2 Novembre

<u>Teo</u> $u_0 \in V = H^1 \cap H$. Esiste c_ν e $T > \frac{c_\nu}{|\nabla u_0|_{L^2}^4}$ t.c. una delle soluzioni di Leray soddisfa

$$u \in L^\infty([0,T], H^1), \quad e \quad \nabla^2 u \in L^2([0,T], L^2)$$

Questa soluzione soddisfa

$$|u(t)|_{L^2}^2 + 2\nu \int_s^t |\nabla u|_{L^2}^2 \, dt' = |u(s)|_{L^2}^2$$

$$0 \leq s < t \leq T.$$

Inoltre $\exists \, \varepsilon_\nu > 0$ t.c.

$|\nabla u_0|_{L^2} \, |u_0|_{L^2} < \varepsilon_\nu \Longrightarrow$ allora ~~le~~ precedenti prop. valgono $\forall \, T > 0$.

Dim $\{u_n\}$

$$\begin{cases} \dot{u}_n + P_n \mathbb{P}\,\mathrm{div}(u_n \otimes u_n) - \nu \Delta u_n = 0 \\ u_n|_0 = P_n u_0 \end{cases}$$

Applicando $-\langle \,,\Delta u_n \rangle$

$$\frac{1}{2}\frac{d}{dt}|\nabla u_n|^2_{L^2} + \cancel{\nu}\,\tfrac{\nu}{2}\,|\Delta u_n|^2_{L^2} = \langle P_n \mathbb{P}\,\mathrm{div}(u_n \otimes u_n), \Delta u_n \rangle$$

$$\leq |\mathrm{div}(u_n \otimes u_n)|_{L^2}\, |\Delta u_n|_{L^2} \leq$$

$$\leq |\nabla u_n|_{L^2}\, |u_n|_{L^\infty}\, |\Delta u_n|_{L^2}$$

$$\lesssim |\nabla u_n|_{L^2}\, |\nabla u_n|^{\frac{1}{2}}_{L^2}\, |\Delta u_n|^{\frac{1}{2}}_{L^2}\, |\Delta u_n|_{L^2}$$

$$= |\nabla u_n|^{\frac{3}{2}}_{L^2}\, |\Delta u_n|^{\frac{3}{2}}_{L^2}$$

$$\leq d_\nu\, |\nabla u_n|^6_{L^2} + \cancel{\frac{\nu}{2}\, |\Delta u_n|^2_{L^2}}$$

$$ab \leq \frac{a^4}{4\lambda^4} + \frac{3}{4}\lambda^{\frac{4}{3}} b^{\frac{4}{3}}$$

$$\frac{a}{\lambda}\, b\lambda \leq \lambda^p \frac{a^p}{p} + \frac{b^{p'}}{p'}\lambda^{p'} \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

$$\frac{d}{dt}|\nabla u_n|^2_{L^2} + \gamma |\Delta u_n|^2_{L^2} \le d_\gamma |\nabla u_n|^6_{L^2}$$

$$\begin{cases} \frac{d}{dt}|\nabla u_n|^2_{L^2} \le d_\gamma |\nabla u_n|^6_{L^2} \\ |\nabla u_n(0)|^2_{L^2} = |P_n \nabla u_0|^2_{L^2} \end{cases}$$

$$\begin{cases} \dot{X} = d_\gamma X^3 \\ X(0) = |\nabla u_0|^2_{L^2} \end{cases}$$

$$\frac{dX}{dt} = d_\gamma X^3$$

$$\int \frac{dX}{X^3} = \int d_\gamma \, dt$$

$$-\frac{1}{2} X^{-2} + C = d_\gamma t$$

$$C = \frac{1}{2}\frac{1}{X^2(0)} = \frac{1}{2}\frac{1}{|\nabla u_0|^4_{L^2}}$$

$$X(t) = \frac{X(0)}{\sqrt{1-2d_\gamma t X^2(0)}} = \frac{|\nabla u_0|^2_{L^2}}{\sqrt{1-2d_\gamma t |\nabla u_0|^4_{L^2}}}$$

$X(t)$ definito per $0 \le t < \frac{1}{2d_\gamma |\nabla u_0|^4_{L^2}}$

Vogliamo dimostrare che $\forall\, n$

si ha $|\nabla u_n(t)|^2_{L^2} \leq X(t)$

$$\left[0, \frac{1}{2d_\nu |\nabla u_0|^4_{L^2}}\right)$$

$$\frac{d}{dt}|\nabla u_n|^2_{L^2} + \nu|\Delta u_n| \leq d_\nu |\nabla u_n|^6_{L^2} \leq \underline{\underline{d_\nu X(t)}}$$

sono vere in $\left[0, \underbrace{\frac{c_\nu}{|\nabla u(0)|^4_{L^2}}}_{T}\right]$ $\forall t$

$$c_\nu < \frac{1}{2d_\nu}$$

$$|\nabla u_n(t)|^2_{L^2} + \nu\int_0^t |\Delta u_n|^2_{L^2}\, dt' \leq \underbrace{C_T |\nabla u_0|_{L^2}}_{C_{(*)}} \quad (*)$$

$$t \in [0, T]$$

$$\nabla u_n \in L^\infty([0,T], L^2)$$

$$\Delta u_n \in L^2([0,T], L^2)$$

$$\nabla u_n \overset{*}{\rightharpoonup} \nabla u$$

$$\Delta u_n \rightharpoonup \Delta u$$

$$|\nabla u(t)|^2_{L^2} + \nu\int_0^t |\Delta u|^2_{L^2}\, dt' \leq C_{(*)}$$

$$\forall n \quad |\nabla u_n|^2_{L^2} \leq X \qquad t < \frac{1}{2d_\nu |\nabla u_0|^4_{L^2}}$$

$$\frac{d}{dt} X = d_\nu X^3$$

$$\frac{d}{dt} |\nabla u_n|^2_{L^2} \leq d_\nu |\nabla u_n|^6_{L^2}$$

1) $X(0) = |\nabla u_0|^2_{L^2} > |P_n \nabla u_0|^2_{L^2}$

$$\frac{d}{dt}\left(X - |\nabla u_n|^2_{L^2}\right) = \dot{X} - \frac{d}{dt}|\nabla u_n|^2_{L^2}$$

$$\geq d_\nu \left(X^3 - |\nabla u_n|^6_{L^2}\right) =$$

$$= d_\nu \underbrace{\left(X - |\nabla u_n|^2_{L^2}\right)}_{>0}\left(X^2 + X|\nabla u_n|^2_{L^2} + |\nabla u_n|^4_{L^2}\right)$$

$$\Rightarrow \quad |\nabla u_n|^2_{L^2} < X \qquad \forall \; t.$$

2) $X(0) = |\nabla u_n(0)|^2_{L^2}$

$$X_\varepsilon(0) = |\nabla u_n(0)|^2_{L^2} + \varepsilon$$

$$\dot{X}_\varepsilon = d_\nu X_\varepsilon^3 \qquad \left[0, (2d_\nu)^{-1}\left(|\nabla u_0|^2_{L^2} + \varepsilon\right)^{-2}\right]$$

$$X_\varepsilon(t) = \frac{X_\varepsilon(0)}{\sqrt{1 - 2d_\gamma t\, X_\varepsilon^2(0)}} \qquad [0, T_\varepsilon)$$

$$T_\varepsilon \xrightarrow{\varepsilon \to 0^+} T_0 = \left(2 d_\gamma |\nabla u_0|_{L^2}^2\right)^{-1}$$

$\forall\ 0 < T < T_0$

$$\left.\begin{aligned} &X_\varepsilon(t) \longrightarrow X(t) \quad \text{in } C^0([0,T]) \\ &X_\varepsilon(t) > |\nabla u_n(t)|_{L^2}^2 \end{aligned}\right| *$$

Se per assurdo in $t_0 < \left(2 d_\gamma |\nabla u_0|_{L^2}^2\right)^{-1}$

avessi $X(t_0) < |\nabla u_n(t_0)|_{L^2}^2$

$$X_\varepsilon(t_0) > |\nabla u_n(t_0)|_{L^2}^2$$

$$\Big\downarrow \varepsilon \to 0^+$$

$$X(t_0) \geq |\nabla u_n(t_0)|_{L^2}^2 \quad \text{///}$$

Dimostriamo $\qquad T = \frac{c\nu}{|\nabla u_0|^2_{L^2}}$

$$\partial_t u \, , \, \mathrm{div}(u \otimes u) \in L^2([0,T), L^2) \; *$$

Dimostriamo che vale

$$\int_0^T \langle \partial_t u - \nu \Delta u + \mathrm{div}(u\otimes u), \overset{u}{w} \rangle dt = 0$$

$$\forall \; w \in L^2([0,T), H)$$

$$w = \chi_{[s,t]} \, u \in L^2([0,T), H)$$

$$\int_s^t \left(\langle \partial_t u, u \rangle + \nu \, |\nabla u|^2_{L^2} \right) dt' = 0$$

$$u \in L^2([0,T), H^1(\mathbb{R}^d))$$

$$\cap \, H^1([0,T), H^{-1}(\mathbb{R}^d))$$

$$\Longrightarrow \quad |u(t)|^2_{L^2} \in AC([0,T])$$

$$\frac{d}{dt} |u|^2_{L^2} = 2 \langle \partial_t u, u \rangle$$

$$\frac{1}{2} |u(t)|^2_{L^2} + \nu \int_s^t |\nabla u|^2_{L^2} = \frac{1}{2} |u(s)|^2_{L^2}$$

$$|\mathrm{div}(u\otimes u)|_{L^2((0,T),L^2)} \leq \big|\, |\nabla u\, u|_{L^2_x}\big|_{L^2(0,T)}$$

$$\leq \big|\, |\nabla u|_{L^2_x}\ |u|_{L^\infty_x}\big|_{L^2(0,T)}$$

$$H^2_x \hookrightarrow L^\infty_x$$

$$\leq |\nabla u|_{L^\infty((0,T),L^2_x)}\ \underline{|u|}_{L^2((0,T),H^2)} < C(t)$$

$$\partial_t u \in L^2((0,T),L^2_x)$$

$$\forall\ \phi_n \in C^\infty_{c\sigma}(\mathbb{R}^3,\mathbb{R}^3)$$

$$\langle u(t);\phi\rangle - \langle u(0);\phi\rangle = \int_0^t dt' \langle \nu\Delta u - \underline{\mathbb{P}\mathrm{div}(u}\otimes u),\phi\rangle$$

$$\forall\ \phi \in L^2(\mathbb{R}^3,\mathbb{R}^3) \qquad \phi \in H$$

$$\phi = \mathbb{P}\phi + \nabla V$$

$$\partial_t u = \underbrace{\nu\Delta u}_{L^2((0,T),L^2)} - \mathbb{P}\overbrace{\mathrm{div}(u\otimes u)}^{\in L^2((0,T),L^2)}$$

$$\mathcal{D}'((0,T),L^2) = \mathcal{L}(\underline{\mathcal{D}(0,T)},L^2)$$

$$\int_0^T \langle \partial_t u - \nu \Delta u + \operatorname{div}(u \otimes u), w \rangle \, dt' = 0$$

$$\forall \ w \in \underline{L^2((0,T), H)}$$

$w = -\Delta u$

$$\int_0^T \langle \partial_t u - \nu \Delta u + \operatorname{div}(u \otimes u), -\Delta u \rangle \, dt = 0$$

$$\int_0^T dt \Big(\underbrace{\langle \partial_t \nabla u, \nabla u \rangle}_{\frac{1}{2}\frac{d}{dt}|\nabla u|_{L^2}^2} + \nu |\Delta u|_{L^2}^2 - \langle \operatorname{div}(u \otimes u, \Delta u \rangle \Big)$$

$= 0$

$$\underline{\frac{1}{2}\frac{d}{dt}|\nabla u|_{L^2}^2 + \nu |\Delta u|_{L^2}^2 = \langle \operatorname{div}(u \otimes u), \underline{\underline{\Delta u}} \rangle}$$

$$\boxed{\partial_t u = \nu \Delta u - \mathbb{P} \operatorname{div}(u \otimes u)}$$

$-\Delta u$

$$\langle \partial_t u, -\Delta u \rangle = -\nu |\Delta u|_{L^2}^2 + \langle \mathbb{P} \operatorname{div}(u \otimes u), \Delta u \rangle$$

$$\frac{1}{2}\frac{d}{dt} |\nabla u|_{L^2}^2 + \nu |\Delta u|_{L^2}^2 = \langle \operatorname{div}(u \otimes u), \Delta u \rangle$$

$$\frac{1}{2}\frac{d}{dt}|\nabla u|_{L^2}^2 + \cancel{\nu}\,\frac{\nu}{2}|\Delta u|_{L^2}^2 = \langle \operatorname{div}(u\otimes u), \Delta u\rangle$$

$$\le |\nabla u|_{L^2}^{3/2}\,|\Delta u|_{L^2}^{3/2} =$$

$$|\nabla u|_{L^2_x} \le |u|_{L^2_x}^{1/2}|\Delta u|_{L^2_x}^{1/2}$$

$$= |\nabla u|_{L^2}^{1/2}\,|\nabla u|_{L^2}\,|\Delta u|_{L^2}^{3/2}$$

$$\frac{a}{\lambda}\,b\lambda \le \frac{a^8}{8\lambda^8} + \frac{7}{8}b^{8/7}\lambda^{8/7}$$

$$= \left(|u|_{L^2}^{1/4}\,|\nabla u|_{L^2}\right)|\Delta u|_{L^2}^{7/4}$$

$$\le C_\nu\,|u|_{L^2}^2\,|\nabla u|_{L^2}^8 + \cancel{\frac{\nu}{2}|\Delta u|_{L^2}^2}$$

$$\frac{d}{dt}|\nabla u|_{L^2}^2 + \cancel{\nu|\Delta u|_{L^2}^2} \le 2C_\nu|u|_{L^2}^2|\nabla u|_{L^2}^8$$

$$\frac{d}{dt}|\nabla u|_{L^2}^2 \le 2C_\nu\,|u|_{L^2}^2\,|\nabla u|_{L^2}^8$$

$$\partial_t u = \nu\Delta u - \mathbb{P}\operatorname{div}(u\otimes u) \qquad \langle\,;u\rangle$$

$$\frac{d}{dt}|u|_{L^2}^2 = -2\nu|\nabla u|_{L^2}^2$$

$$|u|_{L^2}^2 \qquad |\nabla u|_{L^2}^2 \qquad\qquad |\nabla u|_L^2$$

$$\frac{d}{dt}|\nabla u|_{L^2}^2 + \nu |\Delta u|_{L^2}^2 \le C_\nu \left(|u|_{L^2}^4 |\nabla u|_{L^2}^4\right) |\Delta u|_{L^2}^2$$

$$|\nabla u(t)|_{L^2}^2 + \nu \int_0^t |\Delta u|_{L^2}^2 \le$$

$$\le |\nabla u_0|_{L^2}^2 + C_\nu \int_0^t \underbrace{\left(|u|_{L^2}^4 |\nabla u|_{L^2}^4\right)}_{< \boxed{\varepsilon_0}} |\Delta u|_{L^2}^2$$

$$C_\nu \varepsilon_0 < \frac{\nu}{2}$$

$$|\nabla u(t)|_{L^2}^2 + \frac{\nu}{2} \int_0^t |\Delta u|_{L^2}^2 \le |\nabla u_0|_{L^2}^2$$

$$\boxed{\frac{d}{dt} |u|_{L^2}^2 \, |\nabla u|_{L^2}^2 < 0}$$