Equalizer

$E \overset{e}{\dashrightarrow} X \overset{u}{\underset{v}{\rightrightarrows}} Y \qquad \mathcal{C}$

$(E, e : E \longrightarrow X)$ s.that

- $ue = ve$
- universal property

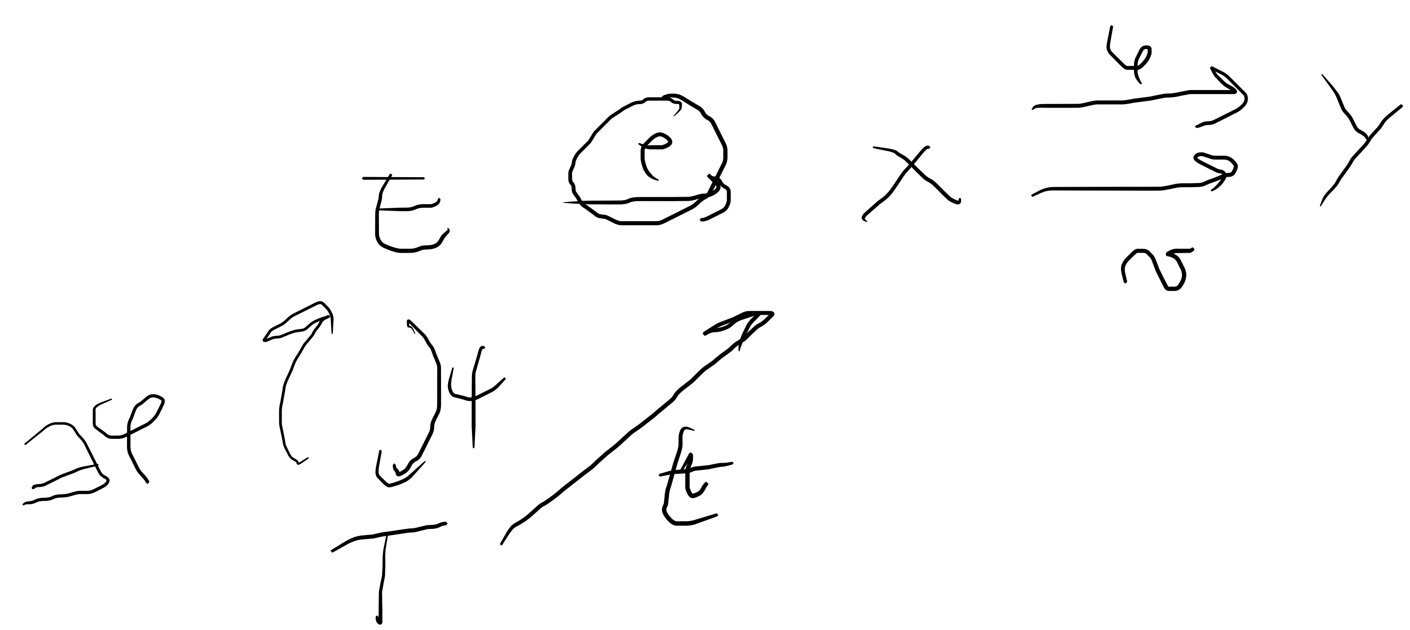
$\forall (T, t : T \longrightarrow X)$ s.t. $ut = vt$

$\exists! \ \varphi : T \longrightarrow E$ s.t

$t = e \varphi$

Theorem. The equalizer if it exists is necess. UNIQUE (up to isomorphism)

Proof



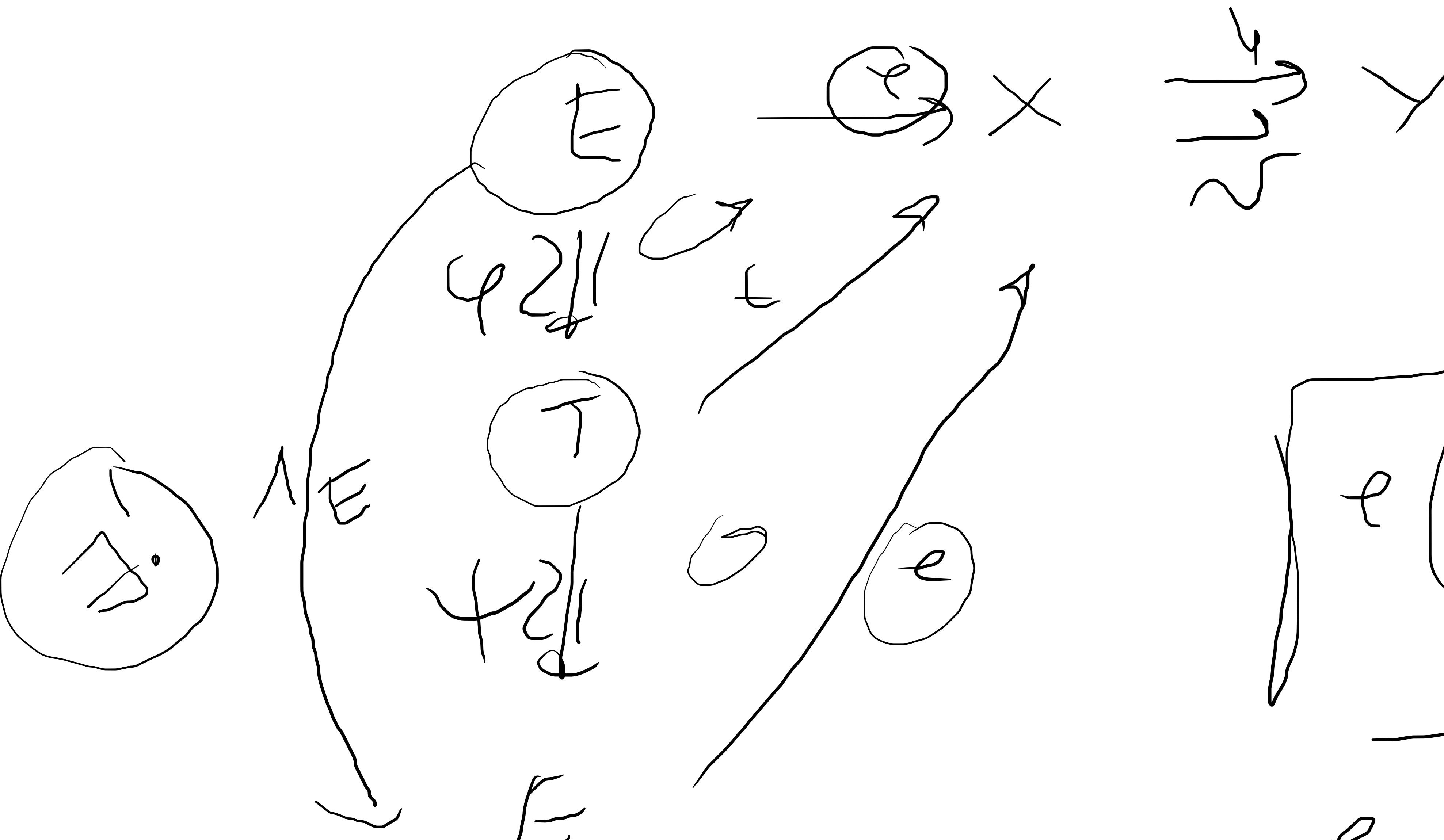
$(E, e) = Eq(\varphi, \nu)$

let (T, t) also be an Equal.

$(T, e) = Eq(\varphi, \nu)$

e univ $\Rightarrow \exists\, \psi : t = e\psi$

t is univ. $\Rightarrow \exists\, \psi' : e = t\psi'$



$$e(\psi \cdot \varphi) = e$$

$$e \cdot 1_E = e$$

$\Rightarrow \quad \psi\varphi = 1_E \quad \Rightarrow \quad \varphi$ is an

iso $\psi = \varphi^{-1}$

$$\varphi \cdot \psi = 1_T$$

Examples:

Set $E \overset{e}{\hookrightarrow} X \underset{v}{\overset{u}{\rightrightarrows}} Y$ $E = \{x \in X : u(x) = v(x)\}$

e is the inclusion of the subset E

Viceversa if $S \subseteq X$ subset

$S \xrightarrow{j} X \underset{v}{\overset{u}{\rightrightarrows}} \{0, 1\}$ $u = \text{const } 1$

$v = \varphi_S$

$j = Eq(u, v)$ $\varphi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$

Grp / Ab / Vett.

equaliser $\equiv$ subalgebra

$$E \xrightarrow{e} G \overset{u}{\underset{v}{\rightrightarrows}} H \qquad E = \{x \in G : u(x) = v(x)\}$$

<u>subgroup</u> E is the equaliser

Viceversa

if $S \leq G$ subgroup of G (Ab)

$$S \xrightarrow[\text{homom.}]{f} G \overset{q}{\underset{\text{cost}_1}{\rightrightarrows}} G/S \qquad S = Eq(q, \text{cost}_1)$$

The result is true also for Group, and for Vect.

<u>Top.</u>

$$E \xrightarrow{e} X \underset{v}{\overset{u}{\rightrightarrows}} Y \qquad E = \{x \in X : u(x) = v(x)\}$$

E has the "<u>initial</u> topology" induced by X

Viceversa $\forall$ subspace $E \subseteq X$

it can be described as an

equalizer

$(X \leq)$ $\qquad x \xrightarrow{\mathrm{id}} x \underset{v}{\overset{u}{\rightrightarrows}} y$

$\nexists \quad u \neq v$

Def a morphism $f : A \to B$ is called a **REGULAR MONOMORPHY** iff $\exists\, (u, v) : B \rightrightarrows C$ s.that

$$f = Eq(u, v)$$

Equalizer

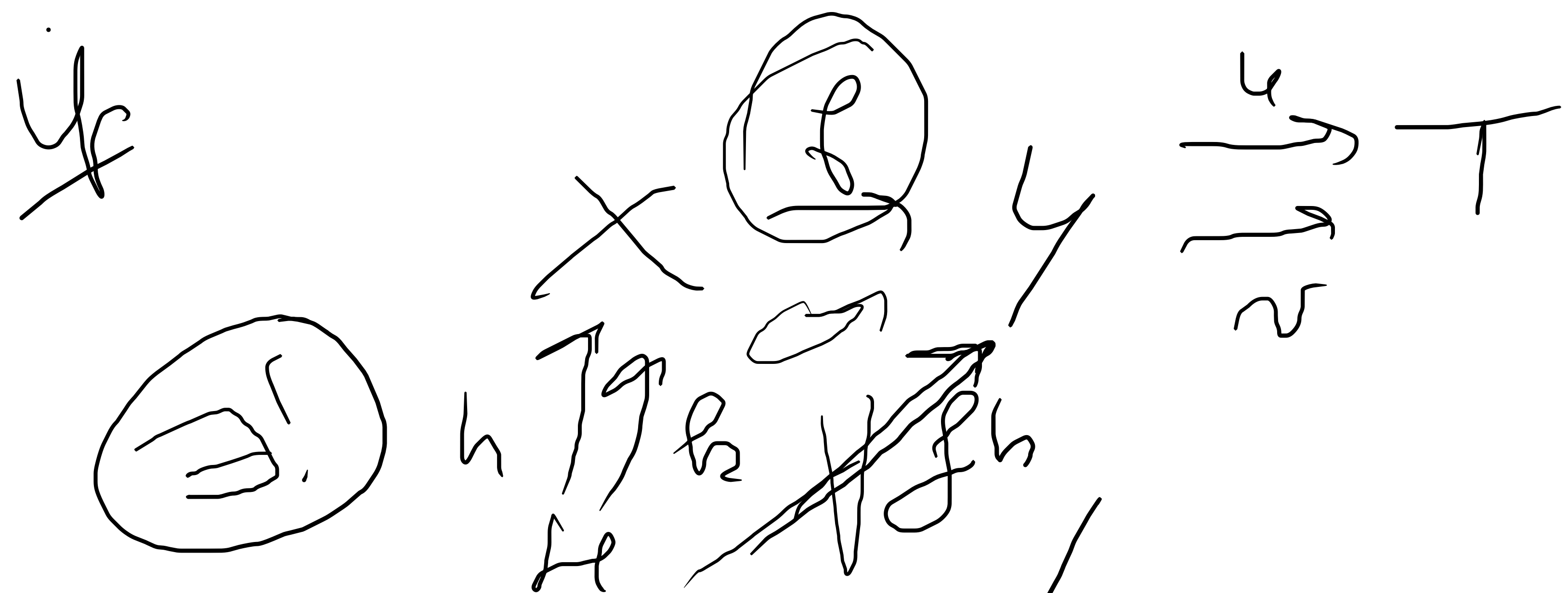
$$A \xrightarrow{f} B \overset{u}{\underset{v}{\rightrightarrows}} C$$

reg mono

$$A \xrightarrow{f} B$$

Prop Any map $f = Eq(u, v)$ is a monomorphism.

Proof $f = \mathcal{E}_q(u, v)$



<u>Th</u> f mono $\iff \left(\forall h, k : \; fh = fk \Rightarrow h = k \right)$

$ufh = vfk$ <u>true</u>.

universal prop. $\Rightarrow \exists!$ $\boxed{h = k}$

Proposit.

> f is reg. mono and f is epimorph
>
> $\Downarrow$
>
> f is an Iso

Proof: $Eq(u,v) = X \xrightarrow{f} Y \overset{u}{\underset{v}{\rightrightarrows}} T$

$uf = vf \Rightarrow$ (f epi) $\boxed{u = v} \to f$ is an Isom.

Properties of morphisms.

Isom → has a left inverse * → rep. mono → mono

DUAL

Isom → has a right inverse → rep. epi → epi

Th. equil. are unique

$\Longrightarrow$ by DUALITY corr. are unique
(up to norm)

Set isom → base change → rep. isom → mono

are the same

↕

bijection — injective funct.

Ab isom → coef univ → rep isom → mono

≠ ≠ = inj. homom.

bijection homom. — DIRECT ADDENDUM — $G \to H$

Top

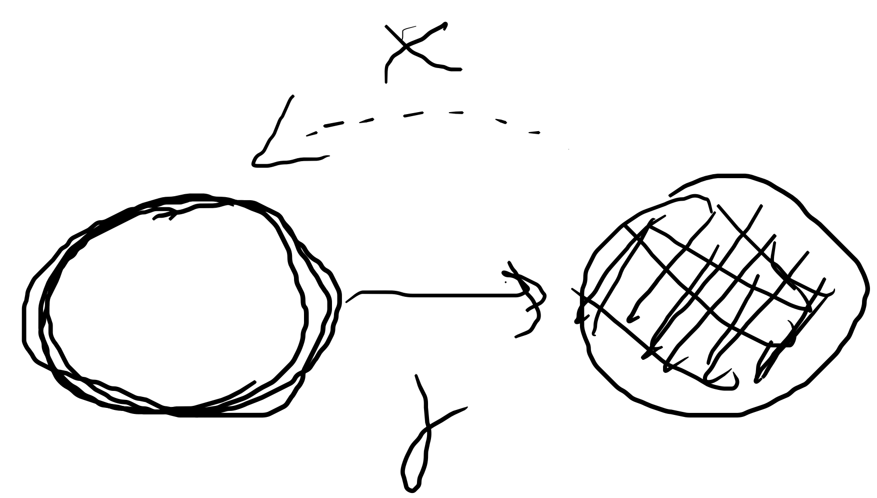
Isom $\Rightarrow$ with left inverse $\Rightarrow$ reg. mono $\Rightarrow$ mono

$\not\Leftarrow$ $\quad\not\Leftarrow$ $\quad\not\Leftarrow$

Isom $\Leftrightarrow$ bijective cont with the cont. inverse

reg. mono: inclusion of subspaces

mono = cont. inj. f...



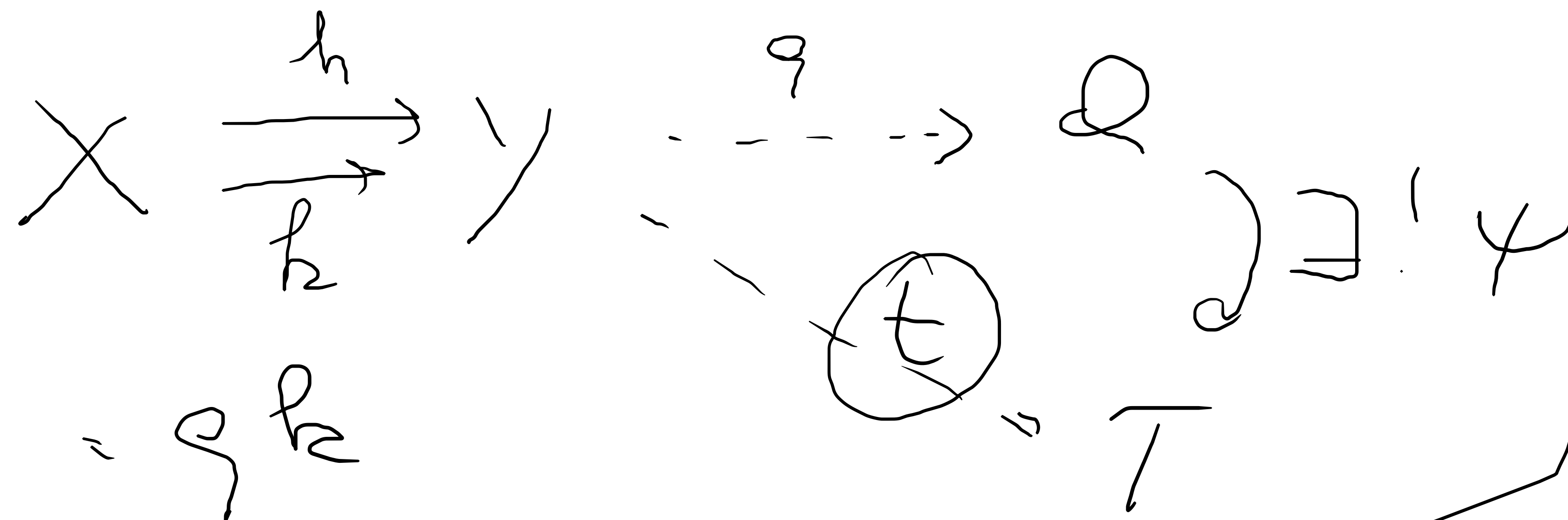
$(X \leq)$ $\quad \cdot \rightrightarrows \cdot$

iso → left inverse → rep mono → mono / any morph.

only the identity

Coequalizer

$(q, Q) = \mathrm{Coeq}(h, k)$



- $qh = qk$
- univ. property

$\forall (T, t) \quad th = tk$

$\exists !\ \psi : Q \to T$ s. that

$t = \psi q$

× DUALITY

- the coeq. is unique (up to iso)
- regep. = coeq. of a pair
- regep. ⇒ ep.

Ex

rep epi + mono = iso

rep mono + epi = iso

mono + epi = bimorph ≠ iso

Set Grp Ab Top (X≤)

// deep interpret the idea of quotients