ES: trovare $Y_{\ell,\ell-1}(\theta,\varphi) = ?$

Conosco: ① $L_\pm |\ell,m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m\pm1)}\ |\ell,m\pm1\rangle$

② $Y_{\ell,\ell}(\theta,\varphi) = \sqrt{\frac{(2\ell+1)!}{2\pi}}\ \frac{1}{2^\ell \ell!}(\sin\theta)^\ell e^{i\varphi\ell}$

$\boxed{m=\ell}$ $L_-|\ell,\ell\rangle = \hbar\sqrt{\cancel{\ell^2}+\ell-\cancel{\ell^2}+\ell}\ |\ell,\ell-1\rangle = \hbar\sqrt{2\ell}\ |\ell,\ell-1\rangle$

$(L_- Y_{\ell,\ell})(\theta,\varphi) = \hbar\sqrt{2\ell}\ Y_{\ell,\ell-1}(\theta,\varphi)$

• $L_- = \hbar e^{i\varphi}\left(i\cot\theta\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta}\right)$

$(L_- Y_{\ell,\ell})(\theta,\varphi) = \hbar e^{i\varphi}\left(i\cot\theta\frac{\partial}{\partial\varphi}Y_{\ell,\ell}(\theta,\varphi) - \frac{\partial}{\partial\theta}Y_{\ell,\ell}(\theta,\varphi)\right) = \hbar\sqrt{2\ell}\ Y_{\ell,\ell-1}(\theta,\varphi)$

$\rightarrow \frac{\partial}{\partial\varphi}Y_{\ell,\ell}(\theta,\varphi) = \sqrt{\frac{(2\ell+1)!}{2\pi}}\ \frac{1}{2^\ell \ell!}(\sin\theta)^\ell\ i\ell\ e^{i\varphi\ell}$

$\rightarrow \frac{\partial}{\partial\theta}Y_{\ell,\ell}(\theta,\varphi) = \sqrt{\frac{(2\ell+1)!}{2\pi}}\ \frac{1}{2^\ell \ell!}\ e^{i\varphi\ell}\ \ell(\sin\theta)^{\ell-1}\cos\theta$

$-\ell\hbar e^{i\varphi(\ell+1)}\sqrt{\frac{(2\ell+1)!}{2\pi}}\ \frac{1}{2^\ell \ell!}\left(\underbrace{\cot\theta(\sin\theta)^\ell + \cos\theta(\sin\theta)^{\ell-1}}_{2\cot\theta(\sin\theta)^\ell}\right) = \hbar\sqrt{2\ell}\ Y_{\ell,\ell-1}(\theta,\varphi)$

$-\ell\cancel{\hbar}e^{i\varphi(\ell+1)}\sqrt{\frac{(2\ell+1)!}{2\pi}}\ \frac{1}{2^\ell \ell!}\ 2\cot\theta(\sin\theta)^\ell = \cancel{\hbar}\sqrt{2\ell}\ Y_{\ell,\ell-1}(\theta,\varphi)$

$$\boxed{Y_{\ell,\ell-1}(\theta,\varphi) = -\frac{\sqrt{2\ell}}{\cancel{2}}\ e^{i\varphi(\ell+1)}\sqrt{\frac{(2\ell+1)!}{2\pi}}\ \frac{1}{2^\ell \ell!}\ \cancel{2}\cot\theta(\sin\theta)^\ell}$$