$\mathrm{Hom}(V, W) = \{ f : V \longrightarrow W \mid f \text{ lineare} \}$

$(f+g)(v) = f(v) + g(v)$

$(\lambda f)(v) = \lambda f(v)$

$V^* = \mathrm{Hom}(V, \mathbb{K})$

(Homomorphisms

omomorfismi)

$\varphi, \psi \in V^*$

$\varphi + \psi \in V^*$

$\lambda \varphi \in V^*$

$(\varphi + \psi)(v) = \varphi(v) + \psi(v)$

$(\lambda \varphi)(v) = \lambda \varphi(v)$

$V^{**} \overset{\text{def}}{=} (V^*)^*$

$\tau \in V^{**} \iff \tau : V^* \longrightarrow \mathbb{K}$ lineare

$\tau : \psi \longmapsto \tau(\psi)$

$\psi : V \longrightarrow \mathbb{K}$

Esempio $\quad \varphi \in (\mathbb{R}^2)^*$

$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ lineare

Poniamo $\varphi(x, y) = x$ è lineare

$$\varphi\big((x,y) + (x',y')\big) = \varphi(x+x', y+y') = x + x' =$$
$$= \varphi(x,y) + \varphi(x',y')$$

$$\varphi(\lambda(x,y)) = \varphi(\lambda x, \lambda y) = \lambda x = \lambda \varphi(x,y)$$

$\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\psi(x,y) = y$

$\varphi, \psi \in (\mathbb{R}^2)^*$

$3\varphi + 2\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(3\varphi + 2\psi)(x,y) = (3\varphi)(x,y) + (2\psi)(x,y) = 3x + 2y$$

$\varphi, \psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad \varphi(x,y) = x \quad , \quad \psi(x,y) = y$

$3\varphi + 2\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(3\varphi + 2\psi)(x,y) = 3x + 2y$

$\mathbb{K} \oplus \mathbb{K} = \mathbb{K}^2 \xrightarrow{\pi_1} \mathbb{K}$ $\qquad \pi_2 : \mathbb{K}^2 \longrightarrow \mathbb{K}$

$(x,y) \longmapsto x$ $\qquad (x,y) \longmapsto y$

Proiezioni

V, W $\mathbb{K}$-sp. vett. $\rightsquigarrow$

$$\begin{array}{ccc} V \oplus W & \xrightarrow{\pi_1} & V \\ \downarrow{\scriptstyle \pi_2} & & \\ W & & \end{array}$$

$\pi_1(v,w) = v$

Suriettive $\longrightarrow$

Lineari

$\pi_2(v,w) = w$

Proiezioni canoniche

$$V \oplus W \xrightarrow{\pi_1} V$$

$$\ker \pi_1 = ?$$

$$\pi_1(v, w) = v$$

$$\ker \pi_1 = \underbrace{\{0_V\} \times W}_{\shortparallel} \subset V \oplus W$$

$$\{(0_V, w) \mid w \in W\}$$

$$\{0_V\} \times W \cong W$$

Def Un'applicazione lineare $f : V \longrightarrow W$ è detta **isomorfismo** se f è biiettiva e f^{-1} è lineare.

Prop. $f : V \to W$ è isomorfismo $\iff$ f lineare e biiettiva

Dim $(\Rightarrow)$ ovvio

$(\Leftarrow)$ mostriamo che f^{-1} lineare

Poniamo $w' = f^{-1}(w)$, $u' = f^{-1}(u)$

$w = f(w')$ $\quad$ $u = f(u')$

$f^{-1} : W \longrightarrow V$ $\qquad$ $w, u \in W$

$$f^{-1}(w + u) = f^{-1}\big(f(w') + f(u')\big) = f^{-1}\big(f(w' + u')\big) = (f^{-1} \circ f)(w' + u')$$
$$= w' + u' = f^{-1}(w) + f^{-1}(u)$$

$$\underline{f^{-1}(\lambda w)} = f^{-1}(\lambda f(w')) = f^{-1}(f(\lambda w')) = \lambda w' = \underline{\lambda f^{-1}(w)}$$

$$w = f(w')$$

$$w' = f^{-1}(w)$$

<u>OSS</u> $f: V \longrightarrow W$ lineare è biettiva $\Longleftrightarrow$ $\operatorname{Im} f = W$ e $\ker f = 0 = \{0_V\}$

Teorema delle dimensione. Se $f : V \longrightarrow W$ è lineare e $\dim V < \infty$

allora $\left[\begin{array}{l} 1) \ \mathrm{Im}\, f \text{ ha dim finita} \\ 2) \ \dim V = \dim(\ker f) + \dim(\mathrm{Im}\, f) \end{array}\right.$

$\dim(\ker f) = \mathrm{null}\, f$

$\dim(\mathrm{Im}\, f) = \mathrm{rk}\, f \ (= \mathrm{rg}\, f)$

$$\boxed{\dim V = \mathrm{null}\, f + \mathrm{rg}\, f}$$

Dim $\ker f \subset V \Rightarrow \dim(\ker f) = t \in \mathbb{N}$

$\rightsquigarrow \{v_1, \dots, v_t\}$ base di $\ker f$ $\rightsquigarrow \{v_{t+1}, \dots, v_n\}$ t.c.

$n = \dim V$

$\{v_1, \dots, v_n\}$ base di V

Facciamo vedere che $\{w_{t+1} = f(v_{t+1}), \dots, w_n = f(v_n)\}$ sono base per Im f

generano, infatti preso $w \in \operatorname{Im} f \Rightarrow \exists\, v \in V$ t.c. $w = f(v)$

$v_1, \dots, v_t \in \ker f$

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n \quad \text{per certi } \lambda_i \in K$$

$$f(v) = f(\lambda_1 v_1 + \dots \lambda_n v_n) \overset{\text{lin.}}{=} \lambda_1 f(v_1) + \dots + \lambda_n f(v_n) =$$

$$= \underbrace{\lambda_1 0_W + \dots + \lambda_t 0_W}_{= 0} + \lambda_{t+1} w_{t+1} + \dots + \lambda_n w_n =$$

$$\Rightarrow \dim(\operatorname{Im} f) < \infty$$

$\{w_{t+1}, \dots, w_n\}$ $\boxed{\text{lin. indip.}}$

$$\boxed{\lambda_{t+1} w_{t+1} + \dots + \lambda_n w_n = 0_W}$$

$$\lambda_{t+1} f(v_{t+1}) + \dots + \lambda_n f(v_n) = 0_W \quad \xRightarrow{\text{linearità di } f}$$

$$\longrightarrow f(\lambda_{t+1} v_{t+1} + \dots + \lambda_n v_n) = 0_W \implies$$

$$\operatorname{Ker} f \ni \lambda_{t+1} v_{t+1} + \dots + \lambda_n v_n = \lambda_1 v_1 + \dots + \lambda_t v_t$$

$$-\lambda_1 v_1 - \dots - \lambda_t v_t + \lambda_{t+1} v_{t+1} + \dots + \lambda_n v_n = 0_V$$

$$\implies \lambda_i = 0 \quad \forall i = 1, \dots, n$$

$\{ w_{t+1}, \dots, w_n \}$ base per $\mathrm{Im}\, f \subset W$

$n = \dim V$, $t = \dim(\ker f)$

$\dim(\mathrm{Im}\, f) = n - t$

$$\dim V = n = t + n - t = \dim(\ker f) + \dim(\mathrm{Im}\, f)$$

$f : V \longrightarrow W$ lineare

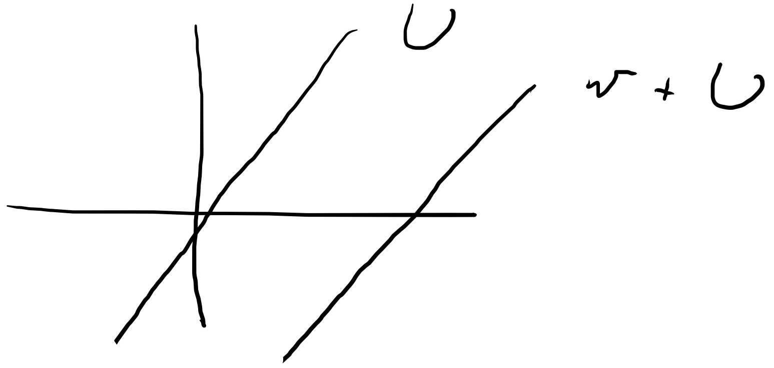
$$\boxed{\dim V = \mathrm{null}\, f + \mathrm{rg}\, f}$$

$U \subset V$, $\dim V < \infty$

V/U $\qquad$ $\underline{[v] = v + U}$ $\qquad$ <u>sottospazio affine di</u> V

$\pi : V \longrightarrow V/U$ lineare e suriettiva

$v \longmapsto v + U$



$\operatorname{Im} \pi = V/U$

$$\boxed{\ker \pi = U}$$

$\dim V = \dim U + \dim V/U$

$$\boxed{\dim V/U = \dim V - \dim U}$$

$$v \in \ker \pi \iff \pi(v) = [v] = \{v + u \mid u \in U\} = [0_V] = \{u \mid u \in U\} = U$$

$$[v] = 0_{V/U} = \pi(0_V)$$

$\pi : V \longrightarrow V/U$ lineare

$\pi(v) = \underline{[v]} = 0_{V/U} = \pi(0_V) = \underline{[0_V]} = \{ 0_V + u \mid u \in U \} =$
$= \{ u \mid u \in U \} = U$

$v \in \operatorname{Ker} \pi \iff v \in [0_V] = U$

$\Rightarrow \operatorname{Ker} \pi = U$

$f, g \in \mathrm{Hom}(V, W) \rightsquigarrow f+g \in \mathrm{Hom}(V, W)$

$(f+g)(v) = f(v) + g(v)$

$f+g$ è lineare $\circledast$

$\lambda \in \mathbb{K} \rightsquigarrow \lambda f : V \longrightarrow W$ lineare $\circledast$

$(\lambda f)(v) = \lambda f(v)$

Teorema Siano V, W $\mathbb{K}$-sp. vett., $\{v_i\}_{i\in I}$ base di V, $\{w_i\}_{i\in I}$ vettori di W (non necessariamente base).

Allora esiste un'unica applicazione lineare

$$f: V \longrightarrow W$$

t.c. $\boxed{f(v_i) = w_i}$ $\forall i \in I$.

Dim 1) unicità. Supponiamo che f esiste

Sia $v \in V \rightarrow \exists!\ \lambda_i \in \mathbb{K}$ di cui al più un numero finito $\neq 0$

t.c. $v = \sum_{i\in I} \lambda_i v_i = \lambda_{i_1} v_{i_1} + \cdots + \lambda_{i_k} v_{i_k}$

$$f(v) = f\left(\sum_{i \in I} \lambda_i v_i\right) = \sum_{i \in I} f(\lambda_i v_i) = \sum_{i \in I} \lambda_i f(v_i) =$$

$$= \sum_{i \in I} \lambda_i w_i$$