Sottocategorie $\boxed{\mathcal{A} \subseteq \mathcal{B}}$

$\mathrm{Ogg}\,\mathcal{A} \subseteq \mathrm{Ogg}\,\mathcal{B}$

$\mathrm{Morf}\,\mathcal{A} \subseteq \mathrm{Morf}\,\mathcal{B}$

<u>Def</u> Una sottocategoria si dice **PIENA** se

$\forall X, Y \in \mathcal{A}$ $\boxed{\mathcal{A}(X,Y) = \mathcal{B}(X,Y)}$

$$\mathrm{Ab} \overset{\text{FULL}}{\subseteq} \mathrm{Grp} \subseteq \underset{\text{cu } 1}{\mathrm{Semig}} \subseteq \mathrm{Semig}$$

<u>piene</u> ?

$\forall X, Y \in \mathrm{Ab} \qquad \mathrm{Ab}(X,Y) = \mathrm{Grp}(X,Y)$

$\forall X, Y \in \mathrm{Grp} \qquad \mathrm{Grp}(X,Y) \underset{\text{ok}}{\overset{?}{=}} \mathrm{Sem}_{\text{cu } 1}(X,Y)$

$X \xrightarrow{f} Y$ $\quad$ X, Y gruppi $(X, \cdot)$ $(Y, \cdot)$

f h. che preserva $\circ$ e 1

$\Downarrow$

f preserva anche gli inversi di X

$$f(x^{-1}) = \underline{(f(x))}^{-1}$$

$$f(\underbrace{x^{-1} \cdot x}) \overset{?}{=} 1_Y$$

$$f(1_X) = 1_Y$$

$$f(x^{-1}) \circ f(x) = 1_Y$$

NON PIENA

Semp con 1 $\subseteq$ Semp.

$X, Y \in$ Semp con 1

$X \xrightarrow{f} Y$

f omom di semp

1'

f preserva l'identità

$$\underline{(\mathcal{P}(A), \cup)} \xrightarrow{f} (\mathcal{P}(A), \cup)$$

$1 = \emptyset$

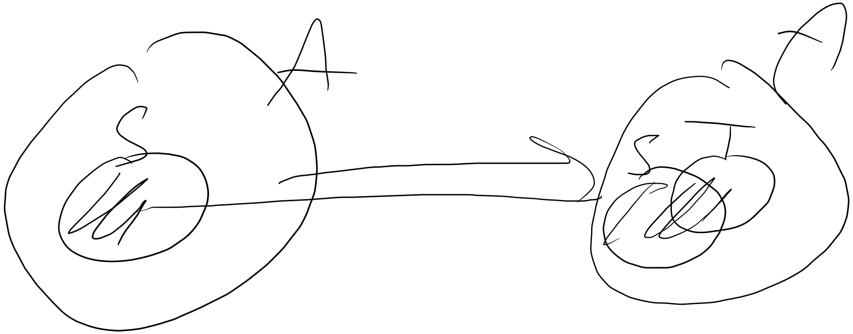
$\boxed{\text{$f$ preserva $\cup$ } \quad \text{non preserva l'identità}}$

$f(S) = S \cup T$

$T \in \mathcal{P}(A)$

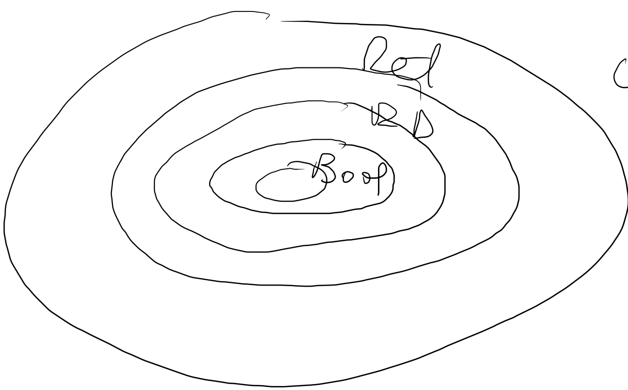
$T \neq \emptyset$

T fissato



$$\mathrm{Ab} \overset{\text{plin}}{\subseteq} \mathrm{Grp} \overset{\text{plin}}{\subseteq} \underset{\text{cu } 1}{\mathrm{Semig}} \overset{\text{nu plin}}{\subseteq} \mathrm{Semig}$$

$$\mathrm{Bool} \subseteq \underset{\text{cu } 0,1}{\mathrm{Latdistr}} \subseteq \underset{\text{distr}}{\mathrm{Lat}} \subseteq \mathrm{Lat} \overset{?}{\subseteq} \underset{\text{ordine}}{\text{Mulțim}}$$



ord

$$\text{Rel} \overset{?}{\subseteq} \text{Pos. con valid}$$

$$X, Y \in \text{Rel}$$

$$\text{Rel}(X, Y) \overset{?}{=} \text{Mons cnd}(X, Y)$$

$$X \xrightarrow{\circ} Y \qquad\qquad X \xrightarrow{\circ} Y$$

Pos. $\vee\, e\, \wedge$ Pos $\leq$

$X, Y \in \mathrm{Retcul}$

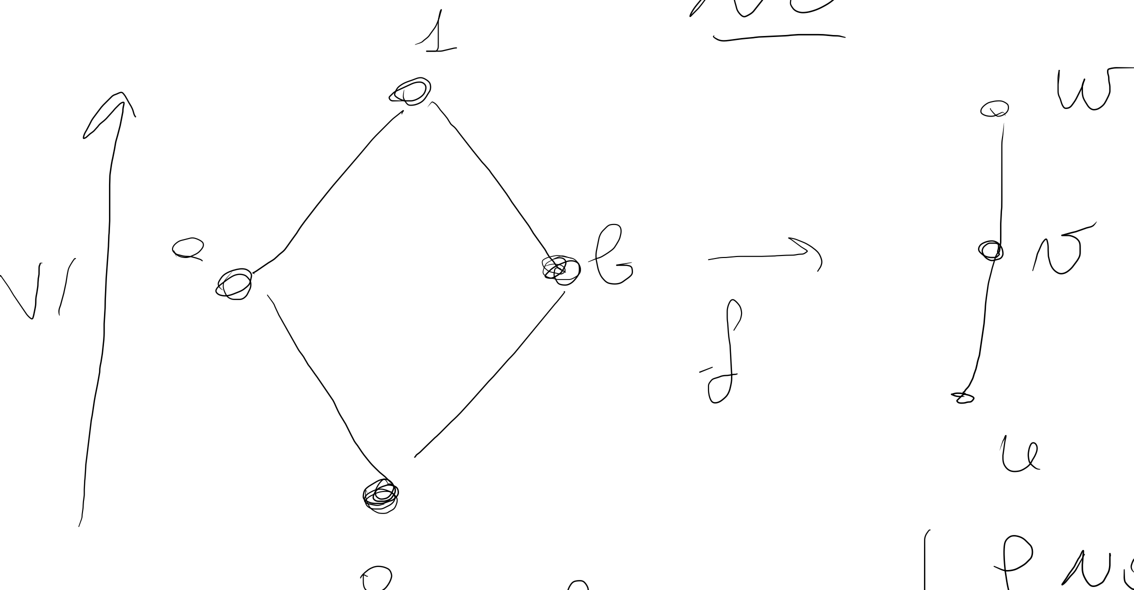
$X \xrightarrow{f} Y \qquad f \in \mathrm{Ord}$

ma f preserva $\vee$ e $\wedge$

$\boxed{\mathrm{Ret} \subseteq \mathrm{Ord}}$ è piena o meno

f preserva $\leq \overset{?}{\Rightarrow} f$ pres. $\vee$ e $\wedge$

NO



$f(1) = w$

$f(a) = f(b) = v$

$f(0) = u$

f pres $\leq$ | f NON pres. $\vee$ e $\wedge$

$a \vee b = 1$ $\quad f(a) \vee f(b) = \nu \neq$ mondo

$$\mathrm{Bool} \overset{\text{PIENA}}{\subseteq} \underset{\substack{\text{DISTR}\\ \text{con } 0,1}}{\mathrm{Ret}} \overset{\substack{\text{NON}\\ \text{PIENA}}}{\subseteq} \underset{\text{DISTR}}{\mathrm{Ret}} \overset{\text{PIENA}}{\subseteq} \mathrm{Ret} \overset{\boxed{\text{NON piena}}}{\subseteq} \mathrm{Ord}$$

$X, Y \in$ Ret distrib.

$f : X \to Y$ $\quad f$ morf di reticoli

$f \in \mathrm{Morf}\,\mathrm{Ret}_{\mathrm{DIST}}$

B, B' alg Boole

$f : B \longrightarrow B'$

f preserva $\vee, \wedge$

f preserva 0 e 1

$\Downarrow$?

SI

f è morf di alg Boole

preserva anche i complementi

x $\neg x$ complemento def. da

$x \vee \neg x = 1$ $x \wedge \neg x = 0$

$f : B \to B'$

x $\neg x$ compl. di x in B

allora $f(\neg x) = \neg(f(x))$ in B'

Sottocat

non piena

$$\boxed{Set \subseteq Rel}$$

Rel — opp — unione

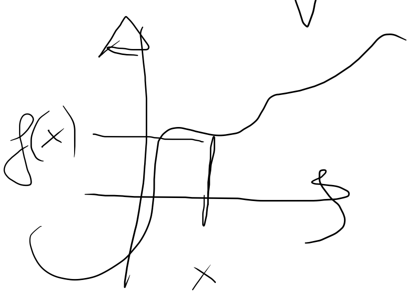
Rel — inv — relaz.

$$X \xrightarrow{R} Y \qquad R \subseteq X \times Y$$

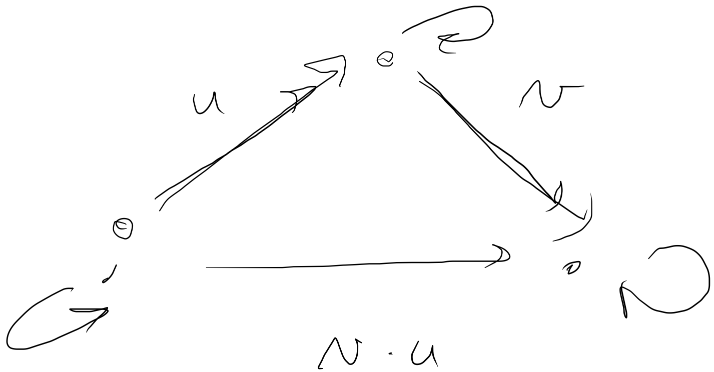
$\mathrm{Set}(X, Y) \subsetneq \mathrm{Rel}(X, Y)$

$\{ f : X \to Y \}$ $\qquad$ $\{ R \text{ relazione } X \text{ ed } Y \}$

ogni funzione è una relazione



grafi come categorie

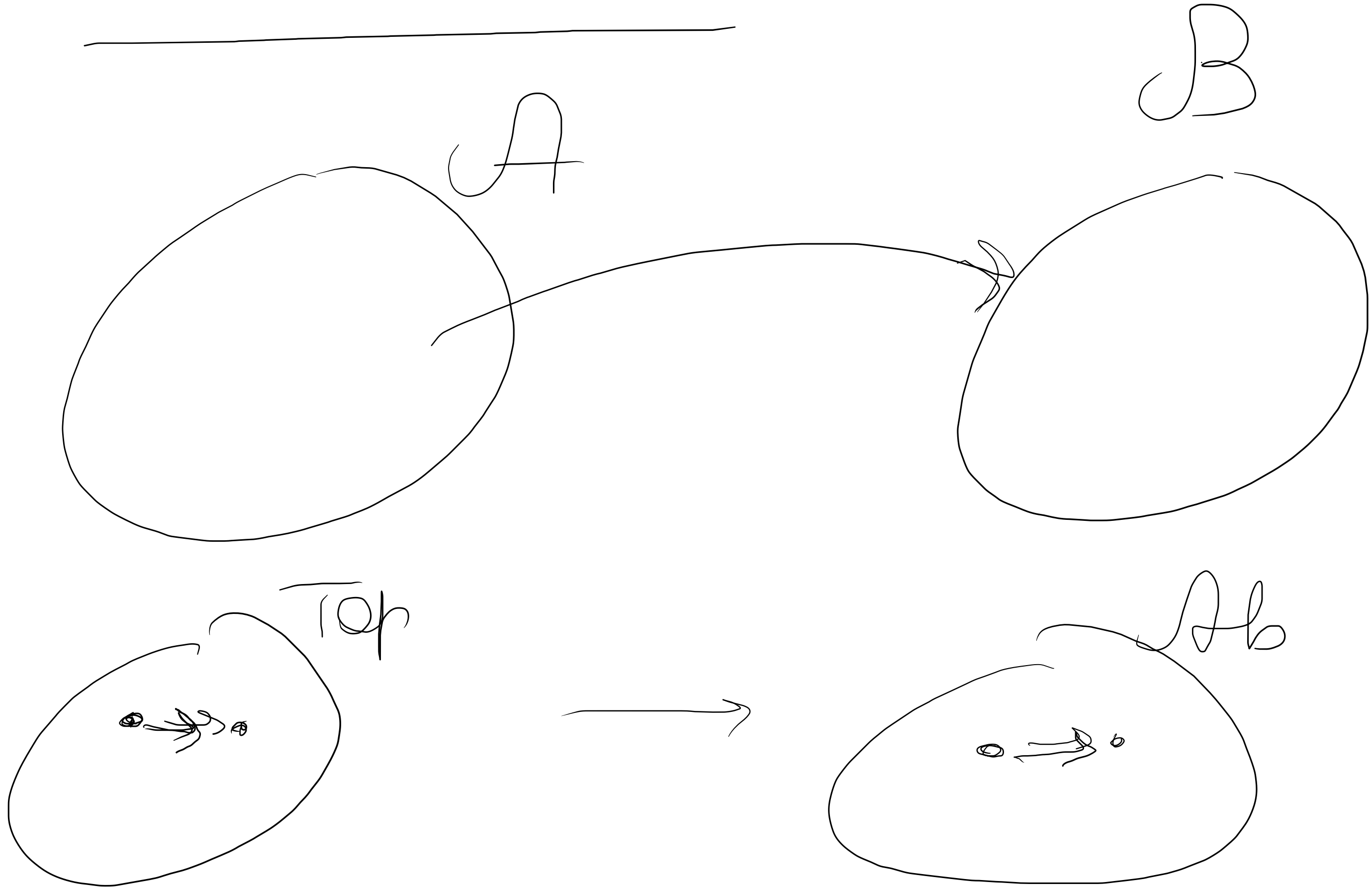


sottocat

$\supseteq$

non sottoc.

FUNTORE



<u>Def</u> $F : A \to B$ <u>funtore</u>

F manda ogg di A in ogg di B

F manda morf di A in morfismi di B

F preserva:

- DOMINI e CODOMINI
- COMPOSTO DI MORFISMI
- IDENTITÀ

$$X \xrightarrow{f} Y \quad \overset{F}{\longmapsto} \quad FX \xrightarrow{Ff} FY$$

in $\mathcal{A}$

F pres. domini e codomini

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & & & & FY \arrow[dr, "Fg"] & \\
X \arrow[ur, "f"] \arrow[rr, "gf"'] & & T \arrow[r, "F"] & {} & FX \arrow[ur, "Ff"] \arrow[rr, "Fg \cdot Ff"] \arrow[rr, bend right, phantom, "\shortparallel"] \arrow[rr, bend right=40, "F(gf)"'] & & FT
\end{tikzcd}$$

$$\mathcal{A} \longrightarrow \mathcal{B}$$

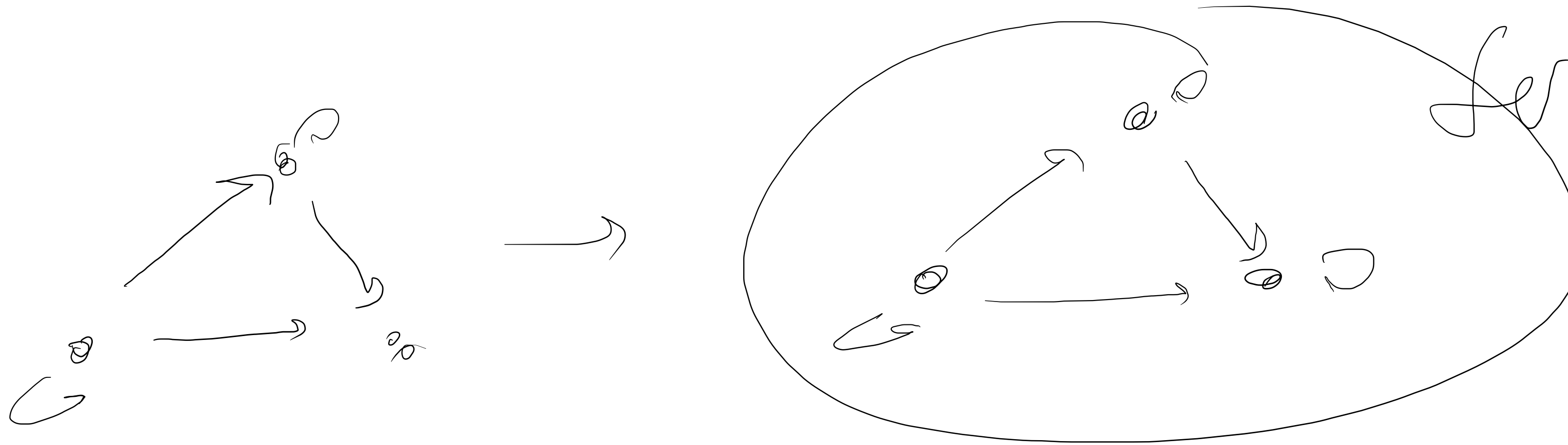
$$\boxed{F(g \circ f) = F(g) \circ F(f)}$$

$\forall\, f, g$ componibile

$\forall\, X \in \mathcal{A} \qquad 1_X : X \to X$ in $\mathrm{Mor}\,\mathcal{A}$

$F(1_X) : FX \longrightarrow FX$ in $\mathcal{B}$

$$\boxed{1_{FX} = F(1_X)}$$



$$X \xrightarrow{F} \mathrm{Set}$$

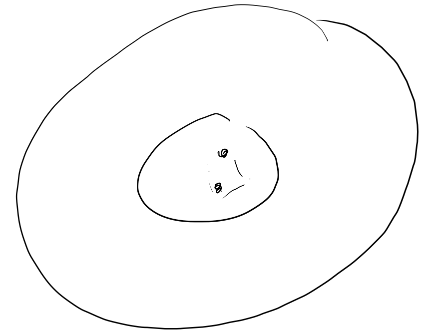
Esempi

funtori DIMENTICANTI

$$\mathrm{Grp} \xrightarrow{U} \mathrm{Set}$$

$$\begin{array}{ccc} G & & |G| \\ \text{omom} \downarrow & & \downarrow |f| \quad \text{funzione} \\ H & & |H| \end{array}$$

$$\mathrm{Vect}_K \xrightarrow{U} \mathrm{Set}$$



$$\mathrm{Top} \xrightarrow{U} \mathrm{Set}$$

$$\mathrm{Grp} \;\cancel{\cong}\; \mathrm{Set}$$

$$\mathrm{Grp} \xrightarrow{U} \mathrm{Set}$$

$$\mathbb{Z}_4 \longrightarrow |\mathbb{Z}_4|$$

$$\not\cong \qquad\qquad\qquad \| \qquad\qquad 4 \text{ elementi}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow |\mathbb{Z}_2 \times \mathbb{Z}_2|$$

$$Ab \xrightarrow{E} Grp$$ funtore inclusione

$\forall \quad A \subseteq B$ sottocategoria

abbiamo il funtore inclusione

$E : A \longrightarrow B$

FUNTORI LIBERI

$$\begin{tikzcd} \mathrm{Smon} \arrow[r, shift right, "U"'] & \mathrm{Set} \arrow[l, shift right, "?F"'] \end{tikzcd}$$

semig. $F(X) \longleftarrow X$ insieme

$X \qquad x_1, x_2\, x_3 \in X$

$x_1 \cdot x_3 \cdot x_3$

$\left\{ \text{parole finite di elementi di } X \right\} = \coprod_{n \in \mathbb{N}^+} X^n$

$\coprod X^n = F(X)$ è un semig.

$(x_1 x_2 x_3) * (x_7 x_6) = x_1 x_2 x_3 x_7 x_6$

è ASSOC.

$$\mathrm{Set} \xrightarrow{F} \mathrm{Semig}.$$

$$\begin{array}{ccc} X & & \coprod X^n = F(X) \\ f \big\downarrow & & \big\downarrow F(f) \\ Y & & \coprod Y^m = F(Y) \end{array}$$

$$F(f)\,(x_1 x_2 x_3 x_4) \stackrel{\text{def}}{=} f(x_1) f(x_2) f(x_3) f(x_4)$$

è un FUNTORE

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, shift right, "U"'] & \mathrm{Set} \arrow[l, shift right, "F"'] \end{tikzcd}$$

$\underline{X \in \mathrm{Set}}$ $\qquad\qquad$ $F(X)$

$X \cup \overline{X}$ $\qquad$ parole finite su $X \cup \overline{X}$

copia di X

$x_1 x_2 \overline{x_3}\, \overline{x_4}\, x_5$

Si presenta l'insieme di frasi rispetto ad una relaz. di equiv. generata da

$$x_1 \bar{x}_1 x_2 \equiv x_2$$

$\forall G \quad \exists$ gruppo libero generato da G

$$\mathrm{Grp} \underset{U}{\overset{F}{\leftrightarrows}} \mathrm{Set}$$

sono funtori <u>AGGIUNTI</u>

$$\boxed{\mathrm{Vett}_K \underset{U}{\overset{F}{\leftrightarrows}} \mathrm{Set}}$$

ex funtori

- dimenticare
- inclusione
- funtori liberi
- funtori identità

$$\begin{tikzcd} \mathrm{Top}_2 \arrow[r, "E"'] & \mathrm{Top} \arrow[l, bend right, "F"'] \end{tikzcd}$$

fun. inclusione

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f \text{ cont.}"'] & X/\sim \arrow[d, "\bar f"] \\ Y & Y/\sim \end{tikzcd}$$

$$\begin{tikzcd} Top \arrow[r, "U"] & Set \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"] \end{tikzcd}$$

I top. indiscreto

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[r] & D(X) = (X, \tau \text{ discreto}) \arrow[d, dashed, "f"] \\ Y & D(Y) \end{tikzcd}$$

f diventa continua

$$\begin{tikzcd} \mathrm{Top\ Conn\ T_2} \arrow[r, "E"'] & \mathrm{Top} \arrow[l, bend right, "?\ S.C."'] \end{tikzcd}$$

funtori AGGIUNTI

vale una prob. UNIVERSALE