Systems Dynamics

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Lecture 3
Stability of Discrete-Time Dynamic Systems

Lecture 3: Table of Contents

3. Stability of Discrete-Time Dynamic Systems

Stability of Discrete-Time Dynamic Systems

When dealing with stability in the context of dynamic systems we consider three different cases (listed in order of decreasing generality):

1. Stability of state movements
2. Stability of equilibrium states
3. Stability of linear systems

Remark. Concerning case 1., we provide definitions and concepts in the context of general abstract dynamic systems so, for example, time-instants belong to any legitimate set of times T.

Stability of State Movements

Stability of State Movements

- Consider a general abstract dynamic system characterised by the state-transition function

$$\varphi(t, t_0, x_0, u(\cdot))$$

- Then, consider a generic **nominal state movement** for a given initial state $\bar{x}_0$ and a given input function $u(\cdot)$:

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

- Now, consider the **perturbed state movement** generated by a **perturbation of the initial state** and a **perturbation of the input function**:

$$\begin{aligned} x(0) &= \bar{x}_0 + \delta\bar{x} \\ u(\cdot) &= \bar{u}(\cdot) + \delta u(\cdot) \end{aligned} \implies \begin{array}{c} \varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot) + \delta u(\cdot)) \\ \text{Perturbed State Movement} \end{array}$$

Stability with Respect to Perturbations of the Initial State

The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **stable** with respect to perturbations of the initial state $\bar{x}_0$ if

$$\forall\, \varepsilon > 0\,,\, \forall t_0 > 0\; \exists\, \delta(\varepsilon, t_0) > 0 \;\; \text{such that if} \;\; \|\delta\bar{x}\| < \delta(\varepsilon, t_0)$$

then, it follows that

$$\|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon\,, \forall\, t \geq t_0$$

Asymptotic Stability with Respect to Perturbations of the Initial State

The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **asymptotically stable** with respect to perturbations of the initial state $\bar{x}_0$ if:

- it is **stable**, that is, if

  $$\forall\, \varepsilon > 0\,,\ \forall t_0 > 0\ \exists\, \delta(\varepsilon, t_0) > 0 \ \text{ such that if } \ \|\delta\bar{x}\| < \delta(\varepsilon, t_0)$$

  then, it follows that

  $$\|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon\,, \forall\, t \geq t_0$$

- it is **attractive**, that is, $\forall t_0 > 0\ \exists\ \eta(t_0) > 0$ such that

  $$\lim_{t \to +\infty} \|\varphi(t, t_0, \bar{x}_0 + \delta\bar{x}, \bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| = 0\,,\ \forall\, \|\delta\bar{x}\| < \eta(t_0)$$

Unstability with Respect to Perturbations of the Initial State

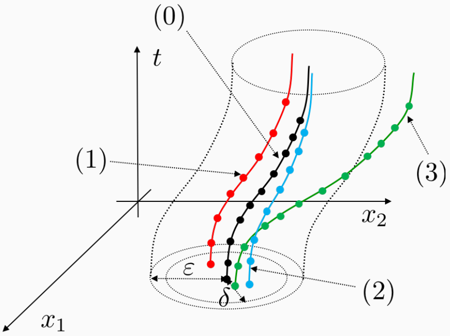
The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **unstable** with respect to perturbations of the initial state $\bar{x}_0$ if it is not stable with respect to such a kind of perturbations.

Geometrical Interpretation

- **(0)**: nominal state movement
- **(1)**: perturbed state movement remaining confined in the ”tube” of radius ε
- **(2)**: perturbed state movement remaining confined in the ”tube” of radius ε and asymptotically converging to the nominal movement
- **(3)**: perturbed state movement crossing the ”tube” of radius ε



Stability with Respect to Perturbations of the Input Function

The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **stable** with respect to perturbations of the input function $\bar{u}(\cdot)$ if

$$\forall \, \varepsilon > 0 \, , \, \forall t_0 > 0 \; \exists \, \delta(\varepsilon, t_0) > 0 \;\; \text{such that if} \;\; \forall \|\delta\bar{u}(\cdot)\| < \delta(\varepsilon, t_0)$$

then, it follows that

$$\|\varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot) + \delta\bar{u}(\cdot)) - \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))\| < \varepsilon \, , \forall \, t \geq t_0$$

Unstability with Respect to Perturbations of the Input Function

The **nominal state movement**

$$\bar{x}(\cdot) = \varphi(t, t_0, \bar{x}_0, \bar{u}(\cdot))$$

is **unstable** with respect to perturbations of the input function $\bar{u}(\cdot)$ if it is not stable with respect to such a kind of perturbations.

Stability of Equilibrium States

Stability of Equilibrium States

- Consider the discrete-time dynamic system

$$\begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = g(x(k), u(k)) \end{cases}$$

  and the equilibrium state $\bar{x}$ corresponding to a constant input sequence $u(k) = \bar{u}, \forall\, k \geq 0$, that is:

$$\bar{x} = f(\bar{x}, \bar{u})$$

- Now, consider a perturbation of the initial state with respect to the equilibrium state $\bar{x}$:

$$\begin{array}{l} x(0) = \bar{x} + \delta\bar{x} \\ u(k) = \bar{u},\ k \geq 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} x(k) \neq \bar{x},\ k \geq 0 \\ \text{perturbed state movement} \end{array}$$

Stability of Equilibrium States (cont.)

The **equilibrium state** is **asymptotically stable** if:

- It is **stable**, that is:

$$\forall \varepsilon > 0 \;\; \exists \delta(\varepsilon) > 0 \;\; \text{such that}: \\ \forall x(0): \; \|\delta \bar{x}\| < \delta(\varepsilon) \quad \Longrightarrow \quad \|x(k) - \bar{x}\| < \varepsilon, \forall k \geq 0$$

- It is **attractive**, that is:

$$\lim_{k \to \infty} \|x(k) - \bar{x}\| = 0$$

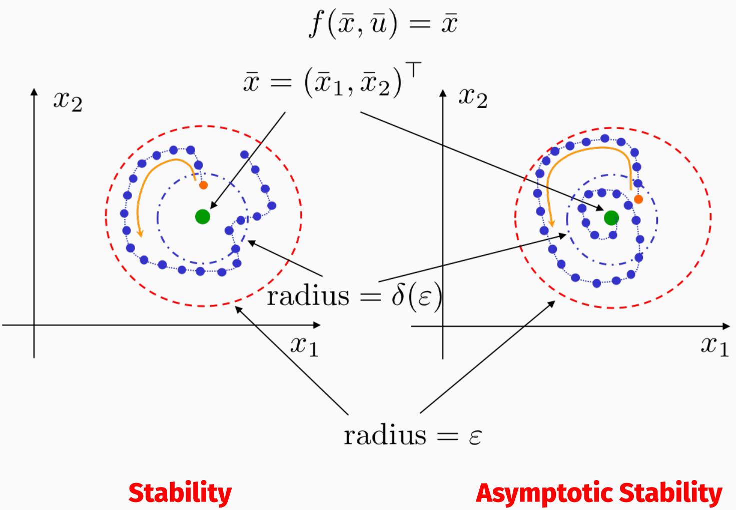
In **qualitative** terms:

when the initial state is perturbed, the state remains ”close” to the nominal equilibrium state and tends to return asymptotically to this equilibrium state.

Stability of Equilibrium States (cont.)

The **equilibrium state** is **unstable** if it is not stable.

Stability of Equilibrium States: Geometric Interpretation



Stability of Equilibrium States

Stability of State Movements and of Equilibrium States

Stability of State Movements and of Equilibrium States

- Consider the general discrete-time dynamic system

$$x(k+1) = f(x(k), u(k), k)$$

and consider a **nominal** state movement

$$\bar{x}(k) = \varphi(k, k_0, \bar{x}_0, \{u(k_0, \ldots, u(k-1)\})$$

starting from the initial state $\bar{x}(k_0) = \bar{x}_0$.
- We analyse the stability of the nominal movement $\bar{x}(k)$ with respect to perturbations of the initial state $\bar{x}_0$, that is, we consider the perturbed state movement

$$x(k) = \varphi(k, k_0, x_0, \{u(k_0, \ldots, u(k-1)\})$$

starting from the perturbed initial state $x_0 \neq \bar{x}_0$.
- Hence, introducing the difference between the perturbed and the nominal state movement $z(k) := x(k) - \bar{x}(k)$, one gets:

$$z(k+1) = x(k+1) - \bar{x}(k+1) = f(z(k) + \bar{x}(k), \bar{u}(k), k) - f(\bar{x}(k), \bar{u}(k), k)$$

Stability of State Movements and of Equilibrium States (cont.)

- Letting:

$$w_{\bar{x},\bar{u}}(z(k),k) := f(z(k)+\bar{x}(k),\bar{u}(k),k) - f(\bar{x}(k),\bar{u}(k),k)$$

  it follows that the dynamics of $z(k)$ can be described by the autonomous (in general time-varying) system

$$z(k+1) = w_{\bar{x},\bar{u}}(z(k),k) \quad (\star)$$

  where the function $w_{\bar{x},\bar{u}}$ is parametrised by the nominal state movement $\{\bar{x}(k)\}$ and the nominal input $\{\bar{u}(k)\}$.
- The function $w_{\bar{x},\bar{u}}$ satisfies:

$$w_{\bar{x},\bar{u}}(0,k) = 0\,, \quad \forall k \geq k_0$$

  Hence, the **constant movement**

$$\tilde{z}(k) = 0, \quad \forall\, k \geq k_0$$

  is an **equilibrium state** of the system $(\star)$.

Stability of State Movements and of Equilibrium States (cont.)

State Movement Stability Analysis

The stability analysis of a generic nominal state movement can always be carried out by analysing the stability of the zero-state as an equilibrium state of a suitable autonomous system.

Therefore:

There is no loss of generality in dealing only with the stability analysis of equilibrium states

Stability of Equilibrium States

The Lyapunov Methodology

Stability of Equilibrium: the Lyapunov Methodology

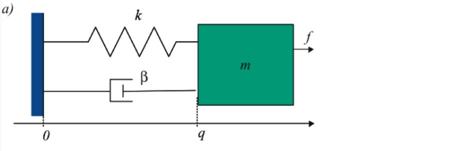
- The **Lyapunov methodology** for stability analysis of equilibrium states is a **direct** technique in that it does not require the determination of the whole perturbed state movement.
- The **Lyapunov methodology** originates from the following observation in physics:

  if the total energy of a mechanical (for example) system is continuously dissipated, then such a system should necessarily evolving over time towards an equilibrium state

- The **Lyapunov methodology** generalises the above observation by associating a suitable **positive scalar function** to the state of the dynamic system. Such a function plays the role of ”energy”

The Lyapunov Methodology: a Mechanical System Example

Consider a **nonlinear** mechanical system (continuous-time)



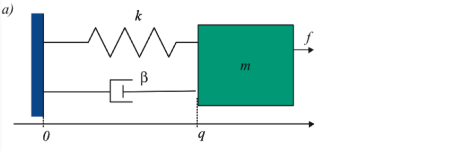
$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot{r}) = b\dot{r}|\dot{r}|$$
$$k_0, k_1, b > 0$$
$$m\ddot{r} + b\dot{r}|\dot{r}| + k_0 r + k_1 r^3 = 0$$

Letting $x_1 := r\,;\ \ x_2 := \dot{r}$ one gets the state equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k_0}{m}x_1 - \dfrac{k_1}{m}x_1^3 - \dfrac{b}{m}x_2|x_2| \end{cases}$$

This is a **free** (or **autonomous**) system where, obviously, $\bar{x} = [0\ 0]^\top$ is equilibrium state.

The Lyapunov Methodology: a Mechanical System Example (cont.)



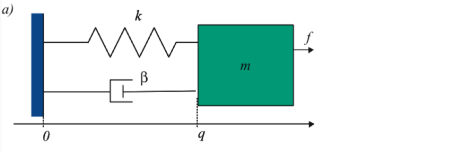
$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot{r}) = b\dot{r}|\dot{r}|$$
$$k_0, k_1, b > 0$$
$$m\ddot{r} + b\dot{r}|\dot{r}| + k_0 r + k_1 r^3 = 0$$

The total mechanical energy is given by the sum of the kinetic energy and of the elastic potential energy:

$$V(x_1, x_2) = \frac{1}{2} m x_2^2 + \int_0^{x_1} k(\xi) d\xi = \frac{1}{2} m x_2^2 + \frac{1}{2} k_0 x_1^2 + \frac{1}{4} k_1 x_1^4$$

Clearly the function $V(x_1, x_2)$ is a **positive scalar function having the state as argument**. Moreover, $V(0,0) = 0$.

The Lyapunov Methodology: a Mechanical System Example (cont.)



$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot r) = b\dot r|\dot r|$$
$$k_0, k_1, b > 0$$
$$m\ddot r + b\dot r|\dot r| + k_0 r + k_1 r^3 = 0$$

How about the time-behaviour of the total mechanical energy?

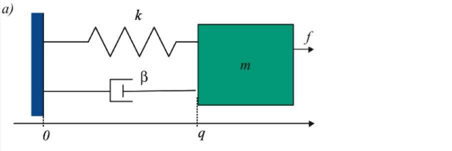
One gets:

$$\dot V(x_1, x_2) = \frac{dV(x_1, x_2)}{dt} = m x_2 \dot x_2 \;+\; k_0 x_1 \dot x_1 \;+\; k_1 x_1^3 \dot x_1 = -b|x_2|^3$$

Clearly:

- The function $\dot V(x_1, x_2)$ is not an explicit function of time but only of the state. Hence, for a given state $x = [x_1 \;\, x_2]^\top$ the rate of variation of $V(x_1, x_2)$ is fixed.
- The mechanical energy is continuously dissipated.

The Lyapunov Methodology: a Mechanical System Example (cont.)



$$k(r) = k_0 r + k_1 r^3$$
$$h(\dot{r}) = b\dot{r}|\dot{r}|$$
$$k_0, k_1, b > 0$$
$$m\ddot{r} + b\dot{r}|\dot{r}| + k_0 r + k_1 r^3 = 0$$

Question:

Is it possible to exploit the condition

$$\dot{V}(x_1, x_2) \leq 0$$

to make conclusions on the stability properties of the equilibrium state $\bar{x} = [0\ 0]^\top$?

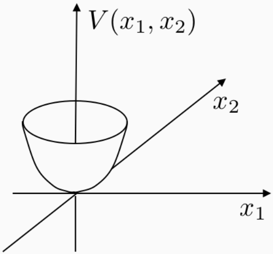
Answer:

YES! $\Longrightarrow$ **Lyapunov Stability Theory**

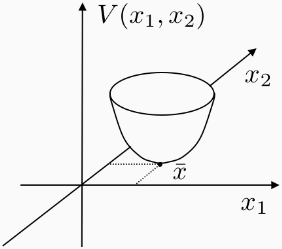
Positive-Definite Functions

A function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is positive-definite in $\bar{x}$ if:

$$V(\bar{x}) = 0 \text{ and } \exists\, \xi > 0 \,:\, V(x) > 0\,, \quad \forall\, x \,:\, \|x - \bar{x}\| < \xi,\, x \neq \bar{x}$$



Positive-definite in 0

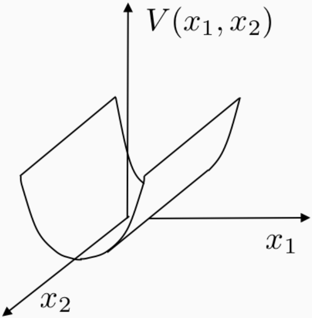


Positive-definite in $\bar{x}$

Positive Semi-Definite Functions

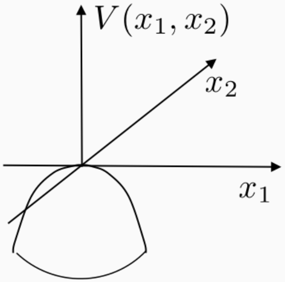
A function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ is positive-semidefinite in $\bar{x}$ if:

$$V(\bar{x}) = 0 \text{ and } \exists\, \xi > 0 \,:\, V(x) \geq 0\,, \quad \forall\, x \,:\, \|x - \bar{x}\| < \xi,\, x \neq \bar{x}$$

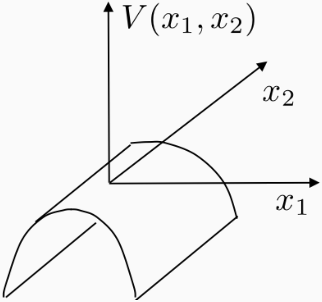


Negative-Definite and Semi-Definite Functions

A function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is negative-definite (negative semi-definite) in $\bar{x}$ if $-V(\cdot)$ is positive-definite (positive semi-definite).



Negative-definite in 0



Negative semi-definite in 0

Quadratic Functions

- The more widely used candidate Lyapunov functions are the **quadratic functions** of the form:

$$V(x) = x^\top A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

  where A is a **symmetric** matrix.
- Matrix A is positive-definite if the quadratic form $V(x) = x^\top A x$ is positive-definite in the origin.
- Analogous definitions can be given for A is positive semi-definite, negative-definite, and so on.
- In case of A not being a **symmetric** matrix, it can be easily shown that only its ”symmetric part” provides a contribution to the quadratic form $V(x) = x^\top A x$.
- In this case, matrix A can be replaced by its ”symmetric part” A^S where its elements are given by:

$$a_{ij}^S = \frac{a_{ij} + a_{ji}}{2}$$

Criteria for Checking the Definiteness of a Matrix

- Recall that all eigenvalues of a **symmetric** matrix are **real**.
- A matrix A is positive-definite **if and only if** all its eigenvalues are strictly positive:

$$\lambda_i > 0\,, \quad i = 1, \ldots, n$$

  where λ_i denotes the i-th eigenvalue of A.
- A matrix A is positive semi-definite **if and only if** all its eigenvalues are non-negative:

$$\lambda_i \geq 0\,, \quad i = 1, \ldots, n$$

- Analogous criteria hold in the other cases.

Time-Behaviour of $V(\cdot)$ Along the Perturbed State Movements

Continuous-Time Case

- Autonomous nonlinear system $\dot{x} = f(x)$
- Analysis of the continuous-time behaviour of $V(\cdot)$:

$$t \to x(t) \to V(x(t))$$

- One has:

$$\dot{V}(x) = \frac{dV(x(t))}{dt} = \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot f(x)$$

where

$$\nabla V(x) = \left[\frac{\partial V(x)}{\partial x_1} \dots \frac{\partial V(x)}{\partial x_n} \right]$$

Discrete-Time Case

- Autonomous nonlinear system $x(k+1) = f(x(k))$
- Analysis of the continuous-time behaviour of $V(\cdot)$:

$$k \to x(k) \to V(x(k))$$

- One has:

$$\Delta V(x) = V(f(x)) - V(x)$$

Stability of Equilibrium States

Lyapunov Theorem

Lyapunov Theorem

Continuous-Time Case

- Given the autonomous nonlinear system $\dot{x} = f(x)$ having the equilibrium state $\bar{x}$.
- Given a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ which is positive-definite in $\bar{x}$ and assuming that $V(\cdot)$ is continuous with continuous partial derivatives.
- Then:
  - $\dot{V}(x)$ **negative semi-definite** in $\bar{x} \implies \bar{x}$ is a **stable** equilibrium state
  - $\dot{V}(x)$ **negative-definite** in $\bar{x} \implies \bar{x}$ is an **asymptotically stable** equilibrium state
  - $\dot{V}(x)$ **positive-definite** in $\bar{x} \implies \bar{x}$ is an **unstable** equilibrium state

Lyapunov Theorem (cont.)

Discrete-Time Case

- Given the autonomous nonlinear system $x(k+1) = f(x(k))$ having the equilibrium state $\bar{x}$.
- Given a function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ which is positive-definite in $\bar{x}$ and assuming that $V(\cdot)$ is continuous.
- Then:
  - $\Delta V(x)$ **negative semi-definite** in $\bar{x} \implies \bar{x}$ is a **stable** equilibrium state
  - $\Delta V(x)$ **negative-definite** in $\bar{x} \implies \bar{x}$ is an **asymptotically stable** equilibrium state
  - $\Delta V(x)$ **positive-definite** in $\bar{x} \implies \bar{x}$ is an **unstable** equilibrium state

Lyapunov Theorem: Remarks

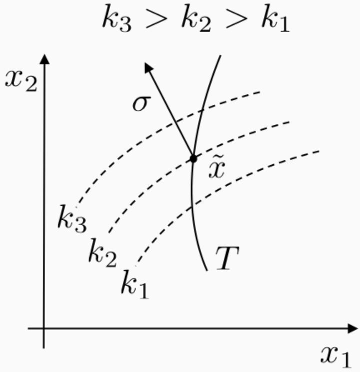
1. The Lyapunov Theorem is of key importance since it allows to analyse the stability of equilibrium states (and hence of generic nominal state movements) **without the need of determining the explicit solutions of the state equations**
2. For the above reason, the Lyapunov Theorem is also called **Direct Lyapunov Method**
3. The Lyapunov Theorem **only provides sufficient conditions** for the stability of equilibrium states.

   In other terms: the construction of a positive-definite function $V(\cdot)$ that does not satisfy any of the conditions on $\dot{V}(\cdot)$ (continuous-time case) or $\Delta V(\cdot)$ (discrete-time case) does not allow to make any conclusion on the stability of the equilibrium state.

Geometric Interpretation - Continuous-Time Case

Consider a second-order nonlinear system:

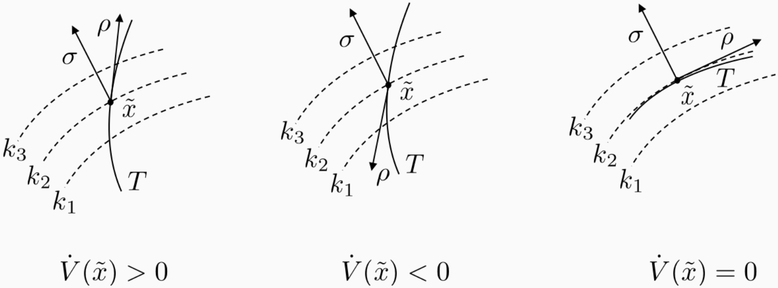
$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$



- Denote by σ the vector orthogonal to the level curve $k_2 = V(\tilde{x})$ evaluated on the state $\tilde{x}$
- Given $\tilde{x}$, the value of $\dot{V}(\tilde{x})$ is determined
- The knowledge of $\dot{V}(\tilde{x})$ allows to determine in which direction the state movement is evolving with respect to the level curves of $V(\cdot)$

Geometric Interpretation - Continuous-Time Case (cont.)

Denoting by ρ the tangent vector to the state trajectory with the direction consistent with the direction in which the state evolves over time on the trajectory T, the following three scenarios may occur:



In the discrete-time case an analogous geometric interpretation can be made with reference to $\Delta V(\cdot)$ instead of $\dot{V}(\cdot)$

Lyapunov Theorem: Example 1

- Consider the second-order nonlinear system:

$$\begin{cases} x_1(k+1) = \dfrac{x_2(k)}{1+x_2^2(k)} \\ x_2(k+1) = \dfrac{x_1(k)}{1+x_2^2(k)} \end{cases}$$

- Clearly, $\bar{x} = [0\ 0]^\top$ is an equilibrium state
- The function $V(x_1, x_2) = x_1^2 + x_2^2$ is continuous and positive-definite
- It follows that:

$$\begin{aligned} \Delta V(x) = V(f(x)) - V(x) &= \left(\frac{x_2}{1+x_2^2}\right)^2 + \left(\frac{x_1}{1+x_2^2}\right)^2 - x_1^2 - x_2^2 \\ &= \frac{-2x_2^2 - x_2^4}{(1+x_2^2)^2}(x_1^2 + x_2^2) \qquad \text{negative semi-definite} \end{aligned}$$

- Thus, the equilibrium state $\bar{x} = [0\ 0]^\top$ is stable

Lyapunov Theorem: Example 2

- Consider the same second-order nonlinear system:

$$\begin{cases} x_1(k+1) = \dfrac{x_2(k)}{1+x_2^2(k)} \\ x_2(k+1) = \dfrac{x_1(k)}{1+x_2^2(k)} \end{cases}$$

- But let us choose a different candidate Lyapunov function:

$$V(x) = (x_1^2 + x_2^2)\left(1 + \frac{1}{(1+x_2^2)^2}\right)$$

- After some algebra, one gets:

$$\Delta V(x) = \underbrace{\frac{x_1^2 + x_2^2}{\left[(1+x_2^2)^2 + x_1^2\right]^2}}_{=0 \text{ for } x=[0\ 0]^\top \text{ and } >0 \text{ elsewhere}} \underbrace{\left\{(1+x_2^2)^2 - \left[(1+x_2^2)^2 + x_1^2\right]^2\right\}}_{=0 \text{ for } x=[0\ 0]^\top \text{ and } <0 \text{ elsewhere}}$$

- Thus, the equilibrium state $\bar{x} = [0\ 0]^\top$ is asymptotically stable
- These two examples show that the Lyapunov Method provides **sufficient but not necessary** stability conditions

Stability of Linear Discrete-Time Systems

Stability of Linear Discrete-Time Systems

- Consider the general discrete-time linear dynamic system

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

  and consider a **nominal** state movement

$$\bar{x}(k) = \varphi(k, k_0, \bar{x}_0, \{u(k_0, \ldots, u(k-1)\})$$

  starting from the initial state $\bar{x}(k_0) = \bar{x}_0$.
- We analyse the stability of the nominal movement $\bar{x}(k)$ with respect to perturbations of the initial state $\bar{x}_0$, that is, we consider the perturbed state movement

$$x(k) = \varphi(k, k_0, x_0, \{u(k_0, \ldots, u(k-1)\})$$

  starting from the perturbed initial state $x_0 \neq \bar{x}_0$.

Stability of Linear Discrete-Time Systems (cont.)

- Hence, introducing the difference between the perturbed and the nominal state movement $z(k) := x(k) - \bar{x}(k)$, one gets:

$$\begin{aligned} z(k+1) &= x(k+1) - \bar{x}(k+1) \\ &= A(k)[z(k) + \bar{x}(k)] + B(k)\bar{u}(k) - A(k)\bar{x}(k) - B(k)\bar{u}(k) \\ &= A(k)z(k) \end{aligned}$$

- It follows that the dynamics of $z(k)$ can be described by the autonomous (in general time-varying) system

$$z(k+1) = A(k)z(k) \quad (\star)$$

- Hence, the **constant movement**

$$\tilde{z}(k) = 0, \qquad \forall\, k \geq k_0$$

is an **equilibrium state** of the system $(\star)$.

Stability of Linear Discrete-Time Systems (cont.)

Summing up:

For **linear systems** the dynamics of the difference between the perturbed and the nominal state movement $z(k) = x(k) - \bar{x}(k)$ satisfies:

$$z(k+1) = A(k)z(k)$$

and:

- The dynamics of $z(k)$ does not depend on the specific initial state $\bar{x}_0$ but on the magnitude of the initial state perturbation $z(k_0) = x(k) - \bar{x}_0$
- All state movements have the same stability properties or, in other terms, stability is not a property of a specific nominal state movement but, instead, is a **global property of the linear dynamic system**

Stability of Linear Discrete-Time Systems

Lyapunov Stability Test

Stability of Linear Systems: Lyapunov Method

- Given the autonomous linear time-invariant discrete-time dynamic system
$$x(k+1) = Ax(k) \quad (\star)$$
- The system $(\star)$ is asymptotically stable **if and only if** $\forall\, Q \in \mathbb{R}^{n\times n}$ symmetric and positive-definite there exists a unique symmetric and positive-definite $P \in \mathbb{R}^{n\times n}$ such that
$$A^\top PA - P = -Q \quad (\circ)$$
- The matrix equation $(\circ)$ is called **Discrete-Time Lyapunov Equation**.

Remark: In the linear case, the Lyapunov method provides a **necessary and sufficient** condition for asymptotic stability

Stability of Linear Systems: Lyapunov Method (cont.)

Sketch of the proof

$(\Longleftarrow)$ (sufficiency)

- Consider two symmetric positive-definite matrices $P, Q \in \mathbb{R}^{n \times n}$ that satisfy the Lyapunov Equation

$$A^\top P A - P = -Q$$

- To show that the system $x(k+1) = Ax(k)$ is asymptotically stable, we let

$$V(x) = x^\top P x$$

- It follows that:

$$\begin{aligned}\Delta V(x) &= (Ax)^\top PAx - x^\top Px = x^\top A^\top PAx - x^\top Px \\ &= x^\top (A^\top PA - P)x = -x^\top Qx < 0\,,\ \forall\, x \neq 0\end{aligned}$$

- Hence $\Delta V(x)$ is negative-definite which implies that the origin is an asymptotically stable equilibrium state of the system $x(k+1) = Ax(k)$.

Stability of Linear Systems: Lyapunov Method (cont.)

$(\Longrightarrow)$ (necessity)

- Suppose that the system $x(k+1) = Ax(k)$ is asymptotically stable
- Consider a symmetric positive-definite matrix $Q \in \mathbb{R}^{n \times n}$ and let us show that a symmetric positive-definite matrix $P \in \mathbb{R}^{n \times n}$ satisfying the Lyapunov Equation

  $$A^\top PA - P = -Q$$

  does exist
- Let:

  $$P := \sum_{i=0}^{+\infty} \left(A^\top\right)^i QA^i$$

  It can be shown that such a matrix does exist (the series converges) and that it is symmetric positive-definite owing to the asymptotic stability assumption

Stability of Linear Systems: Lyapunov Method (cont.)

$(\Longrightarrow)$ (necessity)

- Hence, we just need to show that matrix P is a solution of the Lyapunov Equation:

$$A^\top \left(\sum_{i=0}^{+\infty} \left(A^\top\right)^i Q A^i \right) A - \sum_{i=0}^{+\infty} \left(A^\top\right)^i Q A^i$$
$$= \sum_{i=0}^{+\infty} \left(A^\top\right)^{i+1} Q A^{i+1} - \sum_{i=0}^{+\infty} \left(A^\top\right)^i Q A^i$$
$$= \left[\left(A^\top\right) Q A + \left(A^\top\right)^2 Q A^2 + \dots \right] - \left[Q + \left(A^\top\right) Q A + \dots \right] = -Q$$

which concludes the proof.

Stability of Linear Systems: Lyapunov Stability Test

Lyapunov Stability Test (linear time-invariant systems)

Given the linear system

$$x(k+1) = Ax(k) \quad (\star)$$

- Choose a symmetric and positive-definite $Q \in \mathbb{R}^{n\times n}$ and plug it into the Lyapunov Equation

$$A^\top PA - P = -Q \quad (\circ)$$

- Determine a solution $\bar{P} \in \mathbb{R}^{n\times n}$ of the Lyapunov Equation $(\circ)$
- The system $(\star)$ is asymptotically stable if and only if the matrix $\bar{P}$ is positive-definite

Stability of Linear Systems: Lyapunov Stability Test (cont.)

Remarks

- If the Lyapunov Stability Test returns a positive result, then the quadratic form:
  $$V(x) = x^\top P x$$
  is a Lyapunov function for the system $x(k+1) = Ax(k)$. Hence the Lyapunov Stability Test provides a **systematic procedure** to construct Lyapunov functions - in the linear case.
- For a given symmetric matrix $Q \in \mathbb{R}^{n \times n}$, the Lyapunov Equation
  $$A^\top P A - P = -Q \quad (\circ)$$
  turns out to be an algebraic linear system consisting of $n(n+1)/2$ linear independent equations of the form
  $$\sum_{j=1}^{n} a_{ji} p_{jk} a_{jk} + p_{ik} = -q_{ik}$$
  involving $n(n+1)/2$ unknowns (exploiting the symmetry of P for which $p_{ij} = p_{ji}$).

Stability of Linear Discrete-Time Systems

Analysis of the Free State Movement

Stability of Linear Systems via Analysis of the Free State Movement

- Given the linear time-invariant discrete-time dynamic system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

- In **equilibrium** conditions:

$$x(0) = \bar{x}$$
$$u(k) = \bar{u},\ k \geq 0$$
$$\Longrightarrow x(k) = A^k\,\bar{x} + \sum_{i=0}^{k-1} A^{k-i-1}\,B\bar{u} = \bar{x},\ \forall k \geq 0$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

- **Perturbing the equilibrium** conditions:

$$\begin{array}{lcc} x(0) = \bar{x} + \delta\bar{x} & \Longrightarrow & x(k) \neq \bar{x}, k \geq 0 \\ u(k) = \bar{u},\ k \geq 0 & & \text{perturbed state movement} \end{array}$$

$$\Longrightarrow x(k) = A^k\ (\bar{x} + \delta\bar{x}) + \sum_{i=0}^{k-1} A^{k-i-1}\, B\bar{u}$$

$$= \bar{x} + A^k\, \delta\bar{x}$$

Hence:

$$\delta x(k) = A^k \delta\bar{x}$$

- Also, recall that:

$$x_l(k) = A^k\, x(0)$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Stability and A^k

- The stability properties do not depend on the specific value taken on by the equilibrium state $\bar{x}$
- Hence, the stability properties are a structural property of the linear dynamic system as a whole
- The stability properties depend on the time-behaviour of the $n \times n$ elements of the matrix A^k:
  - Stability $\iff$ all elements of A^k are bounded $\forall\, k \geq 0$
  - Asymptotic stability $\iff$ $\lim_{k \to \infty} A^k = 0$
  - Instability $\iff$ at least one element of A^k diverges

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Recall that (**Part 2**):

- $x(k+1) = Ax(k)$, $x(0) = x_0 \implies x(k) = A^k x_0$
- $T \in \mathbb{R}^{n \times n},\ \det(T) \neq 0 \implies x = T\hat{x}, \quad \hat{x} = T^{-1}x$ Hence $\hat{x}(k+1) = T^{-1}Ax(k) = T^{-1}AT\hat{x}(k)$, $\hat{x}_0 = T^{-1}x_0$ which yields

$$\hat{x}(k) = \left(T^{-1}AT\right)^k T^{-1}x_0$$

- One lets $J := T^{-1}AT$ and considers the transformation such that J takes on the **Jordan Canonical Form** thus obtaining:

$$x(k) = TJ^kT^{-1}x_0$$

- For the stability analysis, the **boundedness of the free-state movement** has to be ascertained. Since matrix T does not depend on k, it suffices to **analyse the boundedness of the elements of the matrix**

$$J^k$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

Specifically:

$$x(k) = TJ^kT^{-1}x_0 = T \begin{bmatrix} J_0^k & \cdots & \cdots & 0 \\ & {J_1}^k & & \\ & & \ddots & \\ 0 & \cdots & \cdots & {J_s}^k \end{bmatrix} T^{-1}x_0$$

where:

$$J_0 = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_r \end{bmatrix} \implies {J_0}^k = \begin{bmatrix} {\lambda_1}^k & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & {\lambda_r}^k \end{bmatrix}$$

Stability of Linear Systems via Analysis of the Free State Movement (cont.)

and

$$J_i = \begin{bmatrix} \lambda_{r+i} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{r+i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \lambda_{r+i} \end{bmatrix}$$

Thus:

$$\begin{aligned} J_i^k &= (\lambda_{r+i} I_i + N_i)^k \\ &= \lambda_{r+i} I + k\lambda_{r+i}^{k-1} N_i + \frac{k(k-1)}{2!}\lambda_{r+i}^{k-2} N_i^2 + \cdots + k\lambda_{r+i} N_i^{k-1} + N_i^k \end{aligned}$$

eventually getting to **discrete-time response modes** of the form

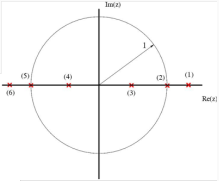
$$\lambda^k, \begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$$

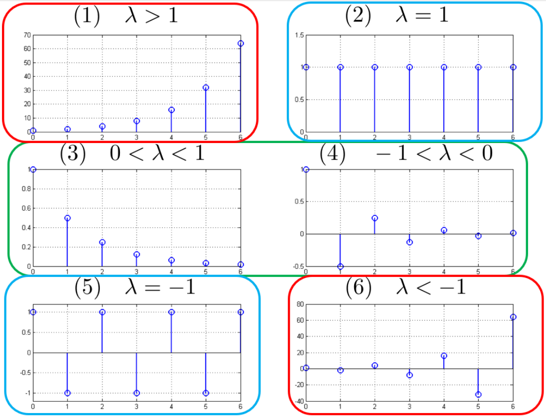
Stability of Linear Discrete-Time Systems

Stability Criterion Based on Eigenvalues

Stability & Qualitative Behaviour of Response Modes

- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{R}$, multiplicity $= 1$





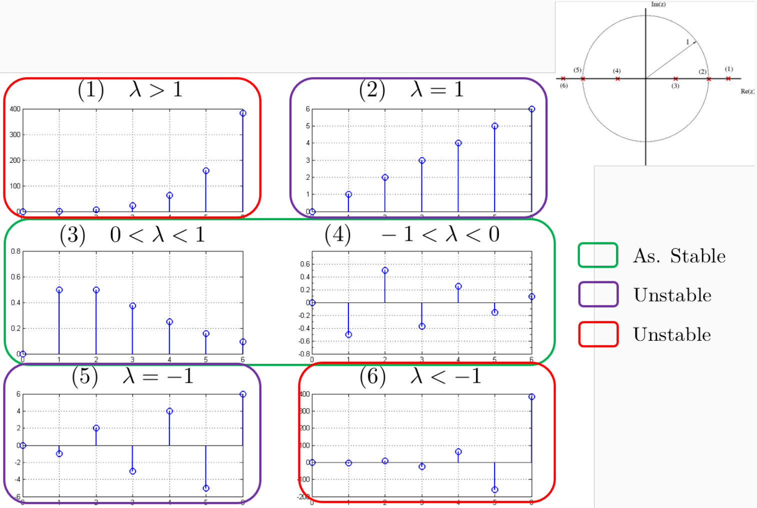
As. Stable

Stable

Unstable

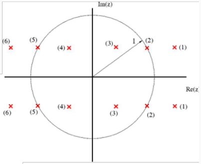
Stability & Qualitative Behaviour of Response Modes

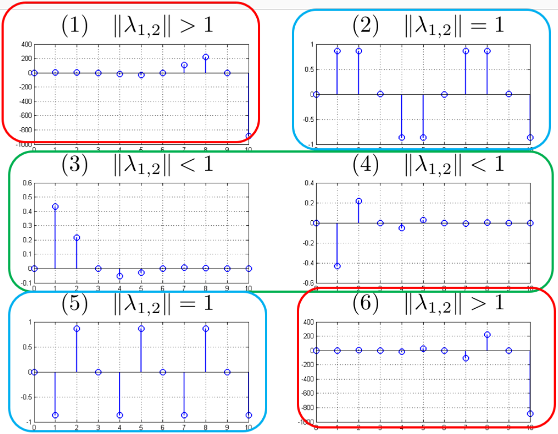
- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{R}$, multiplicity > 1



Stability & Qualitative Behaviour of Response Modes

- $\begin{pmatrix} k \\ n_i \end{pmatrix} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{C}$, multiplicity $= 1$





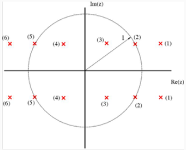
As. Stable

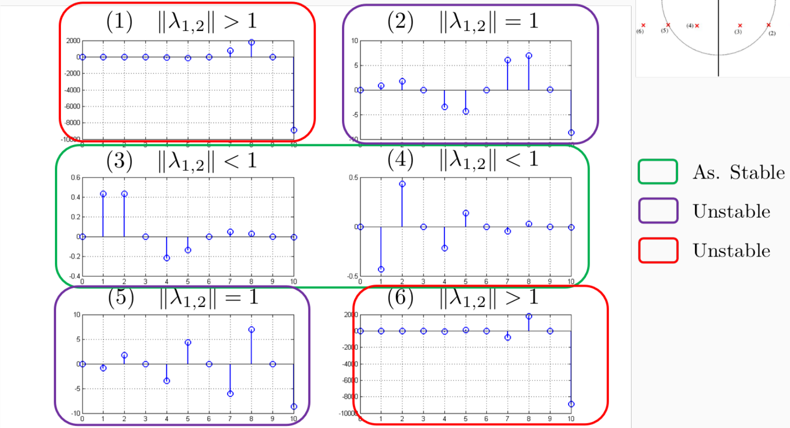
Stable

Unstable

Stability & Qualitative Behaviour of Response Modes

- $\binom{k}{n_i} \lambda_i^{k-n_i}$ with $\lambda \in \mathbb{C}$, multiplicity > 1



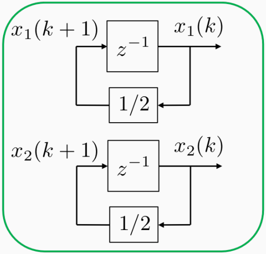


Asymptotically Stable

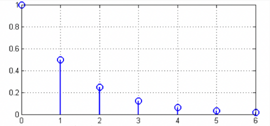
$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}$$

$$A^k = \begin{bmatrix} (1/2)^k & 0 \\ 0 & (1/2)^k \end{bmatrix}$$



Response modes for $x_1(k)$ and $x_2(k)$



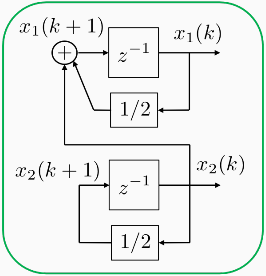
Stability & Behaviour of Response Modes: Example 2

Asymptotically Stable

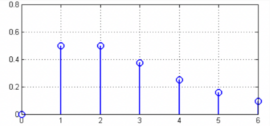
$$A = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = \frac{1}{2}$$

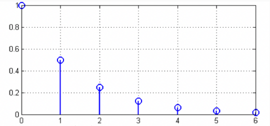
$$A^k = \begin{bmatrix} (1/2)^k & k(1/2)^{k-1} \\ 0 & (1/2)^k \end{bmatrix}$$



Response mode for $x_1(k)$



Response mode for $x_2(k)$



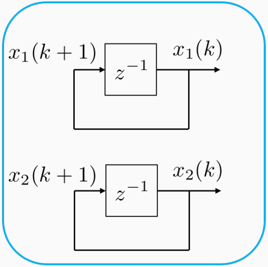
Stability & Behaviour of Response Modes: Example 3

Stable (not asymptotically)

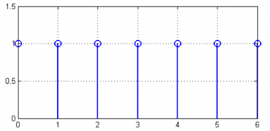
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Response modes for $x_1(k)$ and $x_2(k)$



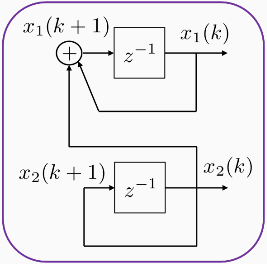
Stability & Behaviour of Response Modes: Example 4

Unstable

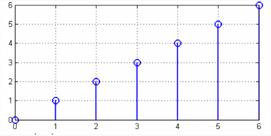
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 1$$

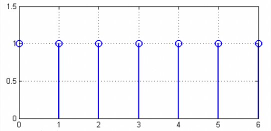
$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$



Response mode for $x_1(k)$



Response mode for $x_2(k)$



Complete Stability Criterion Based on Eigenvalues of A

Stability Criterion

Given the system $x(k+1) = Ax(k)$ and denoting by $\lambda_i,\, i = 1, \ldots n$ the eigenvalues of matrix A.

- $|\lambda_i| < 1,\, \forall\, i = 1, \ldots n \quad \Longleftrightarrow$ The system is as. stable
- $\exists\, i,\, 1 \leq i \leq n : |\lambda_i| > 1 \quad \Longrightarrow$ The system is unstable
- $\left.\begin{array}{l} |\lambda_i| \leq 1,\, \forall\, i = 1, \ldots n \\ \exists\, i,\, 1 \leq i \leq n : |\lambda_i| = 1 \end{array}\right\} \Longrightarrow$ The system is not as. stable
  - $\lambda_i : |\lambda_i| = 1$ have algebraic multiplicity $= 1$, then the system is stable (not as.)
  - $\lambda_i : |\lambda_i| = 1$ have algebraic multiplicity > 1 and all Jordan sub-blocks are of dimension $= 1$, then the system is stable (not as.)
  - $\lambda_i : |\lambda_i| = 1$ have algebraic multiplicity > 1 and at least one Jordan sub-block has dimension > 1, then the system is unstable

Stability of Linear Discrete-Time Systems

Analysis of the Characteristic Polynomial

Stability by Analysing the Characteristic Polynomial

- The previous complete stability criterion requires checking whether the eigenvalues of matrix A belong to the unit circle in the complex plane
- A number of techniques exist to perform the check above **without explicitly calculating** the eigenvalues of matrix A
- Considering the characteristic polynomial

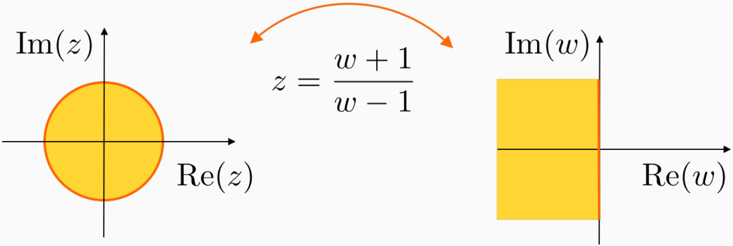
$$p_A(z) = \det(zI - A) = \varphi_0 z^n + \varphi_1 z^{n-1} + \cdots + \varphi_{n-1} z + \varphi_n$$

  a suitable **bi-linear transformation** allows to reduce the problem of checking whether the roots of polynomial $p_A(z)$ belong to the unit circle in the complex plane to an **equivalent problem** of checking whether the roots of a suitable polynomial $q_a(w)$ belong to the complex left half-plane
- This equivalent problem can then be solved by using the **Routh-Hurwitz** technique (see the course *Fundamentals of Automatic Control*)

Use of the Bi-linear Transformation

$$z = \frac{w+1}{w-1},\ z,\, w \in \mathbb{C}$$

$$\begin{aligned} |z| < 1 &\iff \mathrm{Re}(w) < 0 \\ |z| = 1 &\iff \mathrm{Re}(w) = 0 \\ |z| > 1 &\iff \mathrm{Re}(w) > 0 \end{aligned}$$



Use of the Bi-linear Transformation (cont.)

Substitute

$$z = \frac{w+1}{w-1}, \; z, \, w \in \mathbb{C}$$

into

$$p_A(z) = \varphi_0 z^n + \varphi_1 z^{n-1} + \cdots + \varphi_{n-1} z + \varphi_n$$

thus obtaining

$$q_A(w) = \; (w-1)^n \; \Bigg[\varphi_0 \frac{(w+1)^n}{(w-1)^n} + \varphi_1 \frac{(w+1)^{n-1}}{(w-1)^{n-1}} + \cdots + \varphi_{n-1} \frac{(w+1)}{(w-1)} + \varphi_n \Bigg]$$

and hence one gets

$$q_A(w) = q_0 \, w^n + q_1 \, w^{n-1} + \cdots + q_{n-1} \, w + q_n$$

with suitable coefficients $q_0, \, q_1, \ldots, q_n$.

Use of the Bi-linear Transformation. Example 1

Given

$$p_A(z) \ = \ z^3 + 2\,z^2 + z + 1$$

one gets

$$q_A(w) \ = \ (w-1)^3 \ \left[\frac{(w+1)^3}{(w-1)^3} + 2\,\frac{(w+1)^2}{(w-1)^2} + \frac{w+1}{w-1} + 1\right]$$

and after some algebra

$$q_A(w) = 5\,w^3 + w^2 + 3\,w - 1$$

3	5	3
2	1	−1
1	8	
0	−1	

Hence, there is one root of $q_A(w)$ on the complex right-half plane which in turn implies that one of the roots of $p_A(z)$ lies outside the unit circle.

Use of the Bi-linear Transformation. Example 2

Given

$$p_A(z) = z^2 + az + b$$

with $a, b \in R$. Thus, one gets:

$$q_A(w) \;=\; (w-1)^2 \left[\frac{(w+1)^2}{(w-1)^2} + a\,\frac{(w+1)}{(w-1)} + b\right]$$

and after some easy algebra

$$q_A(w) = (1+b+a)w^2 + 2(1-b)w - a + 1 + b$$

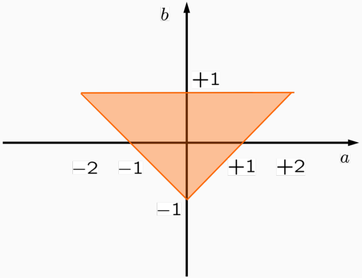
$$\begin{array}{c|cc} 2 & (1+b+a) & (1+b-a) \\ 1 & 2(1-b) & \\ 0 & (1+b-a) & \end{array}$$

$$\left\{\begin{array}{l} 1+b+a>0 \\ 2(1-b)>0 \\ 1+b-a>0 \end{array}\right. \implies \left\{\begin{array}{l} b>-a-1 \\ b<1 \\ b>a-1 \end{array}\right.$$

Use of the Bi-linear Transformation. Example 2 (cont.)

The stability condition has a nice geometric interpretation:

$$\begin{cases} b > -a - 1 \\ b < 1 \\ b > a - 1 \end{cases}$$



Stability of Linear Discrete-Time Systems

Stability of Equilibrium States Through the Linearised System

Stability of Equilibrium States Through the Linearised System - Time-Invariant Systems

Recall from Part 1

- Consider the nonlinear time-invariant system:

$$x(k+1) = f(x(k), u(k))$$

- Moreover, consider an **equilibrium state** $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u},\ k \geq k_0$.
- Let us **perturb** the initial state and the nominal input sequence, thus getting a **perturbed state movement**:

$$x(k_0) = \bar{x}_0 + \delta x_0\,;\ u(k) = \bar{u} + \delta u(k) \implies x(k) = \bar{x} + \delta x(k)$$

- Hence:

$$\begin{gathered} x(k+1) = \bar{x} + \delta x(k+1) = f(\bar{x} + \delta x(k), \bar{u} + \delta u(k)) \\ \simeq f(\bar{x}, \bar{u}) + f_x(\bar{x}, \bar{u})\delta x(k) + f_u(\bar{x}, \bar{u})\delta u(k) \end{gathered}$$

Stability of Equilibrium States Through the Linearised System - Time-Invariant Systems (cont.)

- Since the equilibrium state $\bar{x}$ is the constant solution of the algebraic equation $\bar{x} = f(\bar{x}, \bar{u})$, it follows that

$$\delta x(k+1) \simeq A \delta x(k) + B \delta u(k)$$

where $A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m}$ are **constant matrices** defined as:

$$A = f_x(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

$$B = f_u(\bar{x}, \bar{u}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{x(k)=\bar{x}, u(k)=\bar{u}}$$

Stability of Equilibrium States Through the Linearised System - Time-Invariant Systems (cont.)

Summing up:

The linear time-invariant system obtained by linearization around a given equilibrium state $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u}$, $k \geq k_0$ is

$$\delta x(k+1) = A\delta x(k) + B\delta u(k)$$

The Reduced Lyapunov Method for Discrete-Time Systems

- Consider the nonlinear time-invariant system:
$$x(k+1) = f(x(k), u(k))$$
- Moreover, consider an **equilibrium state** $\bar{x}$ obtained by the constant input sequence $u(k) = \bar{u},\ k \geq k_0$.
- Consider the free linear time-invariant system obtained by linearization around the equilibrium state $\bar{x}$ (the effect of the input is not considered in the stability of the equilibrium) and denote by $\lambda_i,\ i = 1, \ldots n$ the eigenvalues of matrix A:
$$\delta x(k+1) = A\delta x(k)$$
- $|\lambda_i| < 1, \forall\, i = 1, \ldots n \quad \Longrightarrow \quad \bar{x}$ is an **asymptotically stable equilibrium state**
- $\exists\, i,\ 1 \leq i \leq n : \ |\lambda_i| > 1 \quad \Longrightarrow \quad \bar{x}$ is an **unstable equilibrium state**
- In all other situations, **no conclusions** on the stability of the equilibrium state can be drawn from the analysis of the linearised system.

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Lecture 3
Stability of Discrete-Time Dynamic Systems

END