category   functor

Def Natural transformation

$\alpha : F \Rightarrow G$

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{B} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

$\forall A \in \mathcal{A}$

$\boxed{\exists\, \alpha_A : FA \to GA}$

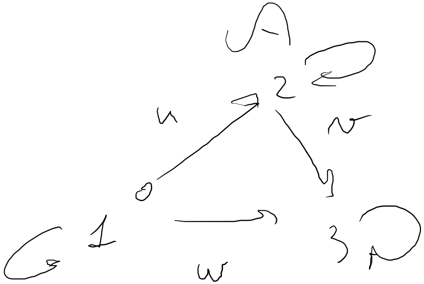
in $\mathcal{B}$

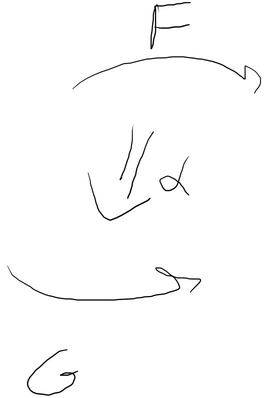
such that

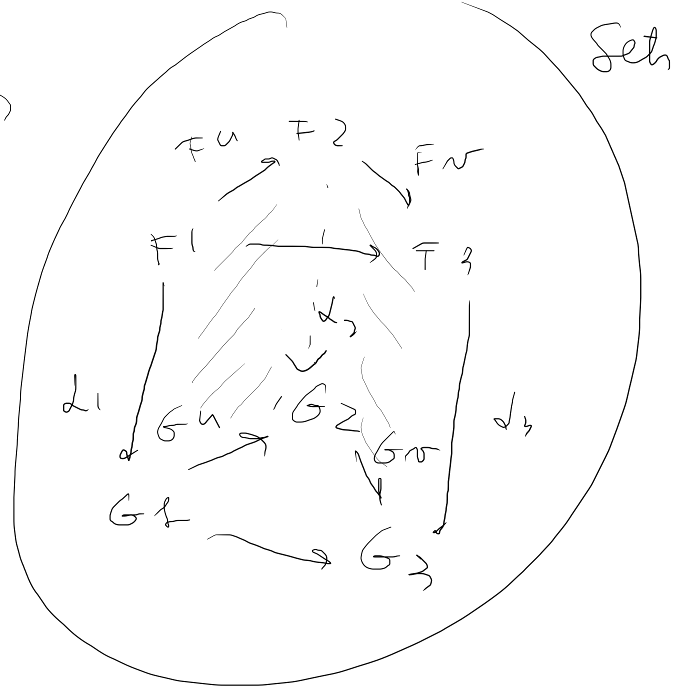
$\boxed{\forall\, u : A \to A'}$

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Fu"'] \arrow[dr, phantom, "\circlearrowleft"] & GA \arrow[d, "Gu"] \\ FA' \arrow[r, "\alpha_{A'}"'] & GA' \end{tikzcd}$$

the diagram must commute







$w = v \cdot u$

Ex of $\boxed{\text{nat transf}}$

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, bend left, "\mathrm{Grp}(\mathbb{Z}{,}-)"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Set} \end{tikzcd}$$

$$\mathrm{Grp}(\mathbb{Z}, G) = \{ f : \mathbb{Z} \to G \text{ hom} \}$$
$$\cong \Downarrow \alpha_G$$
$$U(G) = |G| \quad G \text{ as a set}$$

$$f \longleftrightarrow f(1) = g \in G$$

$\alpha_G \in$ AllMorph Set

$$\boxed{\alpha_G(f) \overset{\mathrm{def}}{=} f(1) \in G}$$

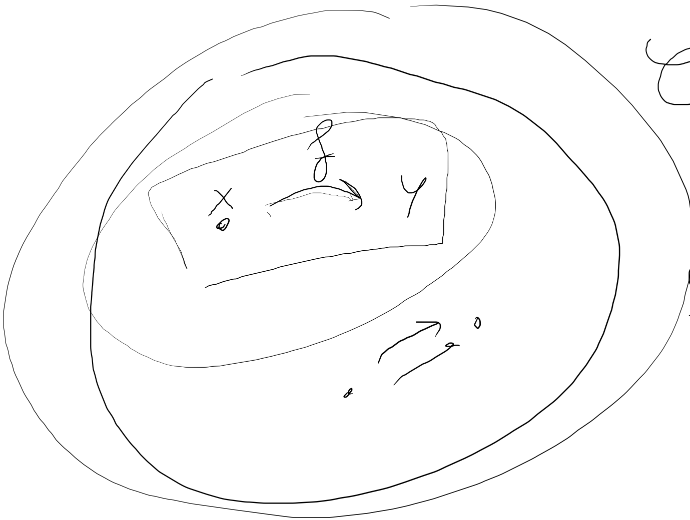
α nat transf.

$u : G \to H$ gp hom

$$\begin{tikzcd} \mathrm{Grp}(\mathbb{Z}, G) \arrow[r, "\alpha_G", "\cong"'] \arrow[d, "\mathrm{Grp}(\mathbb{Z}{,}u)"'] & U(G) \arrow[d, "|u|"] \\ \mathrm{Grp}(\mathbb{Z}, H) \arrow[r, "\cong", "\alpha_H"'] & U(H) \end{tikzcd}$$

$f \mapsto$

properties of objs & arrows in a Category



$\mathcal{C}$

PRODUCTS

colee LIMIT

properties of morphisms

Sets

bijection
injective
surjective

sections

retraction

Def an arrow $f: X \to Y$ in $\mathcal{C}$ is an **ISOMORPHISM** iff $\exists\, g: Y \to X$

s.t. $g \cdot f = 1_X$ $\quad f \cdot g = 1_Y$

$$\begin{tikzcd} X \arrow[loop left, "1_X"] \arrow[r, bend left, "f"] & Y \arrow[loop right, "1_Y"] \arrow[l, bend left, "g = f^{-1}"] \end{tikzcd}$$

$\boxed{\underline{\mathbf{Set}}}$ $\quad$ isom $\equiv$ bijection $\equiv$ inj + surj.

$\boxed{\mathbf{Grp} / \mathbf{Ab} / \mathbf{Vect}_K}$

$$\begin{tikzcd} G \arrow[r, "f"] & H \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

f isom iff f homom. f is a bijection

$$\begin{tikzcd} G \arrow[r, "f"] & H \arrow[l, bend left, dashed, "f^{-1}"] \end{tikzcd}$$

Grp
f homom +
f inj + surj ($\exists\, f^{-1}$ inverse function)

$\Rightarrow$ f^{-1} is again an homom.

$$f^{-1}(h \cdot h') = f^{-1}(h) \cdot f^{-1}(h')$$

f iso = f homom + bijection

Top

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$\boxed{f \text{ cont. st.}}$

$\boxed{f \text{ is an}}$ iff $\boxed{\exists\, g : Y \to X \text{ cont.}}$

st. $gf = 1_X \quad fg = 1_Y$

f cont $+$ f is a bijection (in Set)

$\not\Rightarrow$

f isomorph.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, dashed, "f^{-1}"] \end{tikzcd}$$

f^{-1} inverse function not cont.

$$\begin{tikzcd}(X,\tau_1) \arrow[r, "1_X", "\text{cont}"'] & (X,\tau_2) \arrow[l, bend left, dashed, "1 \text{ inverse in set}"]\end{tikzcd}$$

X set

τ_1, τ_2 : topologies

$$\boxed{\tau_2 \subsetneq \tau_1}$$

NOT a cont. function

1 cont. bijective

not homeo.

abstract example

$(X, \leq)$ $\qquad$ $\leq$ is an order

- reflexive
- transitive
- antisymmetric $\quad x \leq y \quad y \leq x \Rightarrow x = y$

$(X, \leq)$ as a category

$$x \xrightarrow{\leq} y$$

<ins>isom.</ins>

if $\exists$ an reverse arrow then $y \leq X$

$\Rightarrow X = y$

$\Rightarrow$ $\boxed{\text{the only isom. are identities}}$

how to interpret the notions of

- → injective function
- → surjective function

$X \xrightarrow{f} Y$ f inj iff $\forall x, x'$
$f(x) = f(x') \Longrightarrow x = x'$

$X \xrightarrow{f} Y$ f surj iff $\forall y \in Y$ $\exists x \in X$
s.t. $f(x) = y$

Def $f : X \to Y$ in $\mathcal{C}$ is called a $\boxed{\text{MONOMORPHISM}}$ iff

$\forall u, v : T \to X$ if $\cancel{f}u = \cancel{f}v \Longrightarrow u = v$

$$\begin{tikzcd} T \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

f can be cancelled on the left

$\mathcal{C}$

$X \xrightarrow{f} Y$

Ex: $\boxed{\text{monos}}$ are the [illegible] of injective functions

DUAL NOTIONS

MONOM.

$$\cdot \overset{u}{\underset{v}{\rightrightarrows}} \cdot \xrightarrow{f} \cdot \qquad \cancel{f}u = \cancel{f}v$$

EPIMORPH

$$\cdot \overset{h}{\underset{k}{\leftleftarrows}} \cdot \xleftarrow{f} \cdot \qquad h\cancel{f} = k\cancel{f}$$

<u>Def</u> $f : X \to Y$ is an $\boxed{\text{EPIMORPHISM}}$ iff $\forall\, h, k : Y \to T$ if

$$hf = kf \Longrightarrow h = k$$

f can be cancelled on the right

MONOM.

DUALISM

EPIMORPH.

interprets — roles of

infection ft.

——— self function

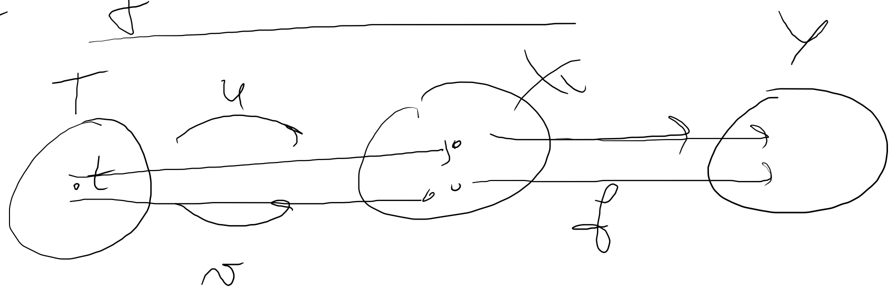
Sets – Grp – Ab – Top.

$\boxed{\text{Set}}$ mono $\longleftrightarrow$ inj. function

$X \xrightarrow{f} Y$

Hp: f is injective

Th: f is mono



$$fu = fv \overset{?}{\Longrightarrow} u = v$$

let us suppose $u \neq v \rightarrow \exists t \in T$ s.t.

$u(t) \neq v(t) \quad \overset{f \text{ injective}}{\Longrightarrow} \quad fu(t) \neq fv(t)$

contrad.

$\mathcal{C}$ concrete category

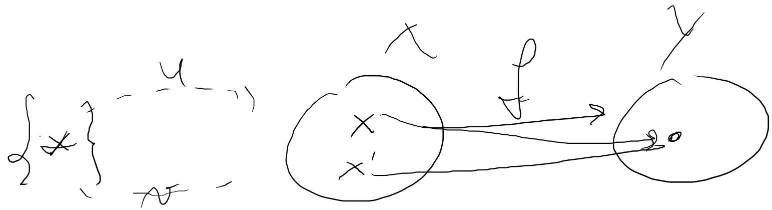
$f : X \to Y$ if f is a morphism and is injective as a function

then f is a monomorphism.

Sets

IP: $\boxed{f \text{ is a monom.}}$

TH: f is injective function



$$\forall\, x, x' \quad \underline{f(x) = f(x')} \Longrightarrow x = x'$$

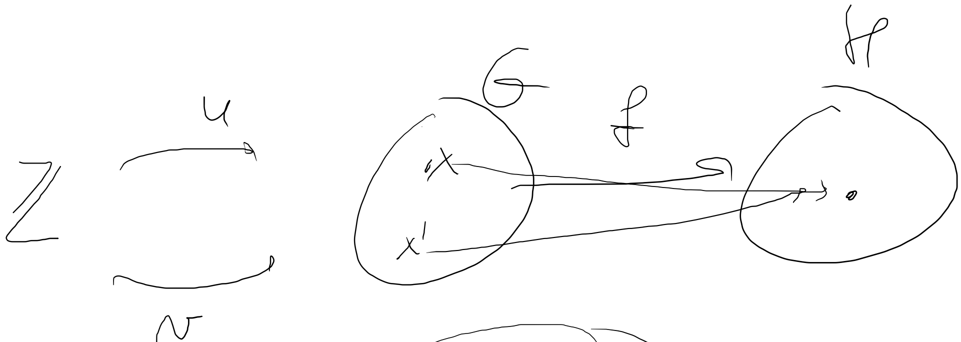
def u and v $\quad u(*) = x \quad v(*) = x'$

$$fu = fv \xrightarrow[\text{f monom}]{} u = v \Longrightarrow \boxed{x = x'}$$

$\boxed{\text{Grp}}$ $f : G \to H$ homom.

Hyp: f monom

Th: f is injective



u, v ?? $\quad$ $fu = fv \Rightarrow u = v$ $\quad$ OK

$\Rightarrow x = x'$

$\boxed{u(1) = x \qquad v(1) = x'}$

$\mathcal{C}$ concrete categ.

$\exists$ $F(1)$ free object on 1

$$F(1) \rightrightarrows X \xrightarrow{f} Y$$

Then MONO $\Rightarrow$ MORPHISM that is injective (on a function)

$\boxed{\mathrm{Top}}$ monomor $\equiv$ continuous and injective function

$\{*\}$

$\boxed{\mathrm{Set} \quad \mathrm{Grp} \quad \mathrm{Ab} \quad \mathrm{Vect} \quad \mathrm{Top}}$

monomor $\equiv$ morphism + injectivity

$(X, \leq)$ $\quad x \xrightarrow{\leq} y$

what is a monomorphism?

$\nexists\; u, v$ with $u \neq v$

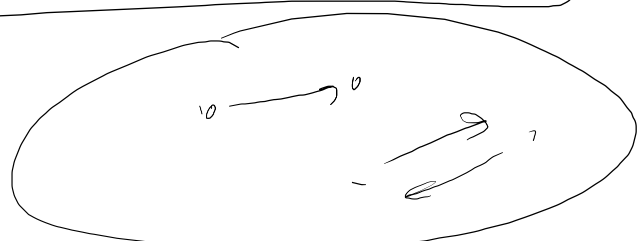
$$\begin{tikzcd} \cdot \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] & x \arrow[r, "\leq"] & y \end{tikzcd}$$

any arrow is a MONOMORP

ISOM = identity

MONOM = ALL

EPIM = ALL



EPIMORPH

$$\begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & \cdot \end{tikzcd}$$

$$h\cancel{f} = k\cancel{f} \Longrightarrow h = k$$

Sets $\boxed{\text{EPIM} \equiv \text{SURJ FUNCTION}}$

if f surj   Th: f epim

$$hf = kf \Longrightarrow h = k$$

$$\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] & T \end{tikzcd}$$

$x \in X$, $y \in Y$

if $h \neq k$ $\exists y : h(y) \neq k(y)$

f surj $\Longrightarrow$ $\exists x : f(x) = y$

$hf(x) \neq kf(x)$ CONTR.

$\boxed{\text{Sets}}$ $\quad X \xrightarrow{f} Y$

IP: f is an EPIM.

Th. f is surjective

$T = \{0,1\}$

$k(y) = 1 \quad \forall y$

$h(y) = \begin{cases} 1 & y \in f(X) \\ 0 & y \notin f(X) \end{cases}$

If f is NOT surjective $\rightarrow \exists \bar{y} \in Y : \bar{y} \notin f(X)$

we must prove that f is NOT an EPIM.

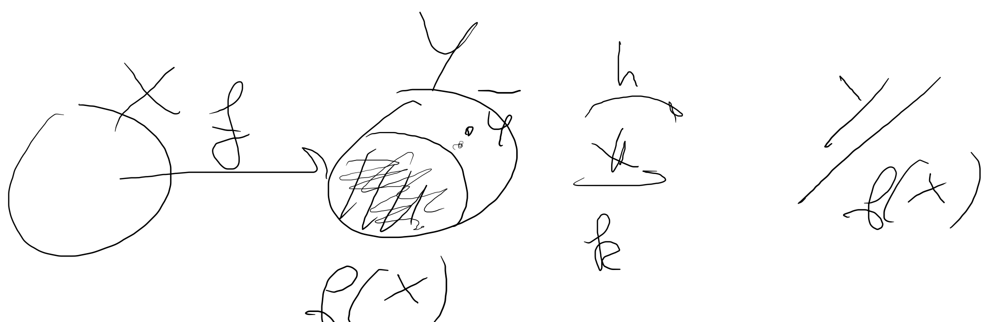
$\exists h, k : Y \to T$ with $h \neq k$ but $hf = kf$

Grp / Ab / Top -/ Vect

$$\boxed{\text{EPIM} \equiv \text{NORM} + \text{SURJECTIVE}}$$

EPIM $\overset{?}{\Longrightarrow}$ SURJ. MORPH.

Grp / Ab



$\boxed{\text{for Ab}}$

f(X) is a NORMAL subgroup

h = quotient homo.

k = constant.

$e_{Y/f(X)}$

It is also true for Groups

$$\cdot \xrightarrow{f} \cdot \underset{g_2}{\overset{g_1}{\rightrightarrows}} \bigcirc$$

$f(x)$ <u>NOT</u> normal

$\boxed{\text{Top}}$ EPI = cont + surj.

$(X, \leq)$ every morphism is an epi.

$\mathcal{C}$ concrete cat.

inj $\longrightarrow$ MONOM.

surj $\longrightarrow$ EPIM.

Exercises :

1. $\boxed{X \xrightarrow{f} Y \xrightarrow{g} T}$

f, g are monom $\longrightarrow$ gf is monom.

f, g are EPIM $\longrightarrow$ gf is an epim.

2. if gf is a monom $\longrightarrow$ f is a monom

if gf is an EPIM $\longrightarrow$ g is an EPIM.

Def $f: X \to Y$ has a left inverse (1)

iff

$\exists\, g: Y \to X$: $gf = 1_X$

$f: X \to Y$ has a right inverse (2)

$\exists\, h: Y \to X$: $fh = 1_Y$

Set (1) $\longleftrightarrow$ injective fund

(2) $\longleftrightarrow$ surj. funct

Def $f : X \to Y$ is called a

BIMORPHISM iff f is

MONOM + EPIM

$$\boxed{\text{ISOM} \Longrightarrow \text{BIMORPH}} \quad \forall \mathcal{C}$$

$$\not\Leftarrow$$

Top

$$(X, \mathcal{G}_1) \xrightarrow{1_X} (X, \mathcal{G}_2)$$

BIM NOT ISO