26 ottobre

$\{u_n\}$ è relativ compatto in

$L^2([0,T]\times K, \mathbb{R}^d)$ $\quad T>0$

$K \subset\subset \mathbb{R}^d$, $\quad d=2,3.$

È sufficiente dimostrarlo per

$\{P_{n_0} u_n\}$ $\qquad P_{n_0} = \chi_{[0,n_0]}(\sqrt{-\Delta})$

Sicuramente $C^0([0,T], L^2(K,\mathbb{R}^d)) \hookrightarrow L^2([0,T]\times K, \mathbb{R}^d)$

$\{P_{n_0} u_n\}$ ha una sottosuccessione conv in

Useremo Ascoli $\quad f_n : K \longrightarrow X$

K compatto, X spazio metrico completo

$\{f_n\}$ è rel comp in $C^0(K,X)$

1) f_n è equicontinua.

2) $\{f_n(k)\}$ rel comp $\forall k \in K$.

$\{P_{n_0} u_n\}$ sono unif Hölderiani cioè che $\exists$

$C \geq 0 \quad \alpha \geq 0 \quad t.c.$

$$* \quad |P_{n_0} u_n(t) - P_{n_0} u_n(s)|_{L^2(K)} < C\, |t-s|^{\alpha} \quad \forall\, t,s \in [0,T]$$

$d = 2, 3 \qquad \frac{4}{d} > 1$

Lemma $\quad |(P_{n_0} u_n)_t|_{L^{\frac{4}{d}}([0,T], L^2(\mathbb{R}^d))} \leq C_T \quad \forall n.$

$t > s$

$$|P_{n_0} u_n(t) - P_{n_0} u_n(s)|_{L^2(\mathbb{R}^d)} \leq \int_s^t |\partial_\tau P_{n_0} u_n(\tau)|_{L^2(\mathbb{R}^d)}\, d\tau$$

$$\leq |\ 1\ |_{L^{(\frac{4}{d})'}([s,t])} \underbrace{|\partial_\tau P_{n_0} u_n|_{L^{\frac{4}{d}}([0,T], L^2(\mathbb{R}^d))}}_{C_T}$$

$$\leq C_T\, |t-s|^{\frac{1}{(\frac{4}{d})'}} \qquad L^{\frac{4}{d}}([0,T], L^2(\mathbb{R}^d))$$

$$\partial_t P_{n_0} u_n = -P_{n_0} P_n \mathbb{P}\, \mathrm{div}(u_n \otimes u_n) + \nu P_{n_0} \Delta u_n$$

$$|P_{n_0} \Delta u_n|_{L^{\frac{4}{d}}([0,T], L^2_x)} \leq n_0^2\, |u_n|_{L^{\frac{4}{d}}([0,T], L^2_x)}$$

$$\leq n_0^2\, T^{\frac{d}{4}}\, |u_n|_{L^\infty([0,T], L^2_x)} \leq$$

$$\leq n_0^2\, T^{\frac{d}{4}}\, |\cancel{P_n} u_0|_{L^2_x}$$

$$\| P_{n_0} P_n \mathbb{P} \operatorname{div}(u_n \otimes u_n) \|_{L^{4/d}([0,T], L^2_x)} \le n_0 \| u_n \otimes u_n \|_{L^{4/d}([0,T], L^2_x)}$$

$$= n_0 \| \; \| u_n \|^2_{L^4_x} \|_{L^{4/d}([0,T])}$$

$$\lesssim n_0 \| \; \| \nabla u_n \|^{\frac{d}{2}}_{L^2_x} \| u_n \|^{2-\frac{d}{2}}_{L^2_x} \|_{L^{4/d}(0,T)}$$

$$\le n_0 \| \; \| u_n \|_{L^2_x} \|^{2-\frac{d}{2}}_{L^\infty([0,T])} \| \; \| \nabla u_n \|_{L^2_x} \|^{\frac{d}{2}}_{L^{\frac{4}{d}\cdot\frac{d}{2}}[0,T]}$$

$$\le n_0 \| \cancel{P_n} u_0 \|^{2-\frac{d}{2}}_{L^2_x} \| \nabla u_n \|^{\frac{d}{2}}_{L^2([0,T], L^2_x)}$$

$$\le n_0 C_\nu \underline{\qquad \| u_0 \|^2_{L^2_x}}$$

Dimostriamo $\{ P_{n_0} u_n(t) \}$ è rel compatto in $L^2(K)$ $\forall\, t \in [0,T]$.

Supponiamo che $\boxed{\{ P_{n_0} u_n(t) \}_n \text{ è limitato}}$ in $H^1(\mathbb{R}^d, \mathbb{R}^d)$

$$\| P_{n_0} u_n(t) \|_{L^2_x} \le \| u_0 \|_{L^2_x} \qquad \| \nabla P_{n_0} u_n(t) \|_{L^2_x} \le n_0 \| u_n(t) \|_{L^2_x}$$

$$\le n_0 \| u_0 \|_{L^2_x}$$

Lemma $H^1(\mathbb{R}^d) \longrightarrow L^2(K)$

$$f \in H^1(\mathbb{R}^d) \longrightarrow f|_K \in L^2(K)$$ è un operatore compatto.

Dim $\mathcal{F}^*\left(\frac{f}{\langle\xi\rangle}\right)$ $H^1(\mathbb{R}^d) \longrightarrow L^2(K)$

$$\begin{array}{ccc} \uparrow & \downarrow \; \uparrow & \\ f & L^2(\mathbb{R}^d) & \mathcal{F}^*\left(\frac{f}{\langle\xi\rangle}\right)\Big|_K \in L^2(\mathbb{R}^d) \\ & f & \end{array}$$

$$L^2(\mathbb{R}^d, d\xi) \longrightarrow H^1(\mathbb{R}^d) \simeq L^2(\mathbb{R}^d, \langle\xi\rangle^2 d\xi)$$

$$f \longrightarrow \cancel{\mathcal{F}^*}\left(\frac{f}{\langle\xi\rangle}\right)$$ è un isomorfism

$$f \in L^2(\mathbb{R}^d, d\xi) \longrightarrow \frac{f}{\langle\xi\rangle} \in L^2(\mathbb{R}^d, \langle\xi\rangle^2 d\xi)$$

$$\left\| \frac{f}{\langle\xi\rangle} \right\|_{L^2(\langle\xi\rangle^2 d\xi)} = \| f \|_{L^2(d\xi)}$$

$$f \longrightarrow \chi_K \, \mathcal{F}^*\left(\frac{f}{\langle\xi\rangle}\right)$$

$$L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$$ è un operatore compatto

$$Tf(x) = \chi_K(x) \int_{\mathbb{R}^d} e^{-ix\xi} \frac{1}{\langle \xi \rangle} f(\xi)\, d\xi$$

$$T \in \mathcal{L}(L^2_x, L^2_x)$$

$$= \int_{\mathbb{R}^d} K(x,\xi) f(\xi)\, d\xi$$

$$T_n \in \mathcal{L}(L^2_x, L^2_x)$$

$$K(x,\xi) = \chi_K(x)\, e^{-ix\xi} \frac{1}{\langle \xi \rangle}$$

$$T_n f(x) = \chi_K(x) \int_{\mathbb{R}^d} e^{-ix\xi} \frac{1}{\langle \xi \rangle} \chi_{B(0,n)}(\xi) f(\xi)\, d\xi$$

$$= \int_{\mathbb{R}^d} K_n(x,\xi) f(\xi)\, d\xi$$

$$K_n(x,\xi) = \chi_K(x)\, e^{-ix\xi} \frac{1}{\langle \xi \rangle} \chi_{B(0,n)}(\xi)$$

$T_n \longrightarrow T$ in norma dentro $\mathcal{L}(L^2_x, L^2_x)$

$$\left\| (T_n - T) f \right\|_{L^2_x} = \left\| \int_{\mathbb{R}^d} e^{-ix\xi} \left(1 - \chi_{B(0,n)}(\xi)\right) \langle \xi \rangle^{-1} f(\xi)\, d\xi \right\|_{L^2_x}$$

$$= \left\| \left(1 - \chi_{B(0,n)}(\xi)\right) \langle \xi \rangle^{-1} f(\xi) \right\|_{L^2_\xi} \le \langle n \rangle^{-1} \|f\|_{L^2_\xi}$$

$$\| T_n - T \|_{\mathcal{L}(L^2_x, L^2_x)} \le \langle n \rangle^{-1} \longrightarrow 0$$

Basta ora dimostrare che T_n sono compatti.

$$T_n f(x) = \int_{\mathbb{R}^d} K_n(x,\xi) \, f(\xi) \, d\xi$$

$$K_n(x,\xi) = \chi_K(x) \, \chi_{B(0,n)}(\xi) \, \langle \xi \rangle^{-1} e^{-ix\xi} \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$$

Sono operatori HS

$$\|T_n\|_{HS} \stackrel{\cdot}{=} \|K_n\|_{L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)}$$

$$\|T_n\|_{\mathcal{L}(L^2_x, L^2_x)} \leq \|T_n\|_{HS}$$

$$\|T_n f\|_{L^2_x} = \left\| \int K_n(x,\xi) \, f(\xi) \, d\xi \right\|_{L^2_x} =$$

$$\leq \left\| \left(\int |K_n(x,\xi)|^2 d\xi \right)^{\frac{1}{2}} \right\|_{L^2_x} \|f\|_{L^2_x}$$

$$= \|K_n\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \underbrace{\|f\|_{L^2}}_{= 1}$$

$$\|T_n\|_{\mathcal{L}(L^2_x, L^2_x)} \leq \|K_n\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}$$

$$\underline{\underline{K_n(x,\xi)}} = \chi_K(x)\,\chi_{B(0,n)}(\xi)\,\langle\xi\rangle^{-1} e^{-ix\xi} \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$$

Dentro $L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ il seguente sottospazio è denso

$$L^2(\mathbb{R}^d_x) \otimes_{\mathbb{R}} L^2(\mathbb{R}^d_\xi)$$

i cui elementi sono le combinazioni lineari finite delle funzioni

$$S_m \quad H_m(x,\xi) = \sum_{j=1}^{N_m} a_{jm}(x)\, b_{jm}(\xi) \xrightarrow[L^2(\mathbb{R}^d \times \mathbb{R}^d)]{m \to +\infty} K_n(x,\xi)$$

$$|H_m - K_n|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \xrightarrow{m\to\infty} 0$$

$$|S_m - T_n|_{HS} \xrightarrow{m\to\infty} 0 \Rightarrow |S_m - T_n|_{\mathcal{L}(L^2_x, L^2_x)} \to 0$$

$$S_m f(x) = \int_{\mathbb{R}^d} H_m(x,\xi)\, f(\xi)\, d\xi =$$

$$= \sum_{j=1}^{N_m} a_{jm} \int_{\mathbb{R}^d} b_{jm} f(\xi)\, d\xi) \quad \text{sono operatori}$$

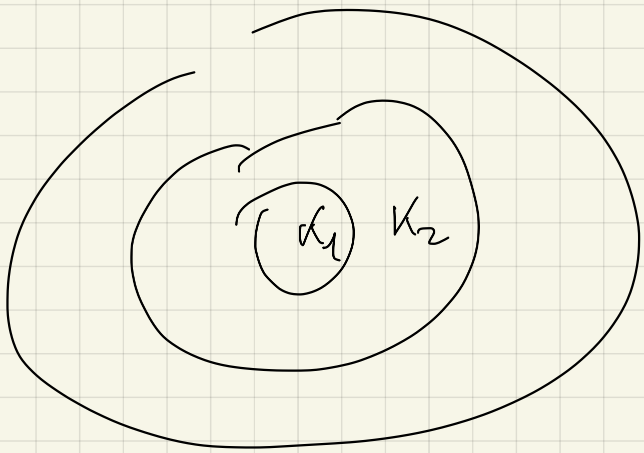
di rango finito $\Rightarrow$ S_m sono compatti

Conclusione $\{P_{n_0} u_n\}$ è rel compatto in $C^0([0,T], L^2_x(K))$

$\Rightarrow$ $\{u_n\}$ ammette una sottosucc. conv in $L^2([0,T]\times K, \mathbb{R}^d)$

Con un argomento diagonale si riesce a trovare una sottosucc. di $\{u_n\}$ che converge in $L^2([0,T]\times K)$ $\forall\, T>0$ e $\forall\, K \subset\subset \mathbb{R}^d$

$T = n \nearrow \infty$ $\qquad$ $\bigcup_{n\geq 1}^{\infty} K_n = \mathbb{R}^d$



$\{K_n\}$ crescente

$$\begin{array}{ll} \begin{pmatrix} u_{11} & u_{12} & \cdots \\ u_{21} & u_{22} & \cdots \\ & & \ddots \end{pmatrix} & \begin{array}{l} [0,1]\times K_1 \\ [0,2]\times K_2 \\ \vdots \\ [0,n]\times K_n \end{array} \end{array}$$

$\{u_{nn}\}$ converge in $L^2([0,n]\times K_n)$ $\forall\, n$.

Preso un qualsiasi $[0,T]\times K \subset [0,m]\times K_m$

Resta definita una $u \in L^2([0,T]\times K, \mathbb{R}^d)$
$\forall\, K$ e $\forall\, T>0$

$$u_n \longrightarrow u \quad \text{in } \mathcal{D}'([0,T)\times\mathbb{R}^d, \mathbb{R}^d)$$

$u_n \cdot \quad L^2([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$

$\cup\!/$

$L^\infty([0,T], L^2(\mathbb{R}^d, \mathbb{R}^d))$

$\exists f$ t.c.

$u_{n_k} \to f$ in $L^2([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$

$\mathcal{D}'([0,T)\times\mathbb{R}^d, \mathbb{R}^d)$

$f = u$

$u_{n_k} \rightharpoonup g$

$u_n \to u$ in $L^2([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$

$\nabla u_n \to \nabla u$

$\mathrm{div}_x\, u = 0$

$$|u_n(t)|^2_{L^2_x} + 2\nu \int_0^t |\nabla u_n(t')|^2_{L^2_x} dt' \le |u_0|^2_{L^2_x}$$

$$\int_0^t |\nabla u(t')|^2_{L^2_x} dt \le \liminf_{n\to\infty} \int_0^t |\nabla u_n|^2 dt'$$

$$|u(t)|^2_{L^2} \le \liminf_{n\to\infty} |u_n(t)|^2_{L^2_x}$$

$$|u(t)|^2_{L^2} + 2\nu \int_0^t |\nabla u|^2_{L^2_x} dt' \le |u_0|^2_{L^2_x}$$

$\psi(t) \in C^0([0,+\infty), L^2_x)$

$$\langle u_n(t), \psi(t)\rangle_{L^2_x} \longrightarrow \langle u(t), \psi(t)\rangle_{L^2_x}$$

$C^0([0,T]) \quad \forall\, T > 0.$