcal{F}\{\Delta_p(x)\} = \frac{1}{p}\Delta_{\frac{1}{p}}(u)$$

→ F.T. of periodic fct: $\hat{F}^{*}(u) \cdot \Delta_{\frac{1}{p}}(u)$

→ Periodic fct is $\hat{f}(x) * \Delta_p(x)$



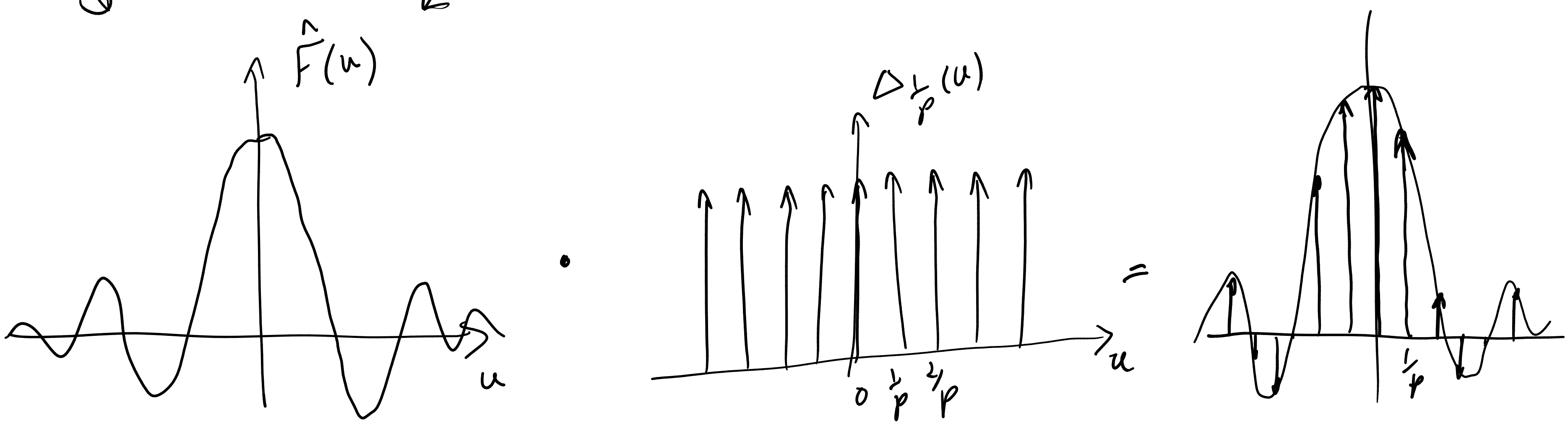
* not the same as $F(u)$ above



$\hat{\ }$ recipe to make any function $\hat{f}(x)$ periodic

$\mathcal{F}$

recipe to sample a function

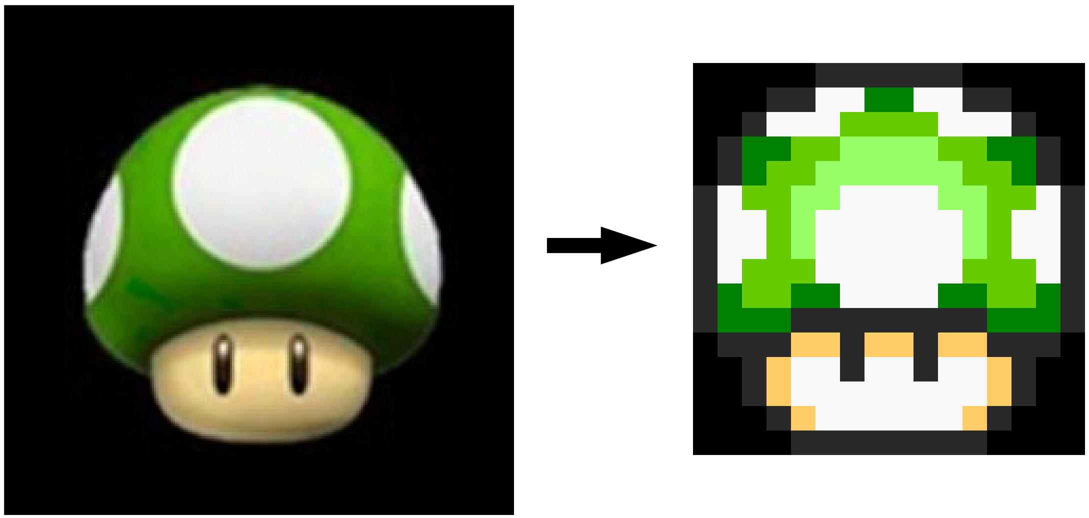


The discrete Fourier transform

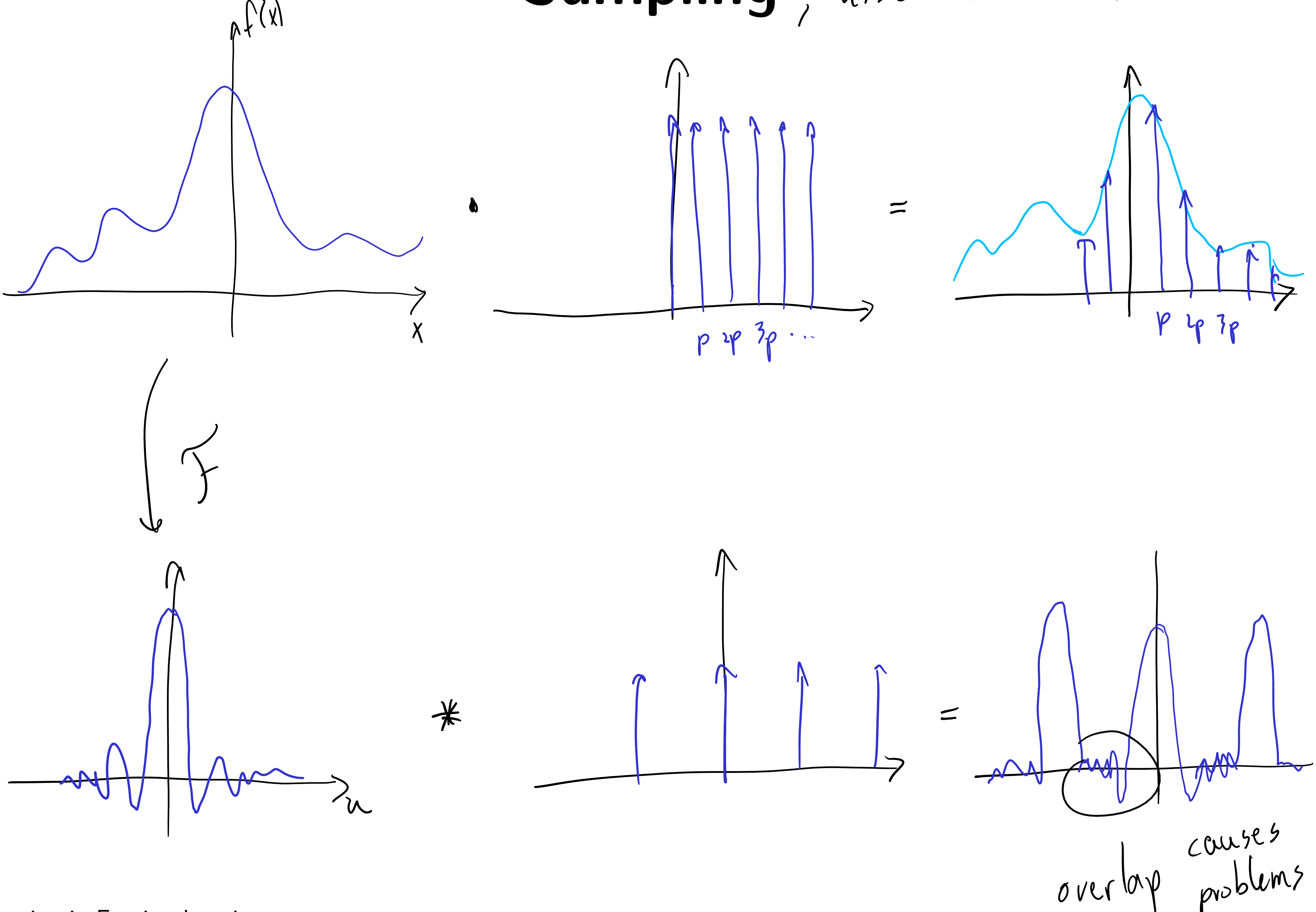
- additional ingredients needed:
  - sampling in space
  - finite field of view in space
  - sampling in frequency domain
  - finite frequency band

periodic

- discrete approximation of some continuous function

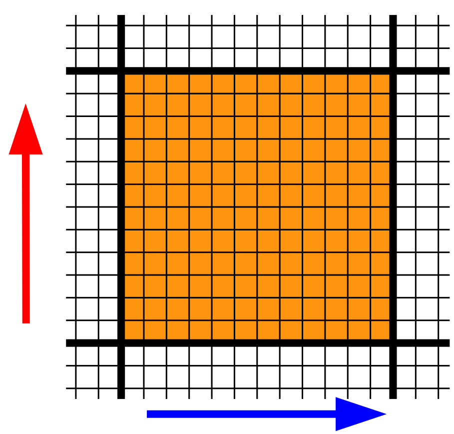


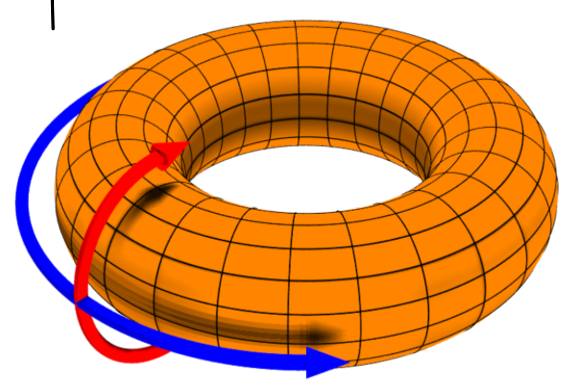
Sampling, discretization



Summary

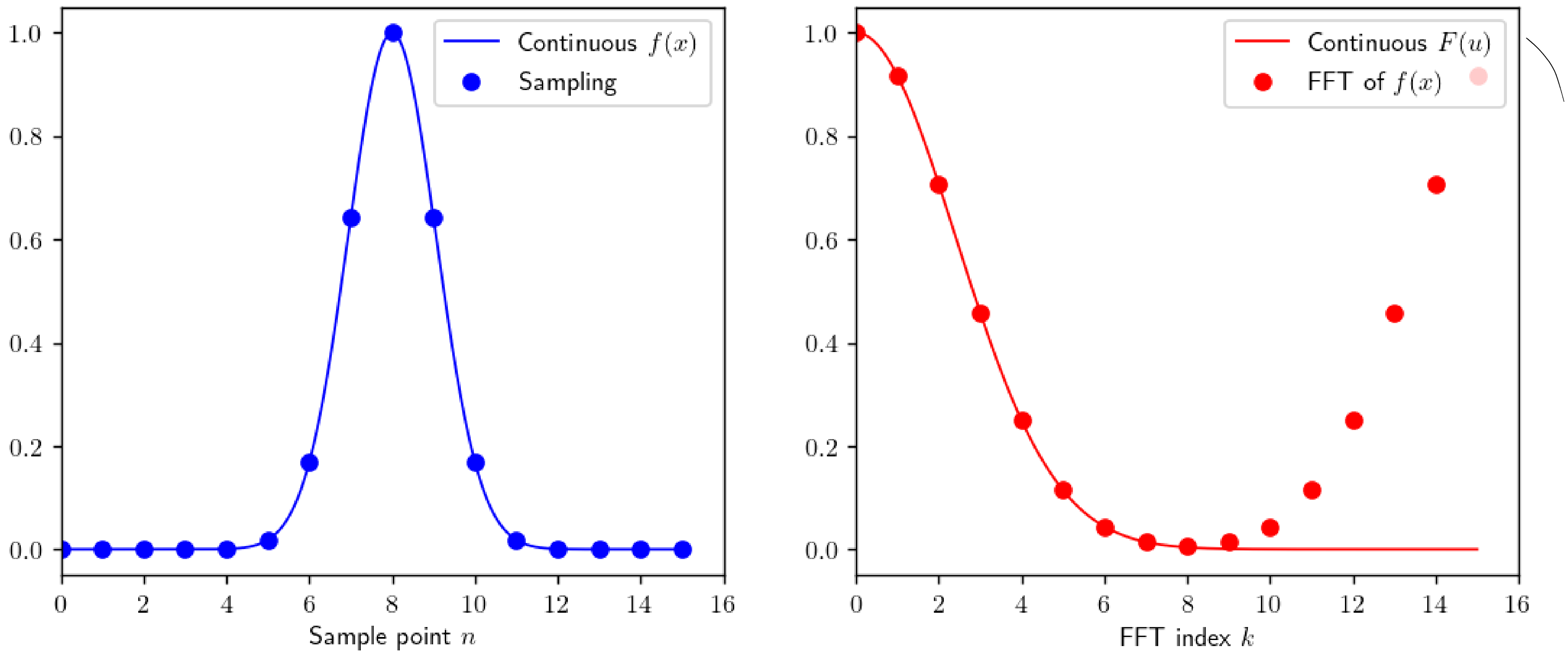
	real space	Fourier space
F.T.	continuous, infinite	continuous, infinite domain
F.S.	continuous, periodic	discrete, infinite
D.F.T.	discrete, periodic	discrete, periodic





DFT example

- Example: relation between space, sampling and frequency

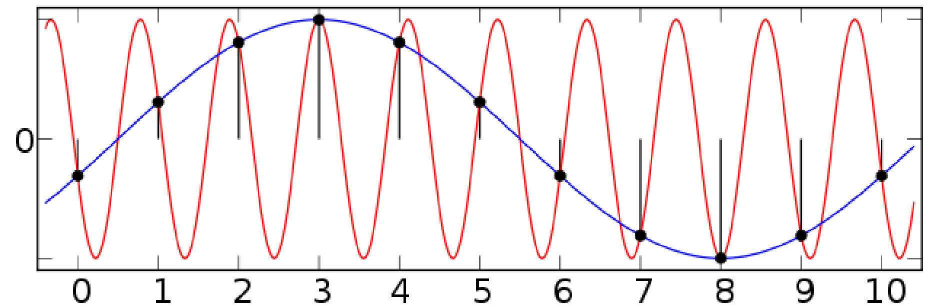


zero frequency component is in the top ~~right~~ left corner output array.

fftshift: brings 0 frequency in the middle.

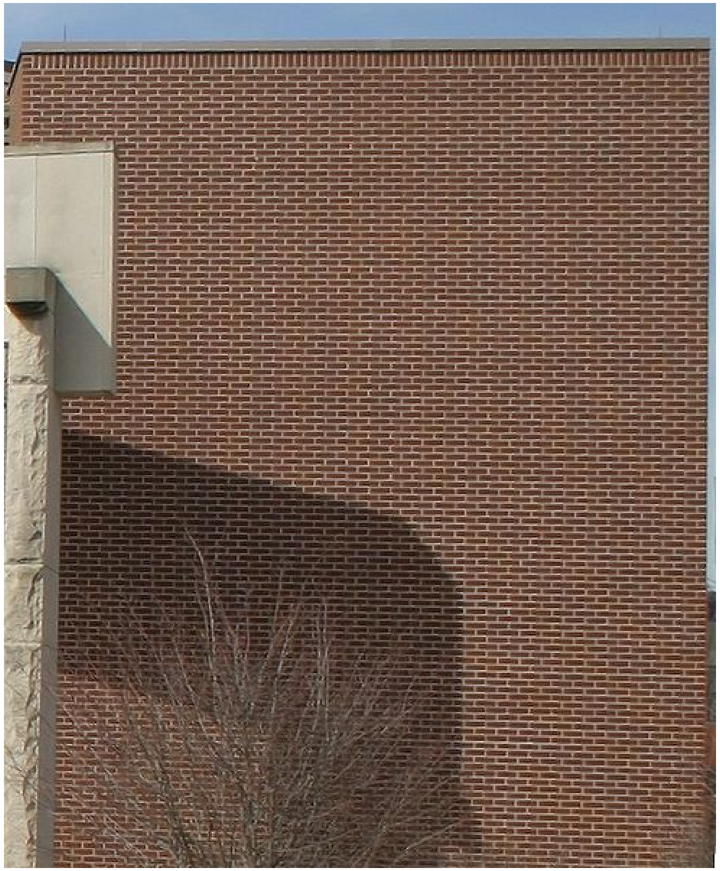
Aliasing

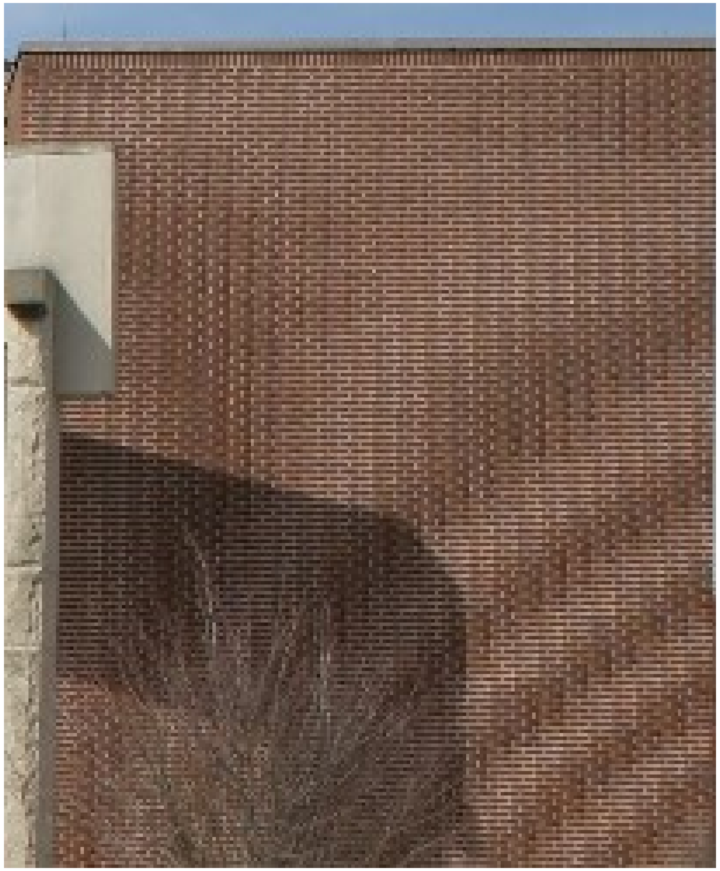
Moiré: after resampling, high spatial frequencies appear as low spatial frequencies



high resolution

low resolution





"moiré"

source: http://wikipedia.org