$$Y_{\ell\ell}(\theta,\varphi) = \sqrt{\frac{(2\ell+1)!}{2\pi}} \; \frac{1}{2^{\ell}\,\ell!} \; \sin^{\ell}\theta \; e^{i\varphi\ell}$$

$$\hat{\ell}_{-} = \hbar e^{-i\varphi}\left(i \operatorname{cotg}\theta \frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta}\right)$$

$$\left(\hat{\ell}_{-} Y_{\ell\ell}\right)(\theta,\varphi) = \underbrace{\hbar e^{-i\varphi} A}_{B} \left(i \operatorname{cotg}\theta \frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta}\right) \sin^{\ell}\theta \, e^{i\varphi\ell}$$

con $A := \sqrt{\frac{(2\ell+1)!}{2\pi}} \; \frac{1}{2^{\ell}\,\ell!}$

$$= B\left(i \operatorname{cotg}\theta \sin^{\ell}\theta \; i\ell e^{i\varphi\ell} - \ell \sin^{\ell-1}\theta \cos\theta \, e^{i\varphi\ell}\right)$$

$$= -B e^{i\ell\varphi} \sin^{\ell-1}\theta \left(\frac{\cos\theta}{\cancel{\sin\theta}} \cancel{\sin\theta}\, \ell + \ell\cos\theta\right)$$

$$= -B e^{i\ell\varphi} \sin^{\ell-1}\theta \; 2\ell\cos\theta$$

$$= -\hbar e^{-i\varphi} A \sin^{\ell-1}\theta \; 2\ell\cos\theta \; e^{i\ell\varphi}$$

$$= -\hbar e^{-i\varphi}\, 2\ell \, \frac{\cos\theta}{\sin\theta} \, A \sin^{\ell}\theta \; e^{i\ell\varphi}$$

$$= -\hbar e^{-i\varphi}\, 2\ell \operatorname{cotg}\theta \; Y_{\ell\ell}(\theta,\varphi)$$

$$Y_{\ell,\ell-1}(\theta,\varphi) = \frac{1}{\hbar\sqrt{2\ell}} \left(\hat{\ell}_{-} Y_{\ell\ell}\right)(\theta,\varphi)$$

$$= -e^{-i\varphi}\sqrt{2\ell} \operatorname{cotg}\theta \; Y_{\ell\ell}(\theta,\varphi)$$

$$= -\sqrt{2\ell}\, \cos\theta \sin^{\ell-1}\theta \sqrt{\frac{(2\ell+1)!}{2\pi}} \; \frac{1}{2^{\ell}\,\ell!} \; e^{i\varphi(\ell-1)}$$