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3. $H' = \lambda x^2 e^{-\gamma x^2}$

$= \langle x | \lambda \hat{x}^2 \psi \rangle$

$= \lambda \langle x | \frac{\hbar}{2m\omega} (\hat{a}^+ + \hat{a})^2 \psi \rangle$

$(\hat{a}^+ + \hat{a})^2 = (\hat{a}^+)^2 + \hat{a}^2 + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+$

$= (\hat{a}^+)^2 + \hat{a}^2 + 2\hat{a}^+\hat{a} + 1$

$\tilde{E}_0^{(1)} = \langle E_0 | H' | E_0 \rangle$

$= \frac{\lambda\hbar}{2m\omega} \left(\langle 0 | \hat{a}^+ \hat{a}^+ \psi 0 \rangle + \langle 0 | \hat{a}\hat{a} \psi 0 \rangle + \langle 0 | 2\hat{a}^+\hat{a} \psi 0 \rangle + \langle 0 | \psi | 0 \rangle \right)$

$= \frac{\lambda\hbar}{2m\omega} \left(\cancel{\langle \hat{a} 0 | \hat{a}^+ \psi 0 \rangle} + \langle 2 | \psi 0 \rangle + 2 \cancel{\langle \hat{a} 0 | \hat{a} \psi 0 \rangle} + \langle 0 | \psi | 0 \rangle \right)$

$\frac{\hat{a}^+ |1\rangle}{\sqrt{2}} = |2\rangle$

$\langle x | 1 \rangle = \sqrt{\frac{2m\omega}{\hbar}} \, x \psi_0(x)$

[illegible]

$\sqrt{\frac{2m\omega}{\hbar}} \left[\sqrt{\frac{m\omega}{2\hbar}} x^2 \psi_0(x) - \frac{\hbar}{\sqrt{2m\omega\hbar}} \left(\psi_0(x) + x \underbrace{\partial_x \psi_0(x)}_{-\frac{m\omega}{\hbar} x \psi_0(x)} \right) \right]$

$= \psi_0(x) \sqrt{\frac{2m\omega}{\hbar}} \left[\cancel{\sqrt{\frac{m\omega}{2\hbar}} x^2} - \sqrt{\frac{\hbar}{2m\omega}} \cancel{- x^2 \sqrt{\frac{m\omega}{2\hbar}} x^2} \right]$

$= -\frac{\psi_0(x)}{\sqrt{2}}$

$\langle 2 | \psi 0 \rangle = -\frac{1}{\sqrt{2}} \langle 0 | \psi 0 \rangle$

$\langle 0 | \psi 0 \rangle = \int_{-\infty}^{+\infty} \langle 0 | x \rangle \langle x | \psi 0 \rangle \, dx = \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{+\infty} e^{-x^2\left(\frac{m\omega}{\hbar} + \gamma\right)} dx$

$= \frac{m\omega}{\pi\hbar} \sqrt{\frac{\pi}{\frac{m\omega}{\hbar} + \gamma}}$

$= \sqrt{\frac{m^2\omega^2\pi}{\pi^2\hbar^2\left(\frac{m\omega}{\hbar} + \gamma\right)}} = \sqrt{\frac{m^2\omega^2}{\pi\hbar m\omega + \pi\hbar^2\gamma}}$

$\Rightarrow \tilde{E}_0^{(1)} = \langle E_0 | H' | E_0 \rangle = \frac{\lambda\hbar}{2m\omega} m\omega \sqrt{\frac{1}{\pi\hbar(m\omega + \hbar\gamma)}} \left(1 - \frac{1}{\sqrt{2}}\right)$

$= \frac{\lambda}{2} \sqrt{\frac{\hbar}{\pi(m\omega + \hbar\gamma)}} \left(1 - \frac{1}{\sqrt{2}}\right)$