28 ottobre

$$|u(t)|_{L^2}^2 + 2\gamma \int_0^t |\nabla u|_{L^2}^2 \, dt' \leq |u_0|_{L^2}^2$$

$u_n \to u \qquad L^2_{loc}(\mathbb{R}_* \times \mathbb{R}^d)$

$\Rightarrow$ $\exists$ una sottosucc t.c.

$u_n \to u$ q.o. in $\mathbb{R}_+ \times \mathbb{R}^d$.

$\Downarrow$

$\Rightarrow$ per q.o. $\underline{t \in \mathbb{R}_+}$ si ha

$u_n(t,x) \to u(t,x)$ per q.o. $x \in \mathbb{R}^d$

$$\Rightarrow \qquad |u(t)|_{L^2} \leq \liminf_{n \to \infty} |u_n(t)|_{L^2}$$

$$\psi \in C^0([0,+\infty), L^2(\mathbb{R}^d, \mathbb{R}^d))$$

$$g_n(t) = \langle u_n, \psi \rangle \longrightarrow \langle u, \psi \rangle = g(t) \text{ loc unif in } C^0([0,+\infty))$$

Dim che in $[0,T]$ si ha conv. ~~uniforme~~. $g_n \to g$ $\quad L^\infty([0,T])$

Cioè $\forall \varepsilon > 0 \ \exists N_\varepsilon$ t.c. $n > N_\varepsilon \Rightarrow |g_n(t) - g(t)| \leq \varepsilon$
$\forall t \in [0,T]$.

Fissiamo un $\varepsilon > 0$

$$g_n = \langle u_n, \psi \rangle = \langle P_{n_0} u_n, \psi \rangle + \langle (1-P_{n_0}) u_n, \psi \rangle$$

Lemma $\forall \varepsilon > 0 \ \exists n_0(\varepsilon)$ t.c. $m \geq n_0(\varepsilon)$

$$\Rightarrow \quad |(1-P_{m})\psi|_{L^\infty([0,T], L^2_x)} < \varepsilon$$

Dim Questo segue dal fatto che $\psi : [0,T] \to L^2_x$ è unif continua, cioè che $\forall \varepsilon > 0 \ \exists \delta > 0$ t.c.

$$|t-s| < \delta \Rightarrow |\psi(t) - \psi(s)|_{L^2_x} \leq \varepsilon$$

Se scompongo $[0,T] = [\overset{t_0}{0}, t_1] \cup [t_1, t_2] \cup \dots \cup [t_{N-1}, \overset{T}{t_N}]$ in intervalli di lunghezza $< \delta$, siccome $\forall t_j$ $j = 0, N$ ho $\lim_{m\to\infty} (1-P_m)\psi(t_j) = 0$

$P_m f \xrightarrow{m\to\infty} f \quad \forall f \in L^2$

$\chi_{[0,m]}(|\xi|) \hat{f} \to \hat{f} \quad \forall f \in L^2$

$(1-\chi_{[0,m]}(|\xi|))^2 |\hat{f}|^2 \to 0$ in L^1

Posso trovare un $n_0(\varepsilon)$ t.c. $m > n_0(\varepsilon)$ ho

$$|(1-P_m)\psi(t_j)|_{L^2_x} < \varepsilon$$

$$\|(1-P_m)\psi(t)\|_{L^2_x} \le \qquad t\in[0,T], \quad t\in[t_{j-1},t_j]$$

$$\le \underbrace{\|(1-P_m)\psi(t_j)\|_{L^2_x}}_{<\varepsilon} + \underbrace{\|\cancel{(1-P_m)}(\psi(t)-\psi(t_j))\|_{L^2_x}}_{<\varepsilon} < 2\varepsilon$$

$\square$

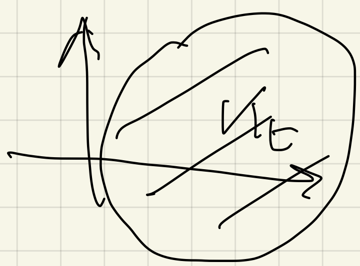
$$g_n(t) = \langle P_{m_0} u_n, \psi\rangle + \langle u_n, (1-P_{m_0})\psi\rangle \qquad \text{per } m \gg 1$$

$$|\langle u_n(t), (1-P_{m_0})\psi(t)\rangle| \le \|u_n(t)\|_{L^2_x}\,\varepsilon \le \|u_0\|_{L^2_x}\,\varepsilon$$

$$\langle P_{n_0} u_n, \psi\rangle$$

<u>Lemma 2</u> $\forall \varepsilon>0 \ \exists K \subset\subset \mathbb{R}^d$ t.c.

$$\|\psi\|_{L^\infty([0,T],L^2(\mathbb{R}^d\setminus K))} < \varepsilon$$



$\mathbb{R}^d$

$$\|\psi(t)\|_{L^2(\mathbb{R}^d\setminus K)} < \varepsilon$$

Quindi fissiamo il $K \subset \mathbb{R}^d$ associato al nostro $\varepsilon>0$

$$\langle P_{n_0} u_n, \psi\rangle = \langle P_{n_0} u_n, \psi\rangle_{L^2(K)} + \langle P_{n_0} u_n, \psi\rangle_{L^2(\mathbb{R}^d\setminus K)}$$

$$\left| \langle P_{n_0} u_n(t), \psi(t) \rangle_{L^2(\mathbb{R}^d \setminus K)} \right| \leq \underbrace{|P_n u_n(t)|_{L^2_x(\mathbb{R}^d)}}_{\leq |u_0|^2_{L^2_x}} \underbrace{|\psi(t)|_{L^2(\mathbb{R}^d \setminus K)}}_{< \epsilon}$$

$\langle P_{n_0} u_n, \psi \rangle_{L^2(K)}$. Ma supponiamo che

$\{P_{n_0} u_n\}$ converge in $C^0([0,T], L^2(K))$

$P_{n_0}(u)$

$P_{n_0} u_{n_k} \longrightarrow v \overset{?}{=} P_{n_0} u$

$u_n \to u$ in $L^2([0,T] \times K)$

Inoltre $u_n \to u$ in $L^2([0,T] \times \mathbb{R}^d)$

$\Rightarrow$ $P_{n_0} u_n \to P_{n_0} u$ $L^2([0,T] \times \mathbb{R}^d)$

$\Rightarrow$ $P_{n_0} u_n \to P_{n_0} u$ in $L^2([0,T] \times K)$

$P_{n_0} u_{n_k} \to v$ $L^2([0,T] \times K)$

$$\left| \langle u_n, \psi\rangle - \langle u,\psi\rangle \right| \le \left| \langle P_{n_0} u_n, \psi\rangle_{L^2(K)} - \langle P_{n_0} u, \psi\rangle_{L^2(K)} \right| +$$

$$+ \underbrace{\left| \langle u-u_n, (1-P_{n_0})\psi\rangle_{L^2_x(\mathbb{R}^d)} \right|}_{<\varepsilon/3} + \underbrace{\left| \langle P_{n_0}(u_n-u), \psi\rangle_{L^2(\mathbb{R}^d\setminus K)} \right|}_{<\varepsilon/3}$$

Concludo $\left|\langle u_n,\psi\rangle - \langle u,\psi\rangle\right| < \varepsilon \quad \forall\, t\in[0,T]$

se $n >> 1$ e se $n_0 >> 1$ e K è grande.

$$\begin{cases} \partial_t u_n - \nu \Delta u_n + P_n \mathbb{P} \operatorname{div}(u_n \otimes u_n) = 0 \\ u_n|_{t=0} = P_n u \end{cases}$$

$\cdot\,\psi$ $\quad \cancel{\langle \partial_t \psi\rangle}_{L^2_x}$

$$\int_{\mathbb{R}^d} u_n(t)\,\psi(t)\,dx = \int_{\mathbb{R}^d} P_n u_0\,\psi(0)\,dx +$$

$$+ \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\,\psi_t(s,x)\,dx - \nu \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\,\Delta\psi(s,x)\,dx$$

$$- \int_0^t ds \int_{\mathbb{R}^d} u_n \otimes u_n : \nabla \mathbb{P} P_n \psi(s,x)\,dx$$

$$\int_{\mathbb{R}^d} u(t)\,\psi(t)\,dx = \int_{\mathbb{R}^d} u_0\,\psi(0)\,dx +$$

$$+ \int_0^t ds \int_{\mathbb{R}^d} u(s,x)\,\psi_t(s,x)\,dx - \nu \int_0^t ds \int_{\mathbb{R}^d} u(s,x)\,\Delta\psi(s,x)\,dx$$

$$- \lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}^d} u_n \otimes u_n : \nabla \mathbb{P} P_n \psi(s,x)\,dx$$

$$\mathbb{P}_n = 1 + (\mathbb{P}_n - 1)$$

$$\int_0^t ds \int_{\mathbb{R}^d} u_n \otimes u_n : \nabla \mathbb{P} \mathbb{P}_n \psi(1,x)\, dx =$$

$$= \int_0^t ds \int_{\mathbb{R}^d} u_n \otimes u_n : \nabla \mathbb{P} \quad \psi(1,x)\, dx \quad \|$$

$$\underline{\int_0^t ds \int_{\mathbb{R}^d} u_n \otimes u_n : \nabla \mathbb{P} (\mathbb{P}_n - 1)\, \psi(1,x)\, dx}$$

Per K un compatto

$$\int_0^t ds \int_{\mathbb{R}^d} u_n \otimes u_n : \nabla \mathbb{P} \quad \psi(1,x)\, dx =$$

$$\int_0^t ds \int_K u_n \otimes u_n : \nabla \mathbb{P} \quad \psi(1,x)\, dx \quad +$$

$$+ \int_0^t ds \int_{\mathbb{R}^d \setminus K} u_n \otimes u_n : \nabla \mathbb{P} \quad \psi(1,x)\, dx$$

$$\int_0^t dt \int_K (u_n \otimes u_n - u \otimes u) : (\nabla \mathbb{P} \quad \psi)\, dx \xrightarrow{n \to \infty} 0$$

$$\leq \underline{|u_n \otimes u_n - u \otimes u|}_{L^1_t L^2_x} \, |\nabla \mathbb{P} \psi|_{L^\infty_t L^2_x}$$

Questo è una conseguenza di

$$\underline{\lim_n u_n \otimes u_n = u \otimes u} \quad \text{in} \quad L^1([0,T], L^2(K))$$

$$\lim_n u_n = u \quad \text{in} \quad L^2([0,T], L^4(K))$$

$$\lim_n u_n = u \quad \text{in} \quad L^2([0,T], L^4(K))$$

Sia f in $H^1(\mathbb{R}^d)$ e sia K un compatto

e sia $K \subset \mathring{\Omega} \subset \Omega \subset\subset \mathbb{R}^d$

Allora $\exists \; C_{K\Omega}$ t.c.

$$|f|_{L^4(K)} \le C_{K\Omega} \, |f|_{L^2(\Omega)}^{1-\frac{d}{4}} \, |f|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \qquad (1)$$

Dim $\exists \; \chi \in C_c^\infty(\mathring{\Omega}, [0,1])$ t.c. $\chi = 1$ in K.

$$|f|_{L^4(K)} \le |\chi f|_{L^4(\mathbb{R}^d)} \le C_d \, |\chi f|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{4}} \, |\nabla(\chi f)|_{L^2(\mathbb{R}^d)}^{\frac{d}{4}}$$

$$\le C_d \, |f|_{L^2(\Omega)}^{1-\frac{d}{4}} \left(|\nabla\chi \, f|_{L^2(\mathbb{R}^d)} + |\cancel{\chi}\nabla f|_{L^2(\mathbb{R}^d)} \right)^{\frac{d}{4}}$$

$$= C_d \, |f|_{L^2(\Omega)}^{1-\frac{d}{4}} \left(|\nabla\chi|_{L^\infty} |f|_{L^2(\mathbb{R}^d)} + |\nabla f|_{L^2(\mathbb{R}^d)} \right)^{\frac{d}{4}}$$

$$\le C_d \, (1 + |\nabla\chi|_{L^\infty})^{\frac{d}{4}} \, |f|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \, |f|_{L^2(\Omega)}^{1-\frac{d}{4}}$$

$$\| u-u_n \|_{L^2([0,T],L^4(K))} = \Big\| \; \| u-u_n\|_{L^4(K)} \Big\|_{L^2[0,T]}$$

$$\lesssim \Big\| \; \| u-u_n\|_{L^2(\Omega)}^{1-\frac{d}{4}} \; \| u-u_n\|_{H^1(\mathbb{R}^d)}^{\frac{d}{4}} \Big\|_{L^2[0,T]}$$

$$\frac{1}{2} = \frac{4-d}{8} + \frac{d}{8}$$

$$\leq \Big\| \; \| u-u_n\|_{L^2(\Omega)}^{\frac{4-d}{4}} \Big\|_{L^{\frac{8}{4-d}}([0,T])} \Big\| \; \| u-u_n\|_{H^1}^{\frac{d}{4}} \Big\|_{L^{\frac{8}{d}}[0,T]}$$

$$= \Big\| \; \| u-u_n\|_{L^2(\Omega)} \Big\|_{L^2([0,T])}^{\frac{4-d}{4}} \Big\| \| u-u_n\|_{H^1} \Big\|_{L^2[0,T]}^{\frac{d}{4}}$$

$$= \| u-u_n\|_{L^2([0,T]\times\Omega)}^{\frac{4-d}{4}} \; \| u-u_n\|_{L^2([0,T],H^1(\mathbb{R}^d))}^{\frac{d}{4}}$$

$$\leq \underbrace{\| u-u_n\|_{L^2([0,T]\times\Omega)}^{\frac{4-d}{4}}}_{\to 0} \; \Big(\underbrace{\| u\|_{L^2[0,T]H^1_x} + \| u_n\|_{L^2[0,T]H^1_x}}_{\leq \| u_0\|_{L^2_x}} \Big)^{\frac{d}{4}}$$

Questo ci trovato una u che oltre a

$$u \in L^\infty(\mathbb{R}_+, L^2_x), \quad \nabla u \in L^2(\mathbb{R}_+, L^2_x), \quad u \in C^0([0,\infty), L^2_w)$$

$$\|u(t)\|^2_{L^2_x} + 2\nu \int_0^t \|\nabla u\|^2_{L^2_x} dt' \leq \|u_0\|^2_{L^2_x} \qquad (1)$$

soddisfa anche $\forall \quad 0 \leq s \leq t$

$$\underbrace{\|u(t)\|^2_{L^2_x} + 2\nu \int_s^t \|\nabla u\|^2_{L^2} dt' \leq \|u(s)\|^2_{L^2_x}}_{\text{Soluzioni di Leray-Hopf}} \qquad (2)$$

Soluzioni di Leray-Hopf $\Leftarrow$

<u>Lemma</u> Sia $u(t)$ di L-H. Allora $\forall s > 0$

$$u(t) \xrightarrow{t \to s^+} u(s) \quad \text{in } L^2(\mathbb{R}^3)$$

<u>Dim</u> Infatti $u(t) \xrightarrow{t \to s^+} u(s)$ in $L^2(\mathbb{R}^3)$.

La (2) implica $\overline{\lim_{t \to s^+}} \|u(t)\|_{L^2_x} \leq \|u(s)\|_{L^2_x}$

$$\|u(s)\|_{L^2_x} \leq \liminf_{t \to s^+} \|u(t)\|_{L^2_x}$$

$$\Longrightarrow \lim_{t \to s^+} \|u(t)\|_{L^2_x} = \|u(s)\|_{L^2_x}$$

$$\Longrightarrow \lim_{t \to s^+} u(t) = u(s) \quad \text{in } L^2_x.$$

Teor Sia $u_0 \in V(\mathbb{R}^3) = H^1 \cap H$. Allora $\exists$ una costante c_ν ed un tempo $T > c_\nu |\nabla u_0|_{L^2}^{-4}$ e una soluzione di Leroy che soddisfa $\underline{u \in L^\infty([0,T], V)}$, $\underline{\nabla^2 u \in L^2([0,T], L^2)}$, e si ha

$$|u(t)|_{L^2}^2 + 2\nu \int_s^t |\nabla u|_{L^2_x}^2 \, dt' = |u(s)|_{L^2}^2$$

$$\forall \quad 0 \le s \le t \le T.$$

$\exists \quad \varepsilon_\nu > 0$ t.c. se $|\nabla u_0|_{L^2} |u_0|_{L^2} < \varepsilon_\nu$

allora quanto sopra vale $\forall T > 0$.