> category
> functor
> natural transformation

Set $\mathcal{C}$ — obj $\mathcal{C}$ — X, Y, T, V
— morphisms

$f : X \longrightarrow Y$

DOMAIN (of X)

CODOMAIN (of Y)

arrows can be composed

$X \xrightarrow{f} Y \xrightarrow{g} Z$ $\quad$ $\mathrm{cod}\, f = \mathrm{Dom}\, g$

composite $\boxed{gf : X \longrightarrow Z}$

AXIOMS

1) composition is associative

2) $\forall X,\ X \in \mathrm{Ob}\,\mathcal{C}$ $\quad$ $1_X : X \longrightarrow X$

identity

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, "f"] & Y \arrow[loop right, "1_Y"] \end{tikzcd}$$

$f \cdot 1_X = f$

$1_Y \cdot f = f$

Concrete categories

$\boxed{\text{obj}}$: sets + structure

$\boxed{\text{morph}}$ - functions that pres. the structure

$$X \xrightarrow{f} Y$$

Abstract cat

$(X, \leq)$ preorder

$x \leq x$

$x \leq y \quad y \leq z \rightarrow x \leq z$

$(X, \le)$ as a category

$x, y \in X$

$x \xrightarrow{\le} y$

$$\underline{X(x, y)} = \{ f : x \to y \}$$

$\{*\}$ $x \le y$

$\emptyset$ $x \not\le y$

obj $X = X$

morphism

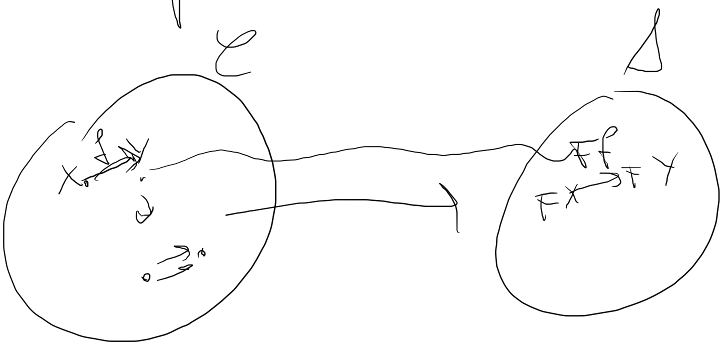
$x \xrightarrow{\le} y$

$1_x \in t$

FUNCTOR

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\mathrm{Top} \longrightarrow \mathrm{Ab}$$



Def F functor

- F sends objects of $\mathcal{C}$ to ob of $\mathcal{D}$
- F sends morph. $\mathcal{C}$ to morph $\mathcal{D}$
- F preserves
  - domain
  - codomain
  - composition
  - identity

$$\underbrace{\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & T \end{tikzcd}}_{\mathcal{C}} \quad \overset{}{\underset{F}{\longmapsto}} \quad \begin{tikzcd} FX \arrow[r, "Ff"] \arrow[rr, bend right, "F(gf)"'] & FY \arrow[r, "Fg"] & FT \end{tikzcd}$$

$$F(gf) = Fg \cdot Ff$$

F preserves compositions:

$$\boxed{F(g \cdot f) = Fg \cdot Ff} \quad \forall f, g \text{ composable}$$

$$X \xrightarrow[1_X]{} X \qquad F(1_X) : FX \to FX \qquad \boxed{F(1_X) = 1_{FX}}$$

Examples

> embedding $\qquad Ab \subseteq Grp$

$$Ab \xrightarrow{E} Grp$$

> $\forall \quad \mathcal{A} \subseteq \mathcal{B}$ $\qquad$ subcategory

$$\mathcal{A} \xrightarrow{E} \mathcal{B}$$

> FORGETFUL $\qquad Grp \xrightarrow{U} Set$

$$\begin{array}{ccc} G & \longmapsto & UG = |G| \\ \text{group hom. } f \downarrow & & \downarrow Uf \text{ function} \\ H & & UH \end{array}$$

- $\mathbf{Top} \xrightarrow{U} \mathbf{Set}$
- $\mathbf{Vett}_k \xrightarrow{U} \mathbf{Set}$
- $\mathbf{Ab} \underset{E}{\overset{F}{\leftrightarrows}} \mathbf{Grp}.$

$$\boxed{\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}}$$

$(G:)$

$G \longmapsto F(G) \in \mathbf{Ab}$

subgroup of COMMUTATORS

$\langle \underline{xyx^{-1}y^{-1}},\ x, y \in G \rangle$

$\boxed{G / \langle xyx^{-1}y^{-1} \rangle}$ is <u>ABELIAN</u>

$$\begin{array}{ccc} G & & G/\langle xyx^{-1}y^{-1}\rangle = F(G) \\ f \downarrow \ \text{omom} & & \downarrow F(f) \quad \text{omom} \\ H & & H/\langle uvu^{-1}v^{-1}\rangle = F(H \end{array}$$

$$\underline{\underline{Gr}} \xrightarrow{\quad F \quad} \underline{\underline{Ab}}$$

$$F(f)([g]) = f[g] = [\underline{\underline{f(g)}}]$$

NOT a functor

$$\mathrm{Grp} \xrightarrow{F} \mathrm{Grp}$$

$$\begin{array}{ccc}
(G, \cdot) & \longrightarrow & F(G) = \text{Centre of } G \\
 & & C_G = \{ x \in G : \forall g \in G \;\; x \cdot g = g x \} \\
\Big\downarrow f \text{ hom} & & \cancel{\Big\downarrow f|_C} \\
(H, \cdot) & \longrightarrow & F(H) = C_H
\end{array}$$

> free functors

$$\mathrm{Set} \xrightarrow{F} \mathrm{Smgr}$$

$$X \longrightarrow F(X)$$ free semigroup

$$\coprod_{n \in \mathbb{N}^+} X^n = \left\{ (x_1, x_2, x_3 \cdots x_n) \cdots \right\}$$

$$\begin{tikzcd}
X \arrow[d, "f"'] & F(X) \arrow[d, "F(f)"] \\
Y & F(Y)
\end{tikzcd}$$

omom

$$(x_1, x_2, x_3) \leadsto f(x_1) f(x_2) f(x_3)$$

> Set $\xrightarrow{F}$ Group

> $\boxed{\text{Set} \xrightarrow{F} \text{Vett}_K}$

X $\qquad F(X) = V_X$

vector space generated by the base X

V_X

$v = \underbrace{\boxed{\sum \lambda_i x_i}}_{\text{finite}}$ $\qquad \lambda_i \in K$, $x_i \in X$

$\boxed{\begin{array}{l} \sum \lambda_i x_i \\ \text{FORMAL WORDS} \end{array}}$

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$

Representable Functors

$\mathcal{C}$ $\qquad X \in \mathcal{C}$ object

$$\boxed{\mathcal{C}(X,-) : \mathcal{C} \longrightarrow \mathrm{Set}}$$

$$\begin{array}{ccc}
A & \qquad & \mathcal{C}(X,A) = \{ f : X \to A \} \\
u \downarrow & & \downarrow \mathcal{C}(X,u) \\
B & & \mathcal{C}(X,B) = \{ h : X \to B \}
\end{array}$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "uf"'] & A \arrow[r, "u"] & B
\end{tikzcd}$$

$$\boxed{\mathcal{C}(X,u)(f) \underset{DEF}{=} u \cdot f}$$

$Grp \longrightarrow Set$ $\qquad\qquad$ $\mathbb{Z} \in Grp$

$(\mathbb{Z}, +)$

$$\boxed{Grp(\mathbb{Z}, -) : Grp \longrightarrow Set}$$

$\xi = f(1) \in G$

$$\{\mathbb{Z} \xrightarrow{f} G,\ f \text{ ow}\}$$

$$\{\mathbb{Z} \to H\ \ g \text{ ow}\}$$

$$\{\mathbb{Z} \to K\ \ t \text{ ow}\}$$

ou

$$\begin{tikzcd}
G \arrow[d, "u"'] \arrow[dd, bend left, "v.u"] \\
H \arrow[d, "v"'] \\
K
\end{tikzcd}$$

$\boxed{\mathrm{Grp}(\mathbb{Z}, -)} \quad \mathrm{Grp} \to \mathrm{Set}$

$U : \mathrm{Grp} \to \mathrm{Set}$

$\mathrm{Grp}(\mathbb{Z}, G) \simeq U(G)$

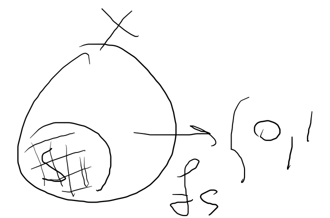
$f \mapsto f(1)$

$$\mathrm{Set}(-,2) : \mathrm{Set} \longrightarrow \mathrm{Set}$$

$$2 = \{0,1\} \in \mathrm{Set}$$

$X \in \mathrm{Set}$

$$\mathrm{Set}(X,2) = \boxed{\{f : X \to 2\} \cong \mathcal{P}(X)}$$



$$\begin{array}{ccc} X & & \mathrm{Set}(X,2) \cong \mathcal{P}(X) \\ f \big\downarrow & & \big\uparrow f^{-1} \\ Y & & \mathrm{Set}(Y,2) \cong \mathcal{P}(Y) \end{array}$$

$$f_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & 2 \\
Y \arrow[ur] &
\end{tikzcd}

$Y \to 2$

Contravariant Functors

$$F : A \longrightarrow B$$

domains & codomains are exchanged

$$\begin{tikzcd}
X \arrow[d, "f"] \arrow[dd, bend right=60, "gf"'] & FX \\
Y \arrow[d, "g"] & FY \arrow[u, "Ff"'] \\
T & FT \arrow[u, "Fg"'] \arrow[uu, bend left=60, "F(gf)"]
\end{tikzcd}$$

preservation of composite map

$$\boxed{F(gf) = Ff \cdot Fg}$$

$$\boxed{F(1_X) = 1_{FX}}$$

$$\mathrm{Top} \xrightarrow{\mathrm{Top}(-,2)} \mathrm{Set}$$

$2 = \{0,1\}$ $\quad 2 \in \mathrm{Top}$

$\{1\}$ is open

contrav. functor

$X \in \mathrm{Top}$ $\qquad \mathrm{Top}(X,2) \cong \mathrm{Open}\,X$

> covariant functors (representables)

> contravariant functors

DUAL CATEGORY

$\mathcal{C}$ $\qquad$ $\mathcal{C}^{op}$ DUAL

obj $\mathcal{C}^{o}$ = obj $\mathcal{C}$

$\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$

$\mathcal{C}^{o}$

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

Set^{op}

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} = \text{function from } Y \to X$$

$\mathcal{C}^{op}$ is a CATEGORY

$F : A \to B$ is a contravariant functor

then

F is a covariant functor

$F : \boxed{A^{o}} \to B$ is

$F : A \to \boxed{B^{o}}$

Duality principle

If a property, theorem, construction ...
is true <u>for any category</u>, then
also the DUAL concept (we change
the directions of arrows) is true
for any category

Ex

$$\begin{tikzcd} & X \\ X \times Y \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl] \\ ? & \\ & Y \arrow[ul] \end{tikzcd}$$

+ univ prop.

Set

Cartesian product

$X \mathring{\cup} Y$ disjoint union.

Functors

Def Full and faithful functors

$F : A \to B$

$\forall x, y \in A$ $\quad \boxed{F(x,y) : A(x,y) \to B(Fx, Fy)}$

$x \xrightarrow{t} y \qquad Fx \xrightarrow{Ft} Fy$

$\boxed{\text{F is faithful iff } \forall x,y \; F(x,y) \text{ is injective}}$

$\boxed{\text{F is full iff} \ldots \; F(x,y) \text{ is surjective}}$

$$\boxed{\mathrm{Ab} \xrightarrow{E} \mathrm{Grp}}$$

FAITH FULL

- $\mathrm{Grp} \xrightarrow{U} \mathrm{Set}$
- $\mathrm{Set} \xrightarrow{F} \mathrm{Smgp}$

$X, Y \in \mathrm{Ab}$

$$\boxed{\mathrm{Ab}(X,Y) \xrightarrow[E_{X,Y}]{} \mathrm{Grp}(EX, EY)}$$

$X \underset{g}{\overset{f}{\rightrightarrows}} Y \rightsquigarrow EX \underset{Eg}{\overset{Ef}{\rightrightarrows}} EY$

$t : EX \to EY \quad t \in \mathrm{Grp}$

$\boxed{\mathrm{Grp} \xrightarrow{U} \mathrm{Set}}$

$\boxed{\text{is FAITHFUL}}$

$G \xrightarrow{f} H \quad \overset{U}{\rightsquigarrow}$ functor $\quad UG \xrightarrow{Uf} UH$

however.

$$\mathrm{Grp}(G,H) \xrightarrow{U_{G,H}} \mathrm{Set}(UG, UH)$$

$$G \overset{g}{\underset{f}{\rightrightarrows}} H \quad \rightsquigarrow \quad UG \overset{|g|}{\underset{|g|}{\rightrightarrows}} UH$$

U is full ??

NO

$$\mathrm{Grp}(G,H) \xrightarrow{U_{G,H}} \mathrm{Set}(UG, UH)$$

f

Ex

ABSTRACT CATEGORIES

category

$(X, \leq)$ $\qquad x \xrightarrow{\leq} y \qquad x, y \in X$

what is a functor?

$$(X, \leq) \xrightarrow{F} (Y, \leq)$$

$$\begin{array}{ccc} x & & Fx \\ \downarrow & \rightsquigarrow & \downarrow F(\leq) \\ y & & Fy \end{array}$$

F functor

if $x \leq y$ in X

then

$Fx \leq F(y)$ in Y

is an order preserving function

$\boxed{\text{abstract category}}$

$(S, \cdot, 1)$ semigroup with identity

it can be seen as a category

$\mathcal{C}$ — obj $\mathcal{C}$ is only one $\{*\}$

Morf $\mathcal{C}$ $\quad \mathcal{C}(*, *) = S$

$x, y \in S$

now

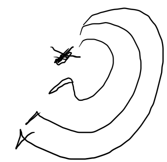
x, y as

morphism

$$(S, \circ, \underline{1}) \xrightarrow{F} (T, \circ, \underline{1})$$ semigroup with 1

F <u>functor</u> is a

<u>semigroup homomorphism</u>



$(X, \le)$