

$$Y_{ll}(\theta, \varphi) = \sqrt{\frac{(2l+1)!}{2\pi}} \frac{1}{2^l l!} \sin^l \theta e^{i\varphi l}$$

$$\hat{L}_- = \hbar e^{-i\varphi} \left(i \cot \theta \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \theta} \right)$$

$$(\hat{L}_- Y_{ll})(\theta, \varphi) = \frac{\hbar e^{-i\varphi} A}{3} \left(i \cot \theta \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \theta} \right) \sin^l \theta e^{i\varphi l}$$

$$\text{or } A = \sqrt{\frac{(2l+1)!}{2\pi}} \frac{1}{2^l l!}$$

$$= 3 \left(i \cot \theta \sin^l \theta i \hbar e^{i\varphi l} - l \sin^{l-1} \theta \cos \theta e^{i\varphi l} \right)$$

$$= -3 \hbar e^{i\varphi l} \sin^{l-1} \theta \left(\frac{\cos \theta}{\sin \theta} \sin \theta l + l \cos \theta \right)$$

$$= -3 \hbar e^{i\varphi l} \sin^{l-1} \theta 2l \cos \theta$$

$$= -\hbar e^{-i\varphi} A \sin^{l-1} \theta 2l \cos \theta e^{i\varphi l}$$

$$= -\hbar e^{-i\varphi} 2l \frac{\cos \theta}{\sin \theta} A \sin^l \theta e^{i\varphi l}$$

$$= -\hbar e^{-i\varphi} 2l \cot \theta Y_{ll}(\theta, \varphi)$$

$$Y_{l, l-1}(\theta, \varphi) = \frac{1}{\hbar \sqrt{2l}} (\hat{L}_- Y_{ll})(\theta, \varphi)$$

$$= -e^{-i\varphi} \sqrt{2l} \cot \theta Y_{ll}(\theta, \varphi)$$

$$= -\sqrt{2l} \cos \theta \sin^{l-1} \theta \sqrt{\frac{(2l+1)!}{2\pi}} \frac{1}{2^l l!} e^{i\varphi(l-1)}$$