Image Processing for Physicists

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Overview

- More on image representations (Fourier-related concepts)
  - DCT → Discrete Cosine Transform
  - WFT → Windowed Fourier Transform
  - WT → Wavelet Transform

Image representations

$$f(x,y) = \sum c_n \mathcal{B}_n(x,y)$$

c_n: coefficients
$\mathcal{B}_n$: basis functions
(most convenient: orthonormal basis)

DFT:

$$f(m,n) = \sum_{k,l} F_{kl} \underbrace{e^{2\pi i\left(\frac{mk}{M} + \frac{nl}{N}\right)}}_{\mathcal{B}_{kl}(m,n)}$$

$$\begin{aligned} f(x) &= \langle x | f \rangle \\ &= \langle x | \mathbb{1} | f \rangle \\ &= \langle x | \Big(\sum_k |k\rangle \langle k| \Big) | f \rangle \\ &= \sum_k \underbrace{\langle x | k \rangle}_{\text{basis } |k\rangle \text{ in representation } |x\rangle} \langle k | f \rangle \end{aligned}$$

1D

$$f_n = \sum_k F_k e^{2\pi i kn/N}$$

$$\omega = e^{2\pi i/N}$$

$$\begin{bmatrix} \\ f \\ \\ \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \\ 1 & \omega^3 & \omega^6 & \cdots & \\ \vdots & & & & \end{bmatrix}}_{\mathcal{F}^{-1}} \begin{bmatrix} \\ F \\ \\ \end{bmatrix}$$

DFT: a simple change of basis

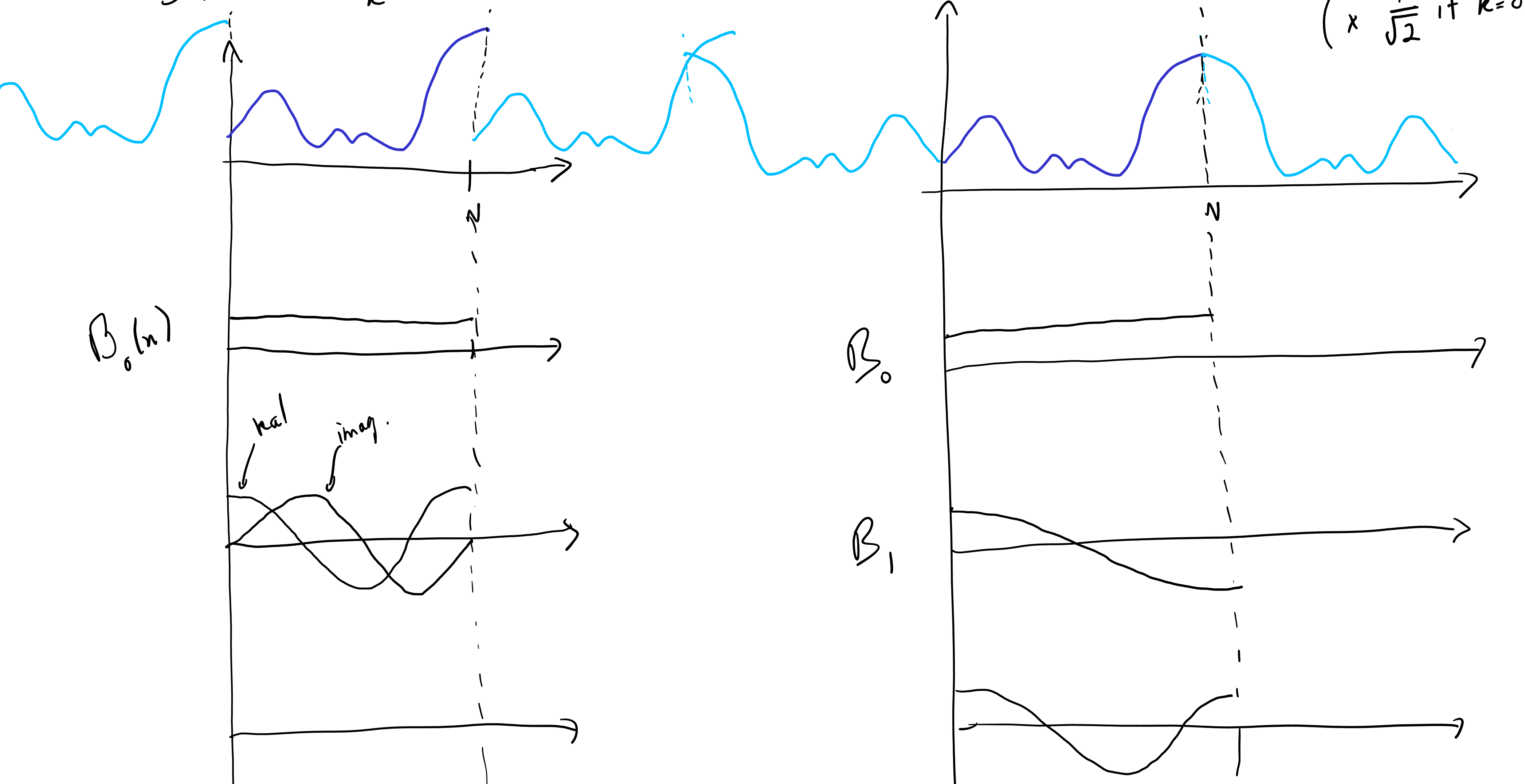
Discrete Cosine Transform

A variation on the theme of DFT

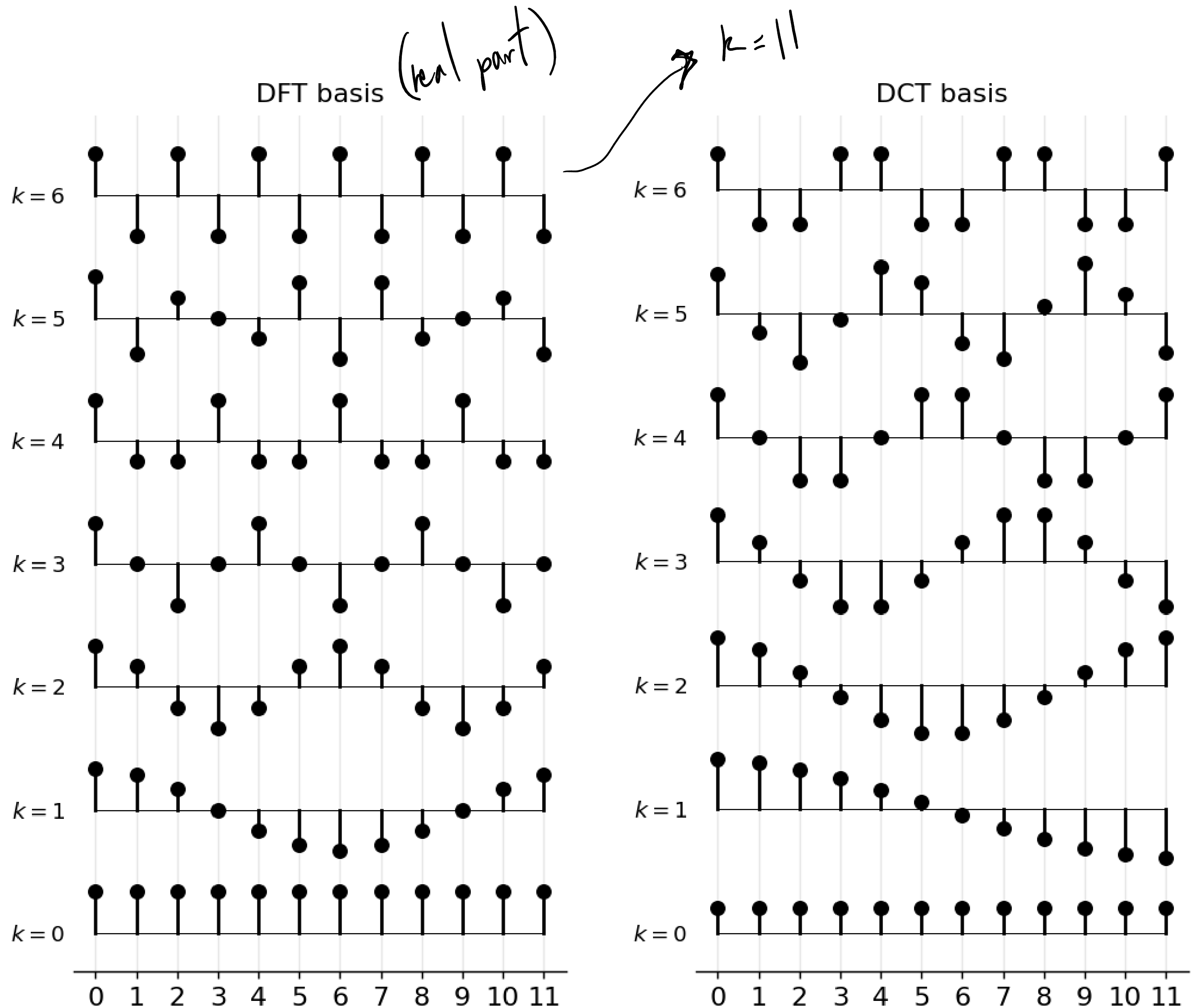
DFT: $B_k(n) = e^{2\pi i kn/N}$

$B_k(n) = \cos\left(\left(n+\frac{1}{2}\right)\frac{k\pi}{N}\right)$

$\left(\times \frac{1}{\sqrt{2}} \text{ if } k=0\right)$

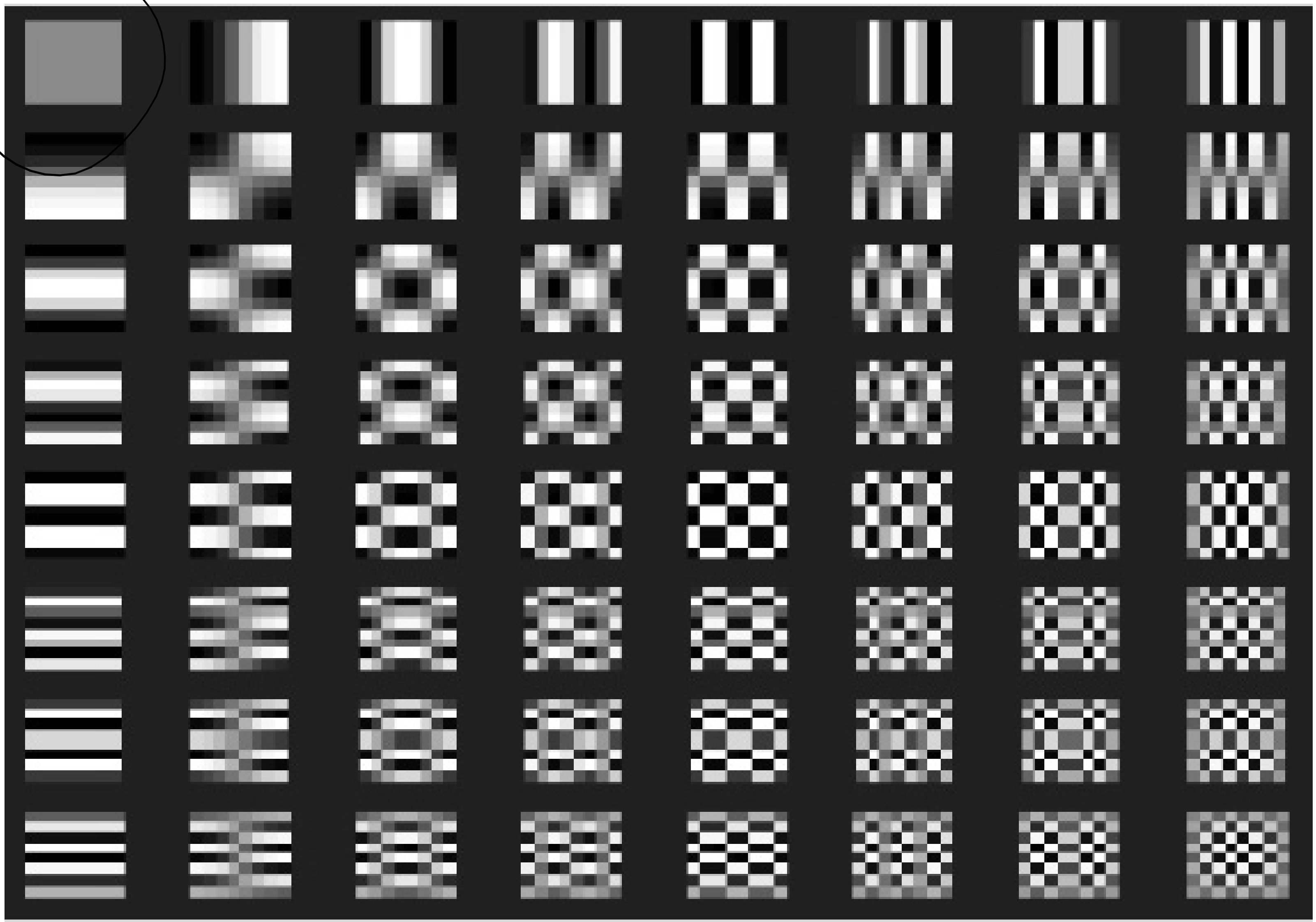


Discrete Cosine Transform



Discrete Cosine Transform

64 DCT basis vectors for 8x8 image



complete orthonormal basis for 8x8

Discrete Cosine Transform

Image compression

1:1 bit rate

8:1 bit rate

"lossy" compression

some information is lost

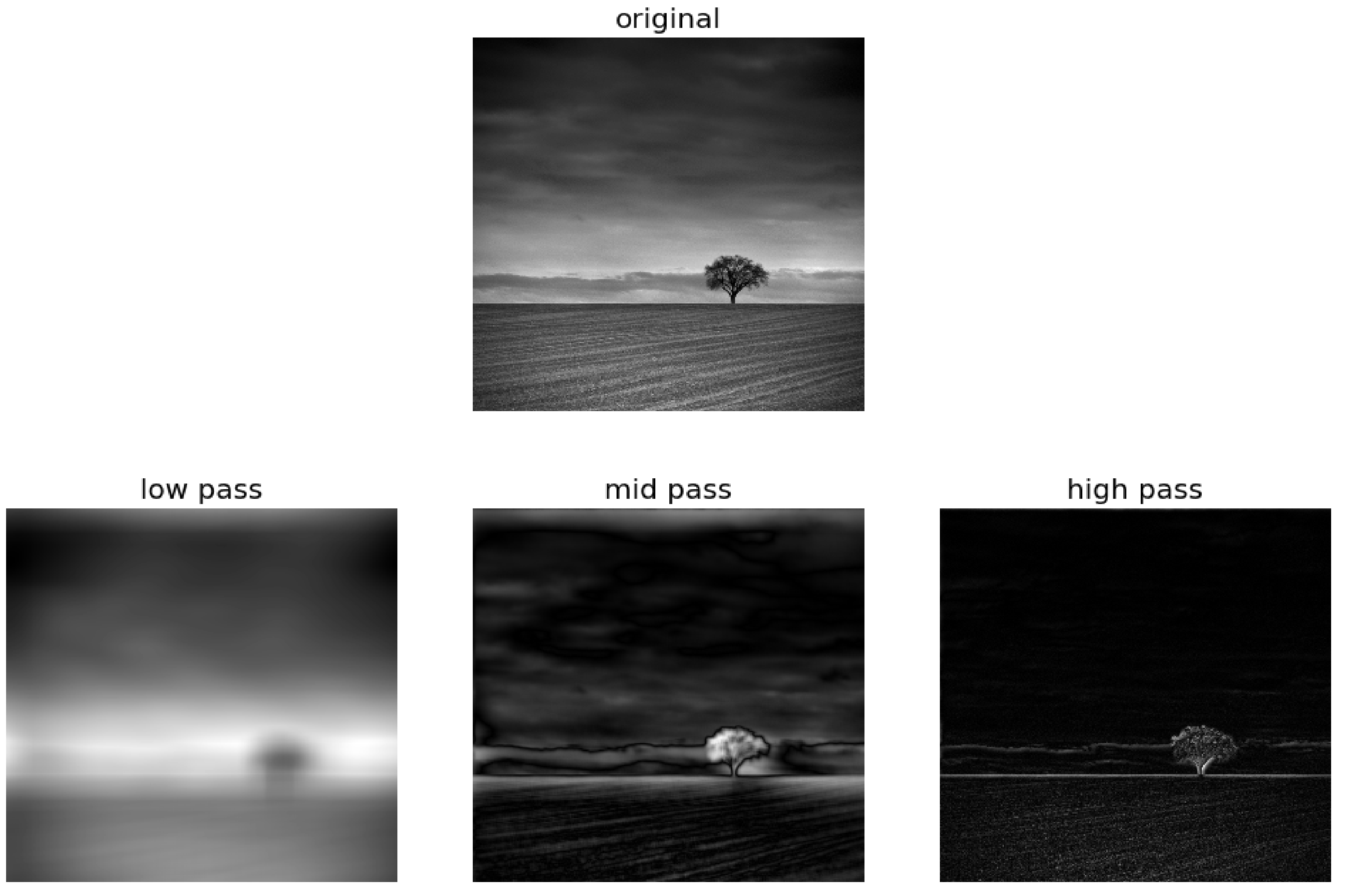
32:1 bit rate

128:1 bit rate

Historical overview

- 1822 Fourier: Fourier transform
- 1946 Gabor: Short-time Fourier transform (STFT)
- 1974 Ahmed, Natarajan & Rao: Discrete Cosine Transform
- 1980s Morlet, Mallat, Daubechies, ... : Wavelets

Bandpass filtering

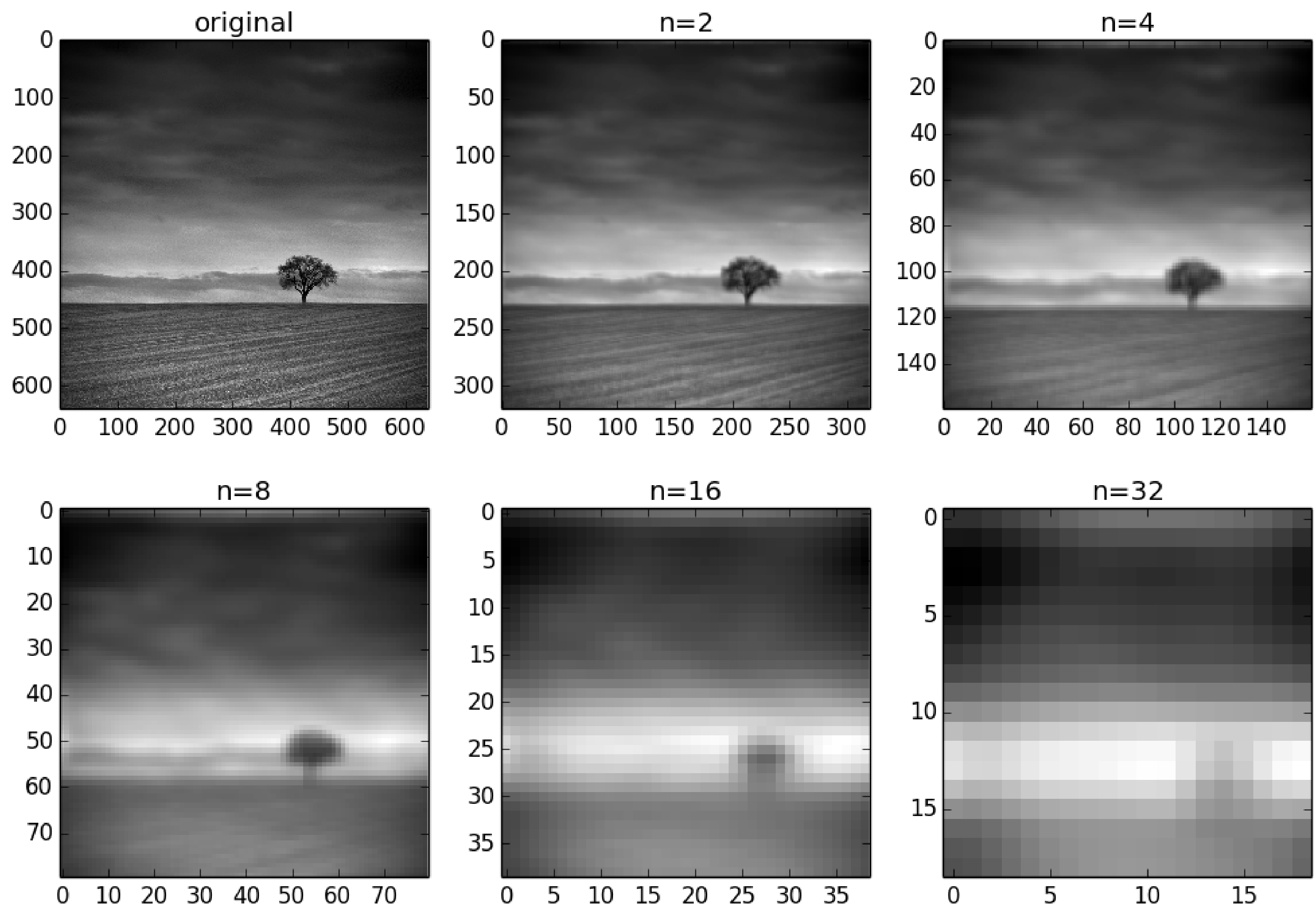


Don't need high spatial resolution

Need high spatial resolution

Multiresolution analysis

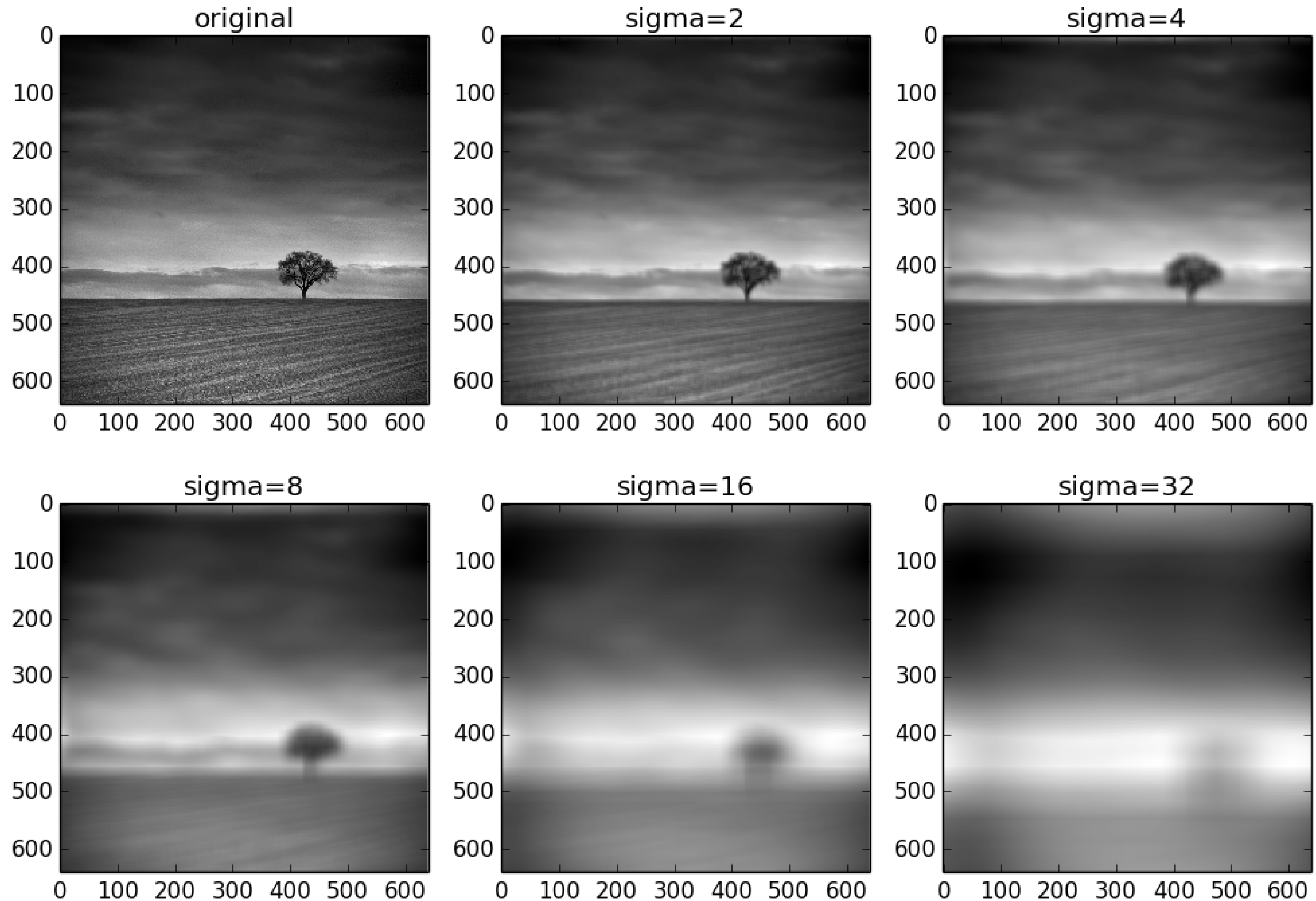
Subsampling (taking every nth pixel) successively reduces high frequency content



power concentrated in low spatial frequencies → can afford fewer samples

Multiresolution analysis

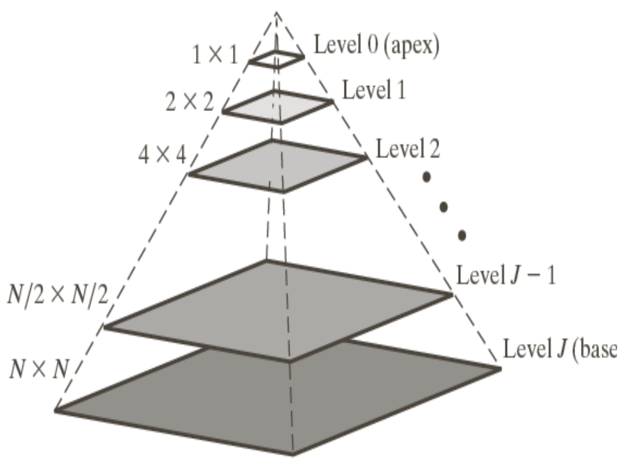
Multiple filtering with Gaussian filters, sigma determines resolution



Pyramid representation

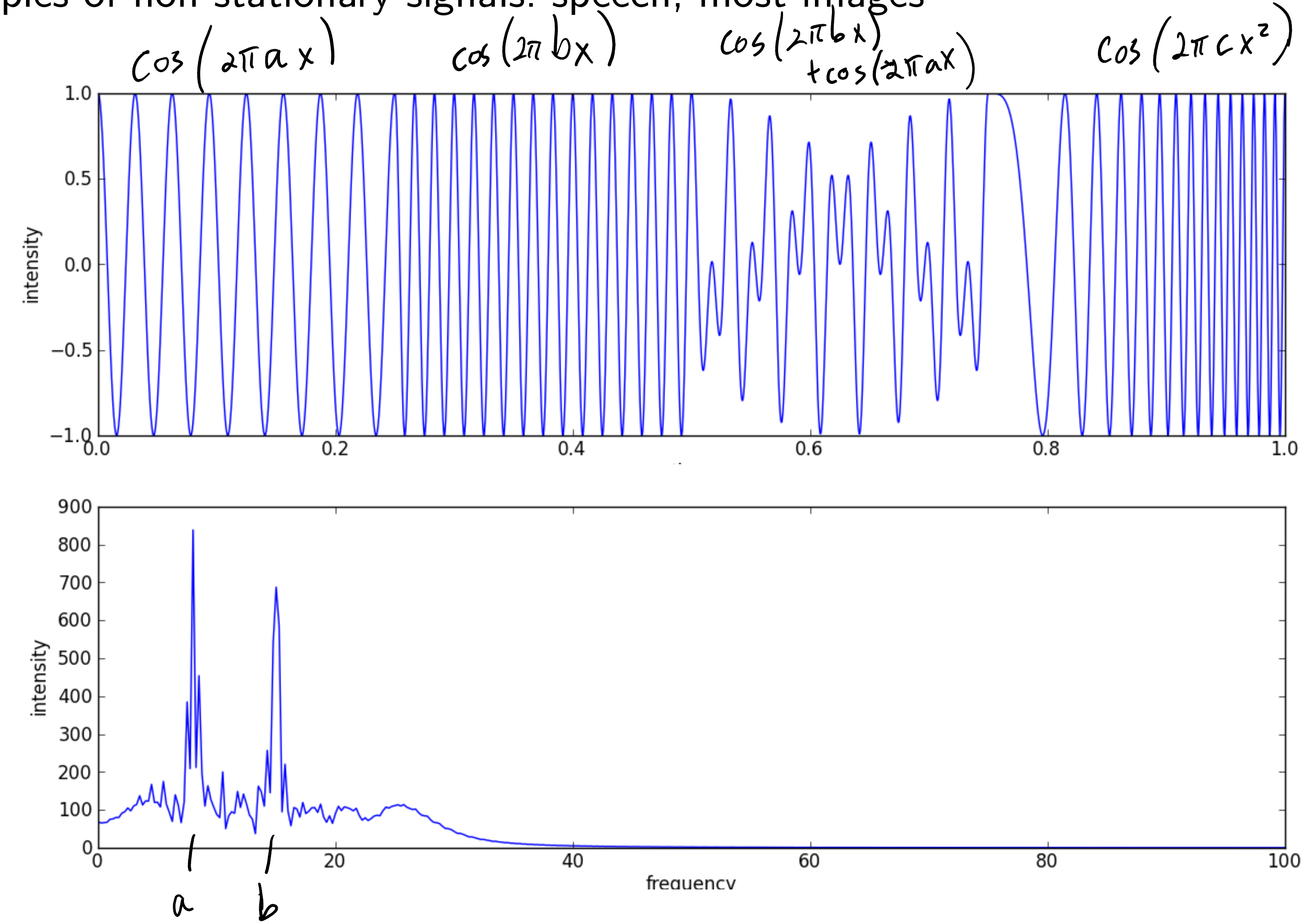
Scale-space representation, pyramidal representation

commonly used in imaging for feature extraction



Stationary vs. non-stationary signals

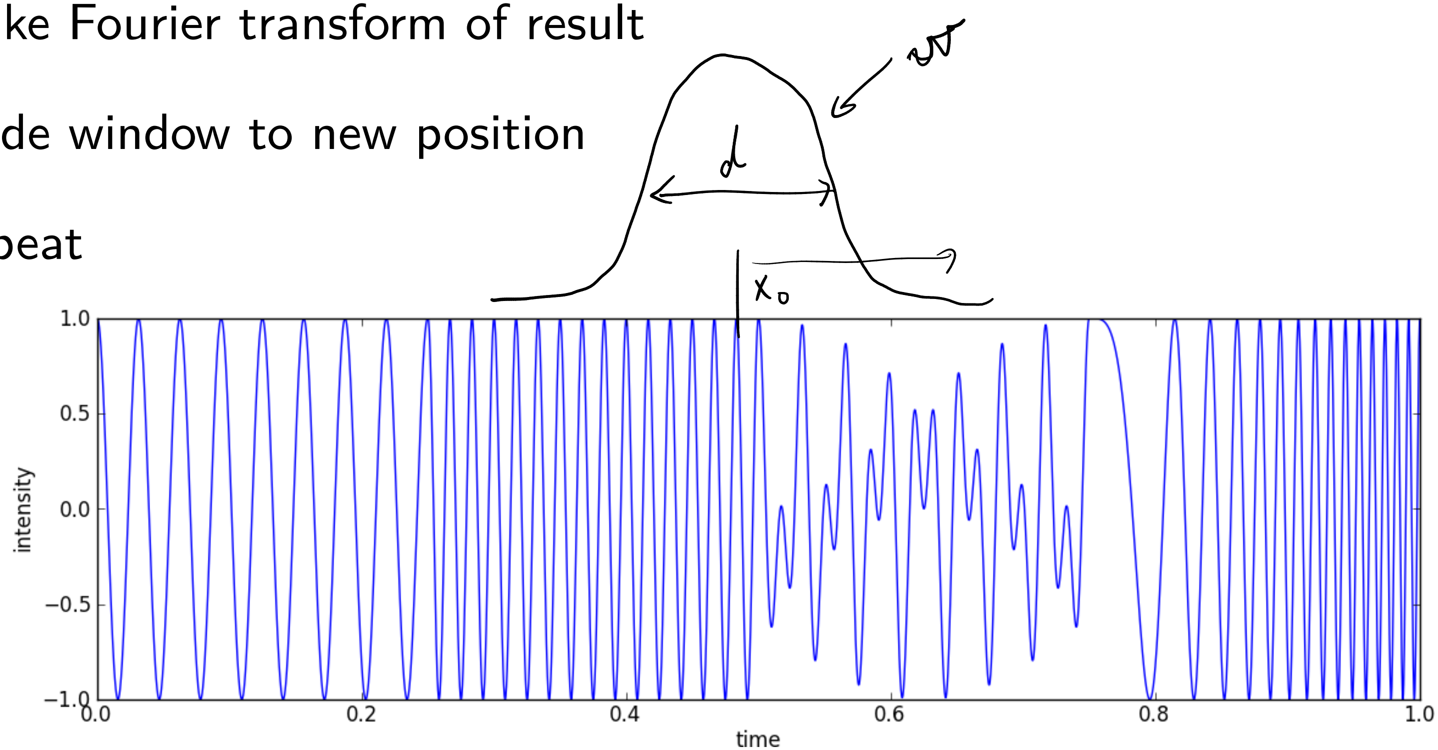
- Stationary signals: frequency doesn't change over time (spatially over the image)
- Non-stationary signals: frequency changes over time (spatially over the image)
- Examples of non-stationary signals: speech, most images



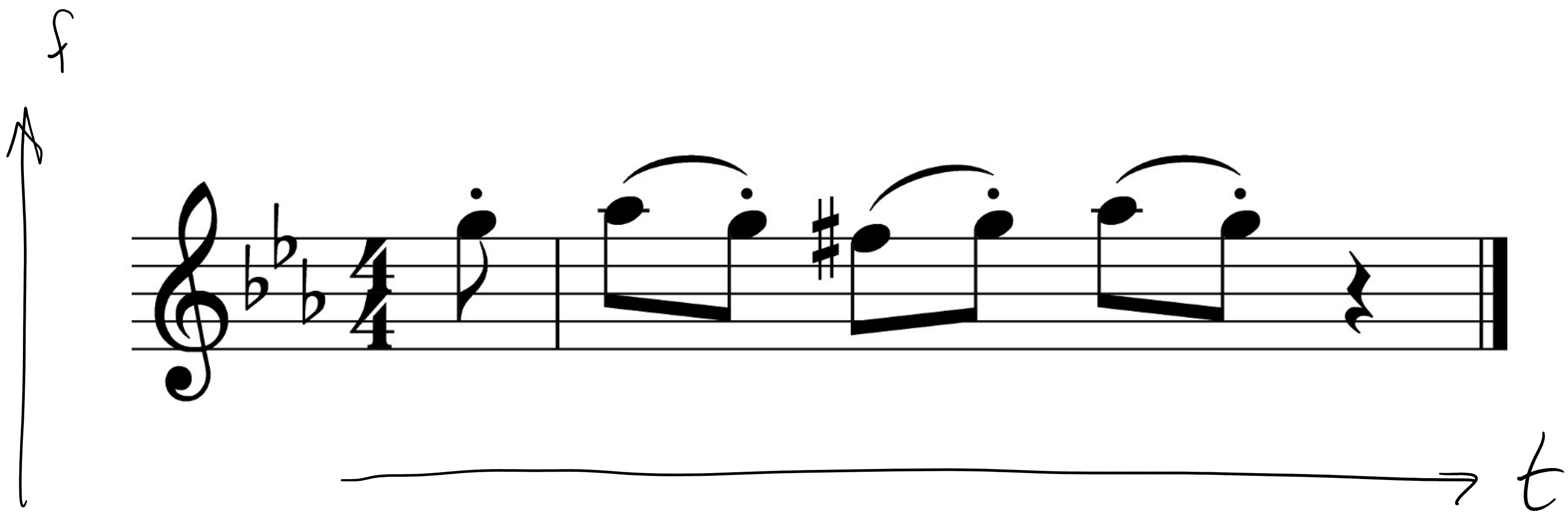
FT insufficient to localize the frequencies in our signal (image)

Windowed Fourier transform

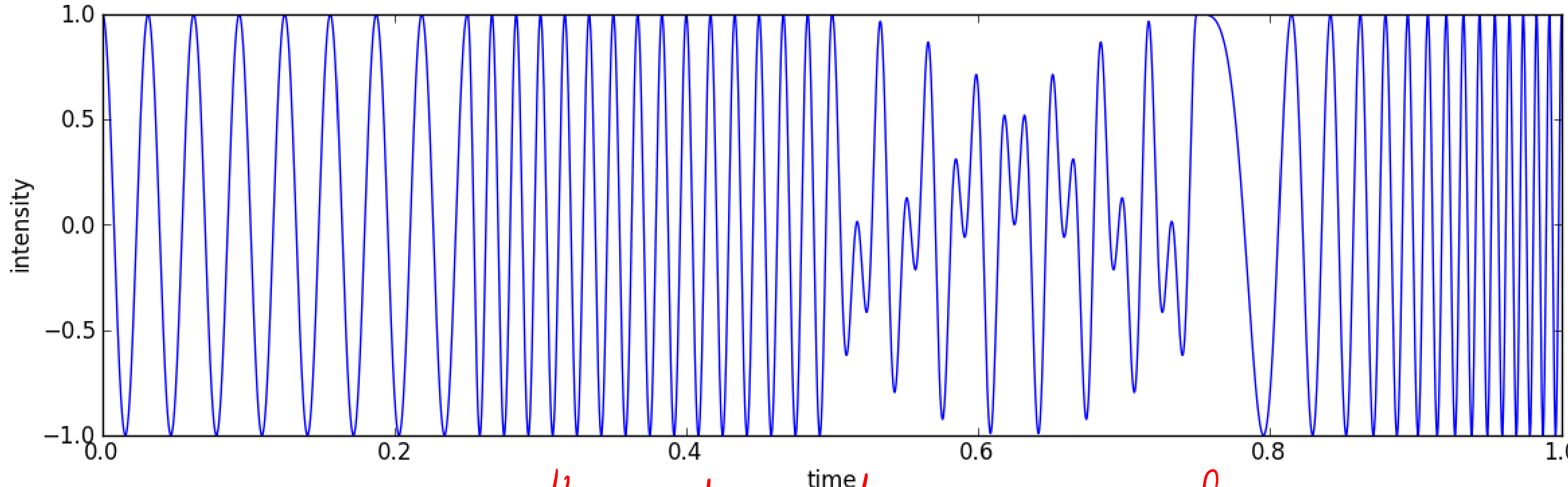
- Windowed Fourier transform is part of the field of "time-frequency analysis"
- Also known as Short-time Fourier Transform (STFT)
- Time-frequency representations are used in many different contexts (Audio, image processing/optics, quantum mechanics)
- Idea: slice up signal into small parts, analyze each separately
  - Multiply with window function w (of width d) at position x0
  - Take Fourier transform of result
  - Slide window to new position
  - repeat



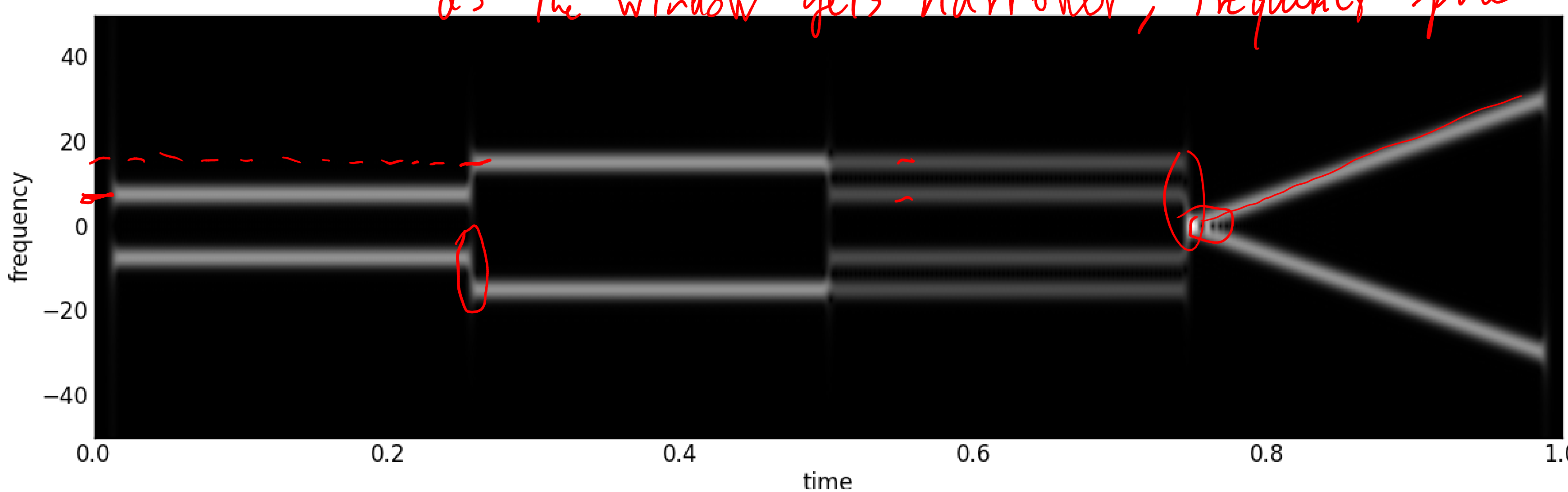
Analogy to audio signals



Spectrogram



as the window gets narrower, frequency space signal gets blurrier

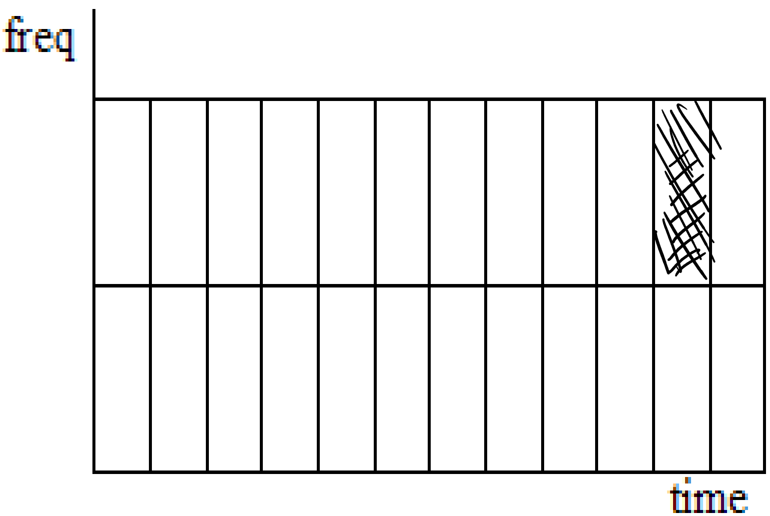


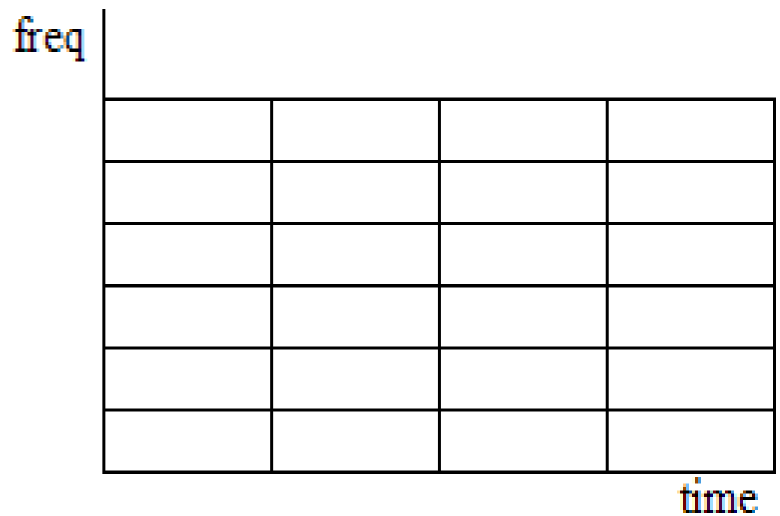
Uncertainty relation

$$\sigma_x \, \sigma_f \geq \frac{1}{4\pi}$$

better frequency resolution $\longrightarrow$ broader window

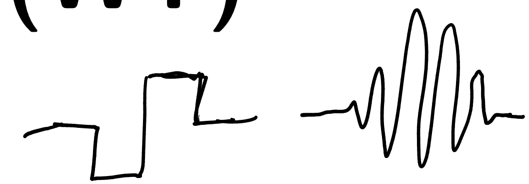
- Finite area in the time-frequency plane





- This is limitation of WFT and hence development of **wavelets**

Continuous wavelet transform (WT)



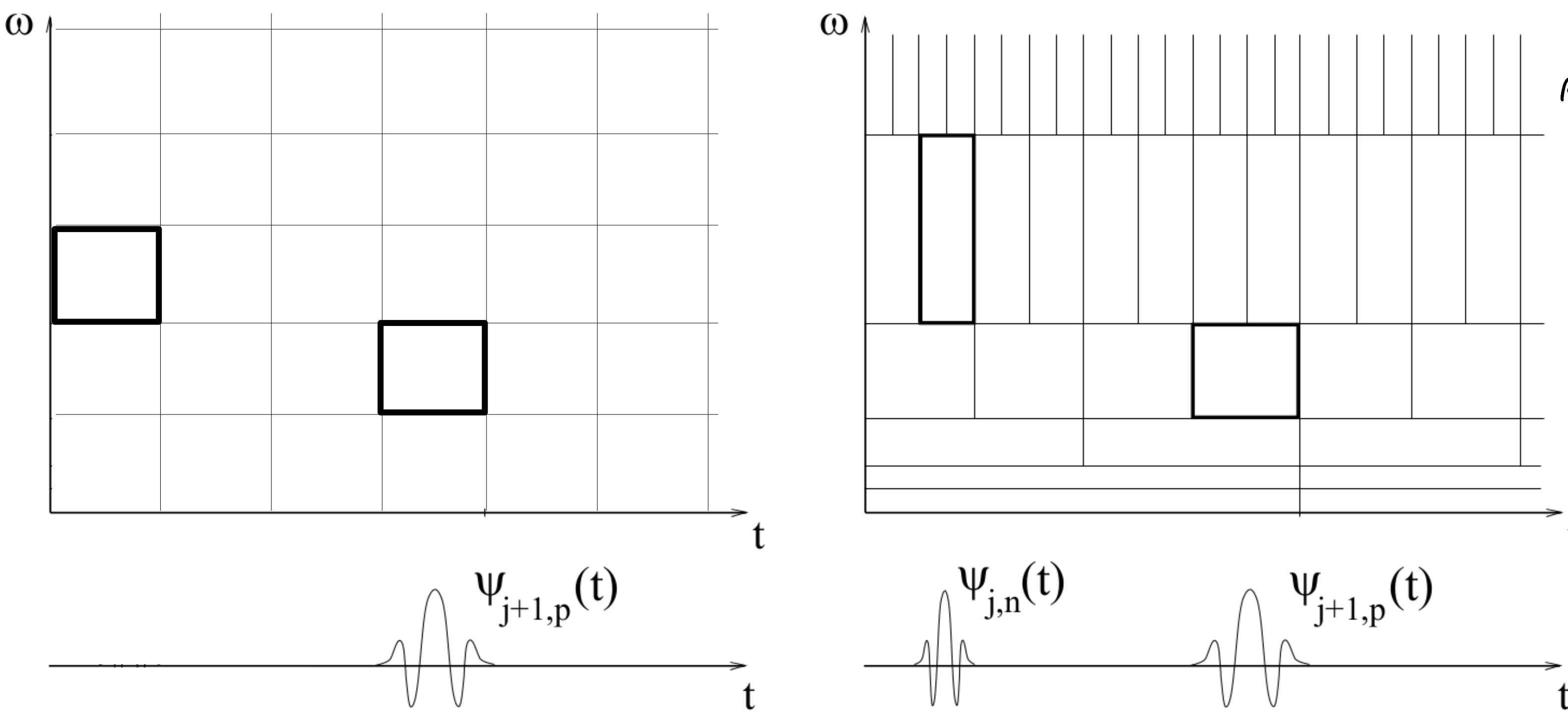
- Parameters: translation and scaling

$$WT(s, x_0) = \int_{-\infty}^{\infty} f(x)\, \psi_{s,x_0}(x)\, dx$$

$$\psi_{s,x_0} = \frac{1}{\sqrt{s}}\, \psi\left(\frac{x - x_0}{s}\right)$$

ψ: mother wavelet

- Analyze signal at <u>different scales</u> instead of <u>different frequencies</u>



better time resolution

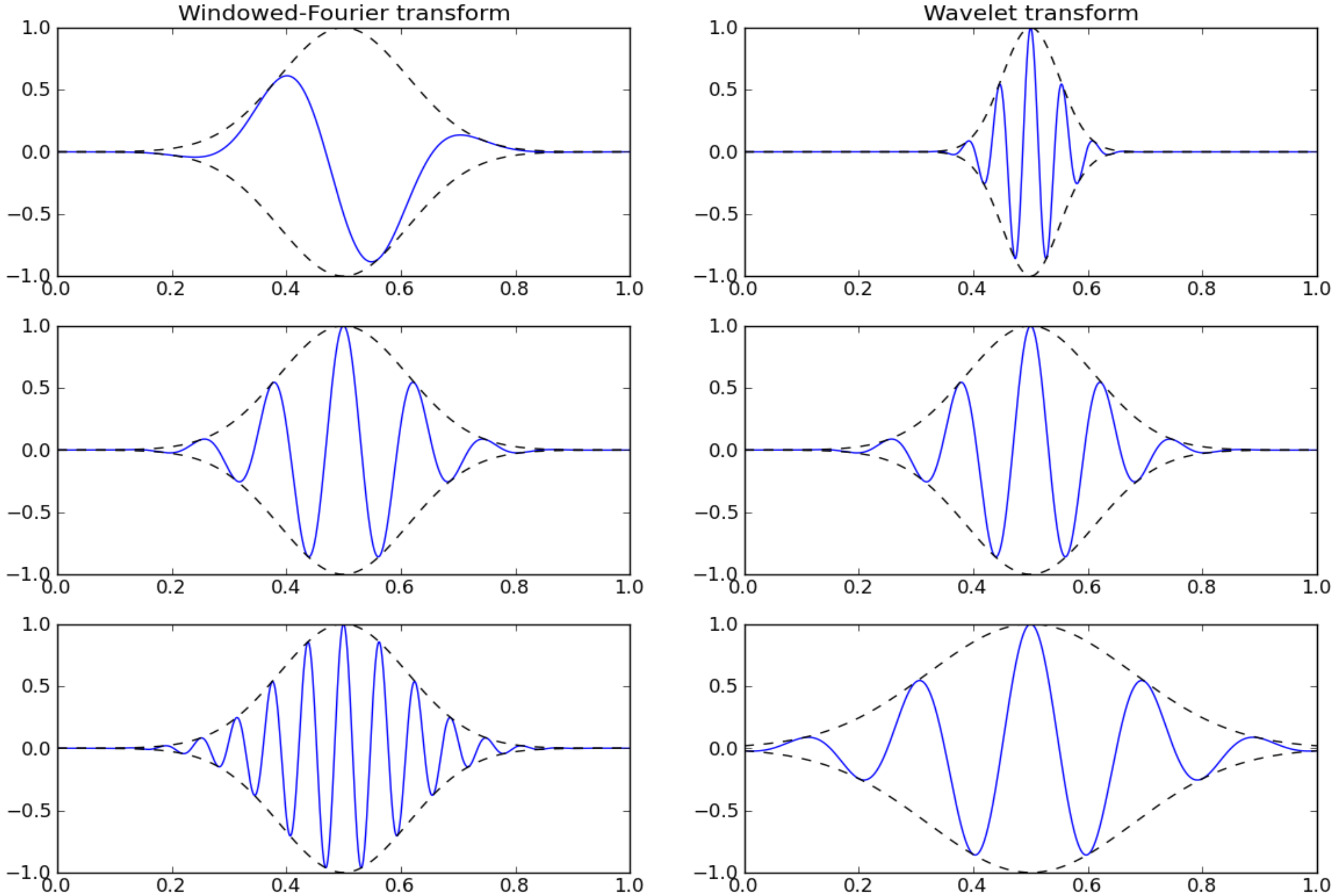
better frequency resolution

Source: Mallat, "A wavelet tour of signal processing"

WFT vs WT

WFT - keep window width constant
- change modulation

Wavelet - keep shape constant
- change scale



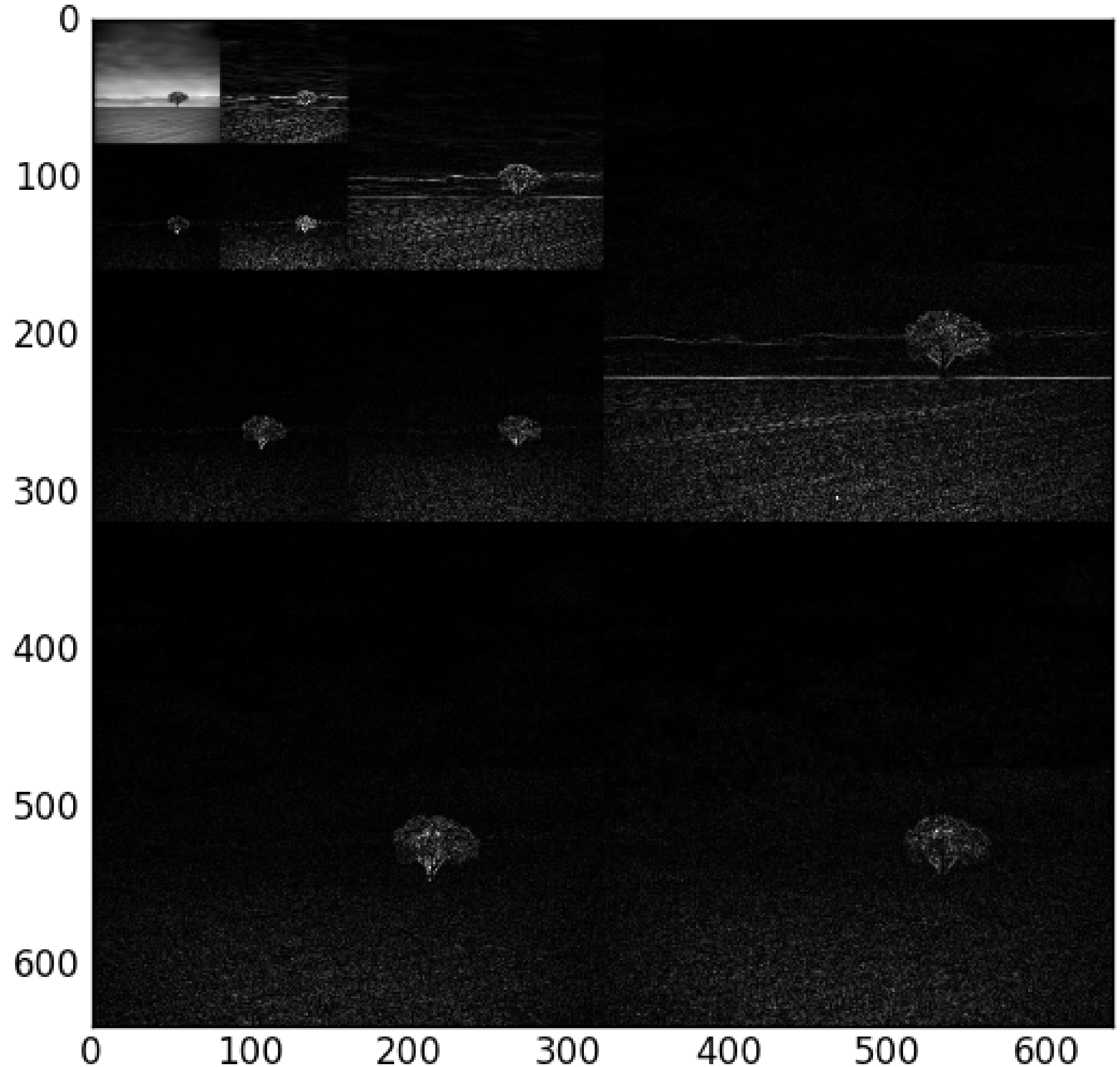
Discrete Wavelet decomposition of image

- Perform each DWT, collect and tile all coefficients

lowest resolution

- Here: 3 level decomposition

JPEG 2000 is based on wavelets



Summary

- Images can be represented by different basis functions.
- Fourier basis: localized in frequency, delocalized in real space.
- Windowed Fourier Transform: localized – to some extent – in both spaces
- Wavelet analysis decomposes a signal in position and scale (instead of position and frequency as for WFT).
- Sparse representations are representations in which the image content is represented by a few relevant coefficients, while the other pixels are close to zero
- Sparse representations have advantages for compression, denoising, ...