Equalizer

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \end{tikzcd}$$

$ue = ve$
+ univ. prop.

f is a regular monom. iff it is the equalizer for a certain pair (h, k)

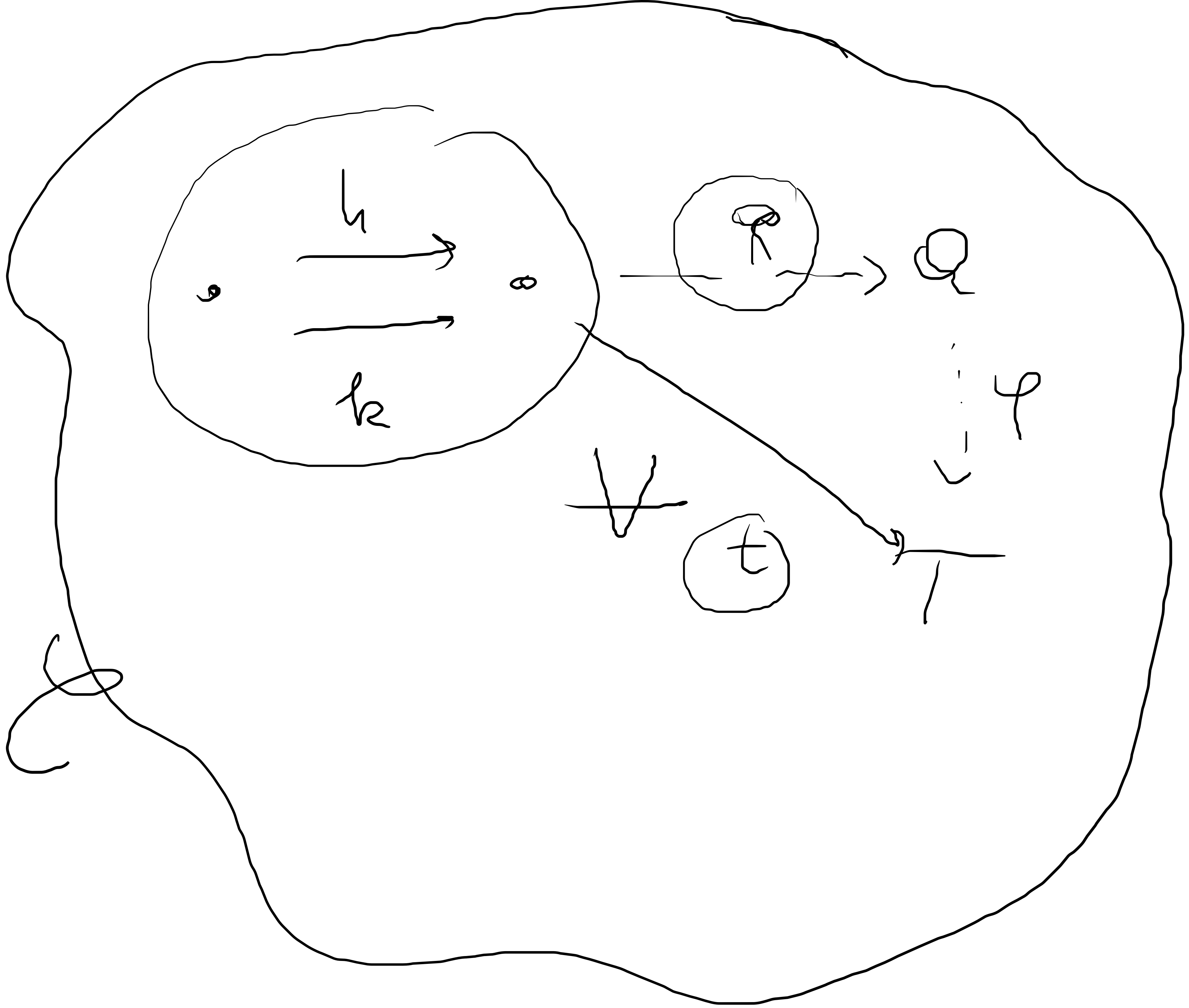
DUAL NOTION

Coequalizer

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & Y \arrow[r, "q"] & Q \end{tikzcd}$$

$(Q, q : Y \to Q)$

- $qh = qk$
- univ. property



univ. property

$\forall T, t : \quad t h = t k$

$\exists! \ \varphi : Q \to T$ s.th.

$t = \varphi \cdot \rho$

Def f is a regular epi if it is the coeq. of a pair

Prop. the coeq. (if it exists) is UNIQUE (up to iso's)

Prop. reg epi $\Rightarrow$ epim.

Prop. reg. epi + mono $=$ isom.

Dual

$\boxed{\text{Coeq}}$ they describe the notion of quotient substructure

$\boxed{\text{Equal}}$

Ex $\boxed{\text{Set}}$

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & Y \arrow[r, dashed, "q"] \arrow[dr, "\forall t"'] & Y/\sim \arrow[d, "\exists!\, \varphi"] \\ & & T \end{tikzcd}$$

$\sim$ equiv relat generated by all pairs

$\forall x \in X \quad (h(x), k(x)) \qquad h(x) \sim k(x)$

$\forall t : \; th = tk \implies \exists!\, \varphi \quad t = \varphi \cdot q$

$\forall \, [y] \in Y/\sim \qquad \varphi([y]) = t(y)$

φ well defined

$[y] = [\bar{y}]$

$y \sim \bar{y}$

$t(y) = t(\bar{y})$

Conversely $\boxed{\mathbf{Set}}$

any quotient map is a $\boxed{\text{coequalizer}}$

$$R \overset{p_1}{\underset{p_2}{\rightrightarrows}} T \xrightarrow{q} T/R$$

R equiv. relat on T

$$R \subseteq T \times T \overset{p_1}{\underset{p_2}{\rightrightarrows}} T$$

$$\Longrightarrow \quad q = \mathrm{Coeq}(p_1, p_2)$$

Grp / Ab / Vett.

quotients $\equiv$ $\boxed{\text{coeq}}$

$$\boxed{\begin{tikzcd} G \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & H \end{tikzcd}} \xrightarrow{q} H/(h-k)(G)$$

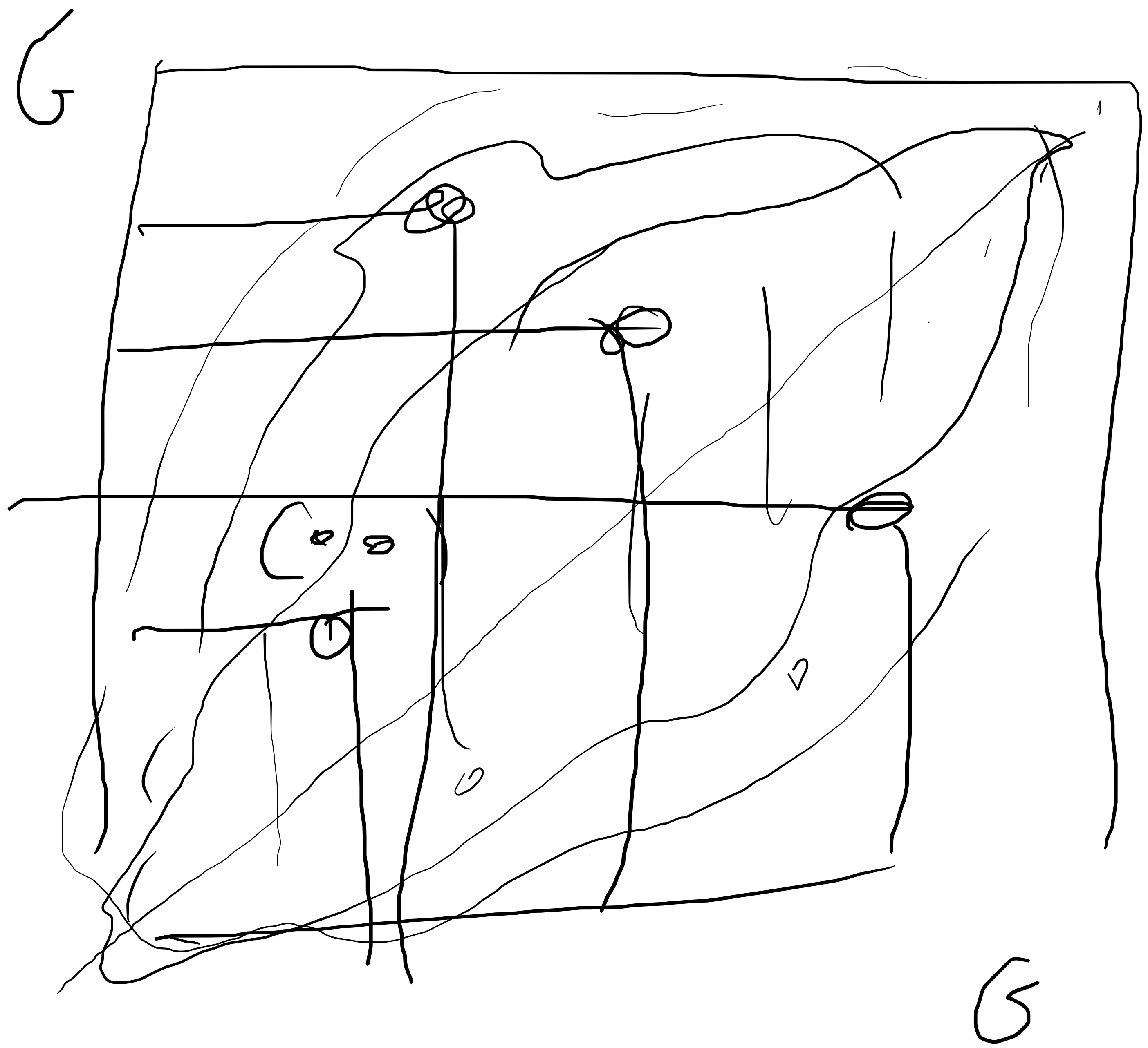
Ab

h, k hom $\Longrightarrow$ $h-k$ still a hom.

$$G \xrightarrow{h-k} H \xrightarrow{q} H/(h-k)(G)$$

$\boxed{h(g) - k(g)}$

true in GROUPS



$G \times G$

generate an equiv. relat. s.t.

$\forall f \quad u(f) \, R \, v(f)$

$$\frac{a \, R \, b \qquad c \, R \, d}{(a+c) \, R \, (b+d)}$$

Ab / Grp / Vect.

if $G \xrightarrow{q} G/R$ is a quotient hom.

$\Longrightarrow$ q it is a coeq.

$$\begin{tikzcd} R \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & G \arrow[r, "q"] & G/R \end{tikzcd}$$

$q = \mathrm{Coeq}(p_1, p_2)$

$$\boxed{R \subseteq G \times G}$$

R cong. equiv. $\Longleftrightarrow$ R is a subgroup of the group product

Top

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y \xrightarrow{q} \boxed{Y/\sim} + \text{final topology}$$

$\rho = \mathrm{Coeq}(u, v)$

as much open to make ρ continuous.

Coeq. $\exists$ and is a top. quotient.

Conversely any quotient is a coeq.

undal top.. $\boxed{\sim} \rightrightarrows Y \longrightarrow Y/\sim$

$$\text{equali} \xleftrightarrow{\ \text{DUAL}\ } \text{coeq}$$

"substructure"   "quotient structure"

$$\underline{\text{Iso}} \;\underset{\not\Leftarrow}{\Rightarrow}\; \exists \text{ right inverse} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{reg epi} \;\underset{\not\Leftarrow}{\Rightarrow}\; \text{epi}$$

$(X, \leq)$ abstract cat. $G \;\underset{\not\Leftarrow}{\longrightarrow}\; G/R$ Grap $\underline{\text{Top}}$

Universal properties of objects

PRODUCTS & COPRODUCT

Set $\quad X \times Y \quad$ cartesian product

$$X \times Y = \{ (x, y),\ x \in X,\ y \in Y \}$$

$\prod X_i$ $\quad$ Group / Ab / Vect.

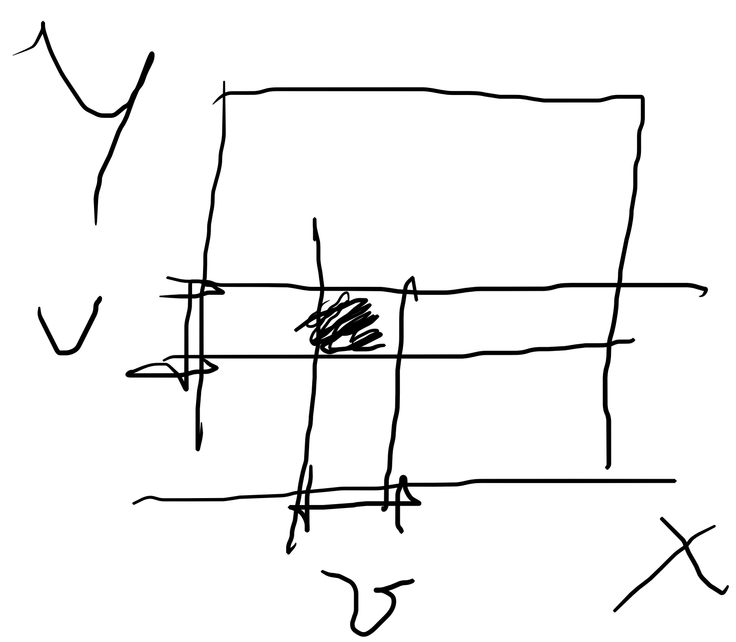
$$G \times H = \{ (g, h),\ g \in G,\ h \in H \}$$

$G \times H$

$$(g, h) \cdot_{G\times H} (\bar{g}, \bar{h}) \overset{Def}{=} (g \cdot_G \bar{g},\ h \cdot_H \bar{h})$$

is a group

Top. $\quad X \times Y \quad$ topological product



There exists a unique definition for these products?

$\mathcal{C}$ category

Def

$X, Y \in \mathcal{C}$ $\boxed{\text{PRODUCT}}$ of X and Y

$X \times Y$ is given by

$$\begin{tikzcd}
P = X \times Y \arrow[rr, "p_x"] \arrow[drr, "p_y"'] & & X \\
& & Y \\
T \arrow[uu, "\exists!\, \varphi"] \arrow[uurr, "t_x"] \arrow[urr, "t_y"'] & &
\end{tikzcd}$$

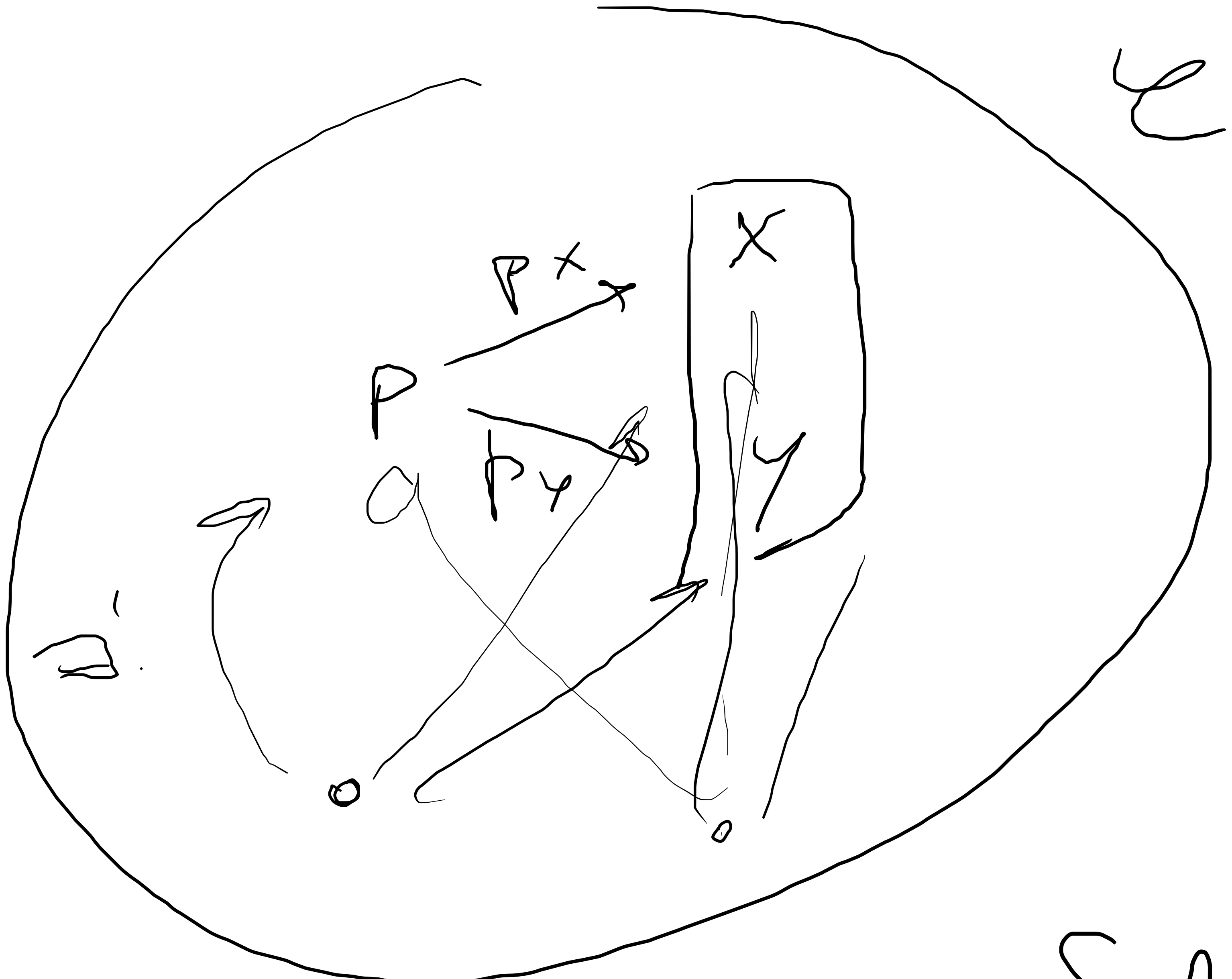
(P, p_x, p_y)

$+$

univ. property

$\longrightarrow$ $\forall (T, t_x, t_y)$ $\exists!\, \varphi : T \to P$

s.t. $t_x = p_x \cdot \varphi$ $\quad t_y = p_y \cdot \varphi$

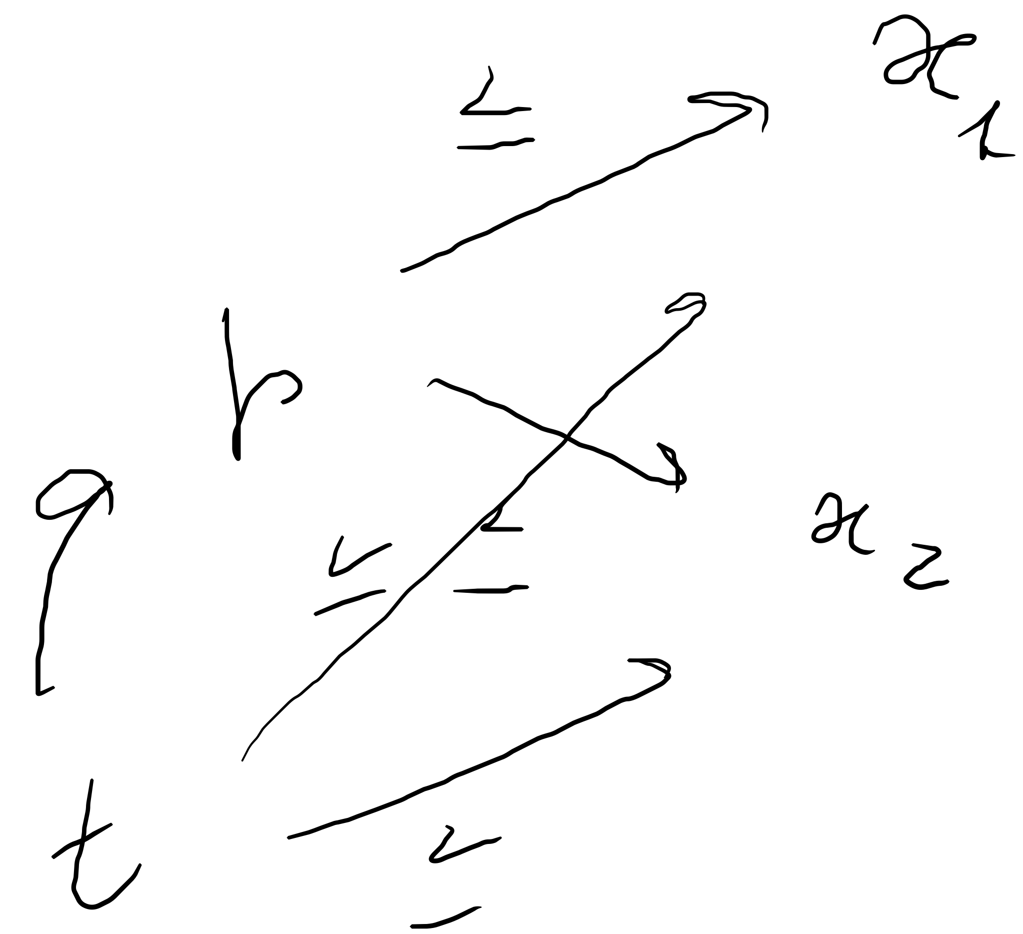


The categorical notion of <u>product</u> gives the classical products in

Set, Grp, Top

$(X, \leq)$ abstract cat.

$x_1, x_2 \in X$



product of x_1 and x_2

$p \in X$ $\quad p \leq x_1$ $\quad p \leq x_2$

univ. prop.

$\forall t,$ $\quad t \leq x_1$ $\quad t \leq x_2$

$\exists \dots$ $\quad t \leq p$

$\boxed{p = \inf(x_1, x_2)}$

DUAL NOTION

$\mathcal{C}$ $\quad X, Y$

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[rd, "t_X"'] & Q \arrow[d, "\varphi \, \exists!"] \\
Y \arrow[ru, "i_Y"'] \arrow[r, bend right, "\forall \; t_Y"'] & T
\end{tikzcd}$$

$\boxed{\text{COPRODUCT}}$

(Q, i_X, i_Y) s.t.

universal prop.

$\forall (T, t_X, t_Y)$

$\exists! \, \varphi : Q \to T$

$t_X = \varphi i_X$

$t_Y = \varphi \cdot i_Y$

Set coproduct = disjoint sum

$X, Y \in \mathbf{Set}$ $\qquad \underline{X \amalg Y} = X \dot\cup Y$

coproduct

Ab $G, H \in \mathbf{Ab}$ $\qquad G \amalg H = G \times H$

Grp $\underline{G \amalg H}$ free product

Top

$$\begin{array}{l} X \searrow \\ \quad\;\; X \dot\cup Y + \text{final top.} \\ Y \nearrow \end{array}$$

$\prod_{i \in I} X_i$ $\qquad$ $\coprod_{i \in I} X_i$

$I = \emptyset$ $\qquad$ product on $(X_i)_{i \in I}$ $\qquad$ $I = \emptyset$

X

P

$\exists!\, \varphi$ $\qquad$ Y

univ. property

$\forall Y, \ \exists \varphi : Y \to P$

$\boxed{\text{P is a Terminal object} \iff \forall Y, \ \exists!\ Y \to P}$

Set Terminal object $\{*\}$

$\forall X \xrightarrow[\exists!]{} \{*\}$

DUAL NOTION INITIAL OBJ.

I $\forall X$ $\exists!$ arrow $I \to X$

Set $\emptyset$ initial obj

Grp

$\{e\}$ both init & terminal

Zero obj.

Def A $\boxed{\text{LIMIT}}$ in $\mathcal{C}$ for a diagram

$\mathbb{D}$ $\left((D_i)_{i \in I} \text{ obj.} \quad \text{arrows } u : D_i \to D_j \right)$

is given by:

$\left(L \in \mathcal{C}, \ \text{cone } l_i : L \to D_i \right)$

Cone means $\forall\, u : D_i \to D_j$ in the diagram

then

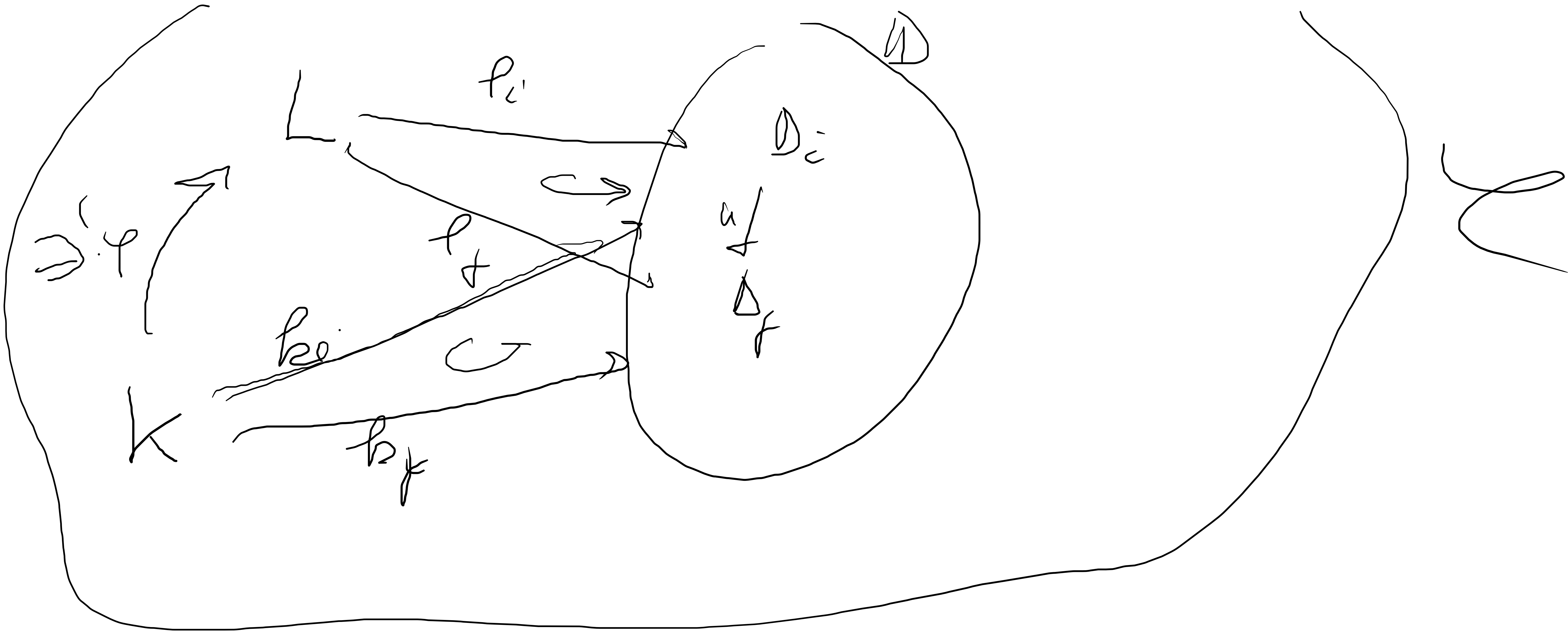
$$\begin{tikzcd}
 & D_i \arrow[d, "u"] \\
L \arrow[ur, "l_i"] \arrow[r, "l_j"'] \arrow[ur, phantom, "\circlearrowleft" near end, shift right=3] & D_j
\end{tikzcd}$$

$u\, l_i = l_j$

$+$ UNIVERS. PROP.

$\forall K, \ k_i : K \to D_i$ cone

$\exists! \varphi : K \to L$ s.t. $k_i = l_i \cdot \varphi$



DUAL NOTION $\underline{\text{COLIMIT}}$

$\underline{\text{Ex}}$ Equaliser.

$$\begin{tikzcd} & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \\ L \arrow[ur, "l_X"] \arrow[urr, "l_Y"'] & & \end{tikzcd}$$

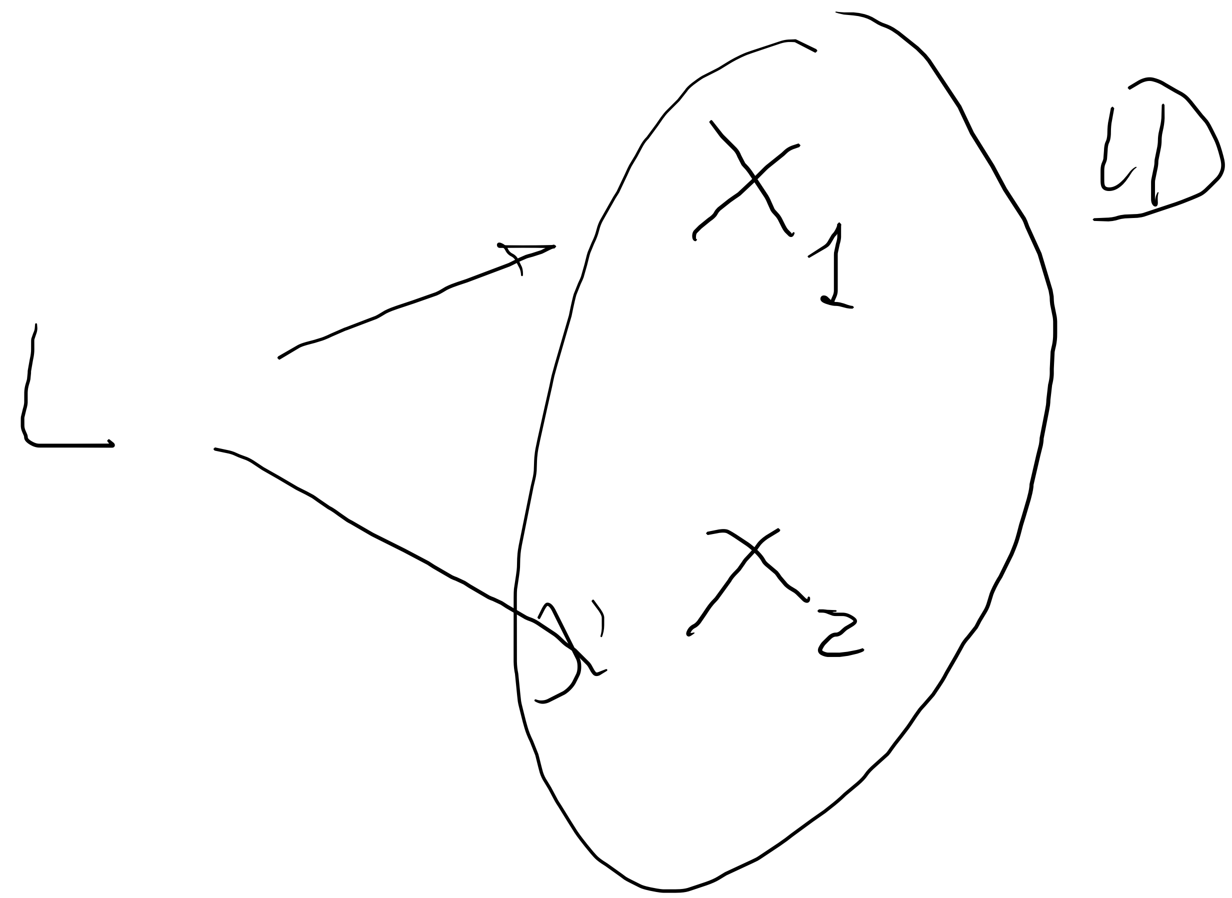
D is the diagram.

$\text{limit} = \text{CONE} = (L, l_X, l_Y)$

$\forall u \quad \underline{u\, l_X = l_Y} \qquad \underline{v\, l_X = l_Y} \iff$

cone $\boxed{u\, l_X = v\, l_X}$

PRODUCTS are LIMITS



Thm: Any limit can be constructed by products and equalizers

Theorem: The limit of D exists is unique (up to iso)

Example

diagram.

$$\begin{tikzcd} & A \arrow[d, "u"] \\ B \arrow[r, "v"'] & C \end{tikzcd}$$

limit?

$$\begin{tikzcd} L \arrow[rr, bend left, "p_A"] \arrow[rrd, "p_C"] \arrow[rd, bend right, "p_B"'] & & A \arrow[d, "u"] \\ & B \arrow[r, "v"'] & C \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend left] \arrow[dr, bend right] & L \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "u"] \\ & B \arrow[r, "v"'] & C \end{tikzcd}$$

(L, p_A, p_B, p_C) cone

$$\begin{aligned} u\,p_A &= p_C \\ v\,p_B &= p_C \end{aligned} \quad \longleftrightarrow \quad \boxed{u\,p_A = v\,p_B}$$

+ univ. property

PULL BACK

PRODOTTI FIBRATI

Set

$$\begin{tikzcd} P \arrow[r, dashed, "h"] \arrow[d, "k"'] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

pull-back

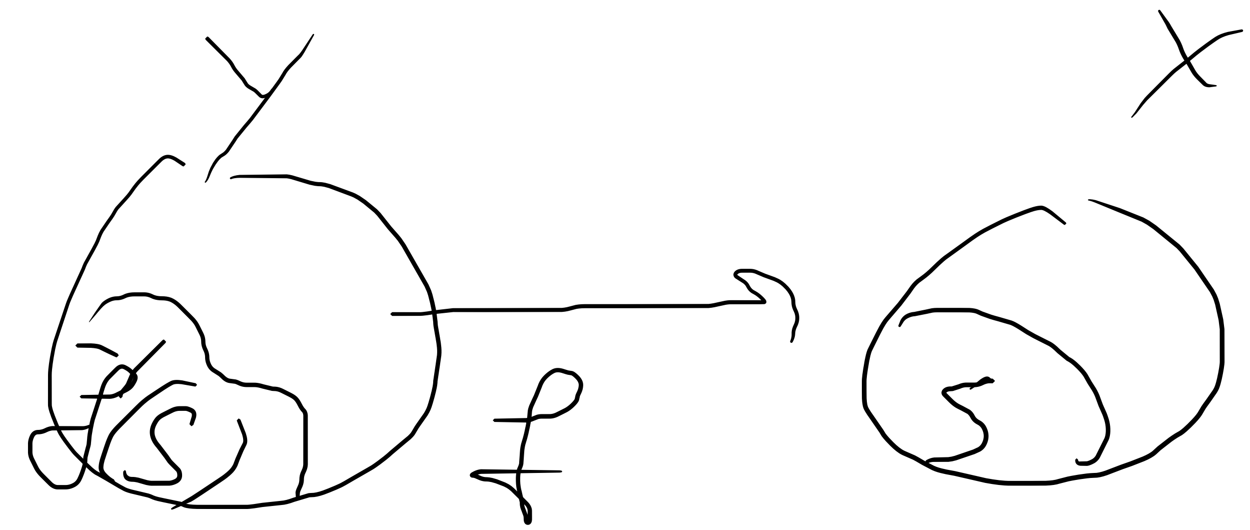
$$P = \{ (a, c),\ a \in A,\ c \in C : f(a) = g(c) \}$$

$X \in Set \qquad S \subseteq X \qquad T \subseteq X$

$$\begin{tikzcd} S \cap T \arrow[r] \arrow[d] & S \arrow[d, hook, "r"] \\ T \arrow[r, hook, "i"'] & X \end{tikzcd}$$

the p.b is the intersection $S \cap T$

$$\begin{tikzcd}
f^{-1}(S) \arrow[r, "f|"] \arrow[d] \arrow[dr, phantom, "pb"] & S \arrow[d, hook, "r"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$



p.b. of r and f

is the inverse image of f

$f^{-1}(S)$

how to construct a p.b.

Set

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr] & X \arrow[d, "f"] \\
X \times T \arrow[ur, "p_X"] \arrow[d, "p_T"'] & \\
T \arrow[r, "g"'] & Y
\end{tikzcd}$$

$$\mathrm{Eq}(f p_X, g p_T) = P$$

$$P = \{ (x,t) : f(x) = g(t) \}$$