$$\begin{tikzcd}
 & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \\
E \arrow[ur, "e"] & & \\
T \arrow[uur, "t"] \arrow[uurr, "t_y"'] & &
\end{tikzcd}$$

$ue = ve$ + prop. univ.

$t : ut = vt \longleftrightarrow$ conv $\;\; t_x = t$

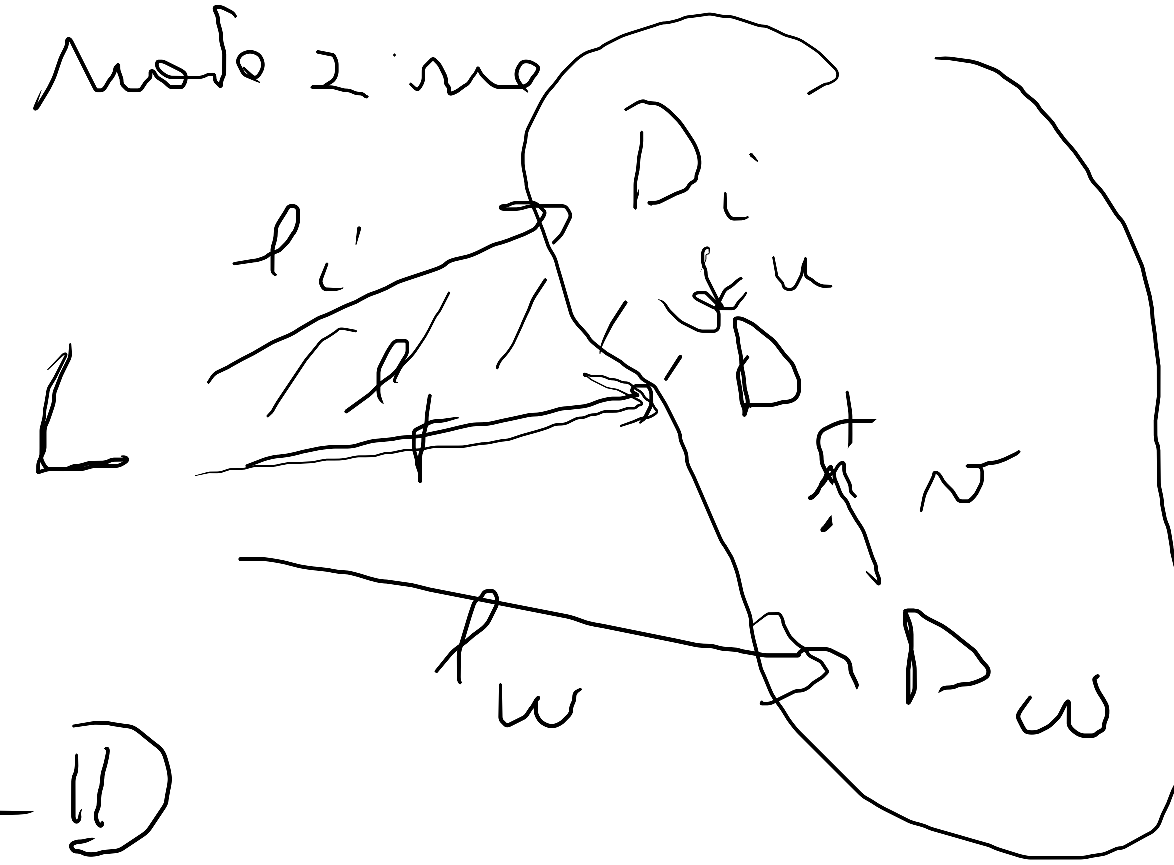
$t_y = ut = vt$

$$\begin{tikzcd}
 & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \\
L \arrow[ur, "\ell_x"] & & \\
\psi \quad T \arrow[uur, "t_x"] \arrow[uurr, "t_y"'] & &
\end{tikzcd}$$

Def **LIMITE** in $\mathcal{C}$ per un **DIAGRAMMA**

$\mathbb{D}$

notazione



(1) è dato da

$(L \in Ob\,\mathcal{C},\ l_i : L \to D_i)\ \forall i\ D_i \in \mathbb{D}$

CONO cioè $\forall u : D_i \to D_j$

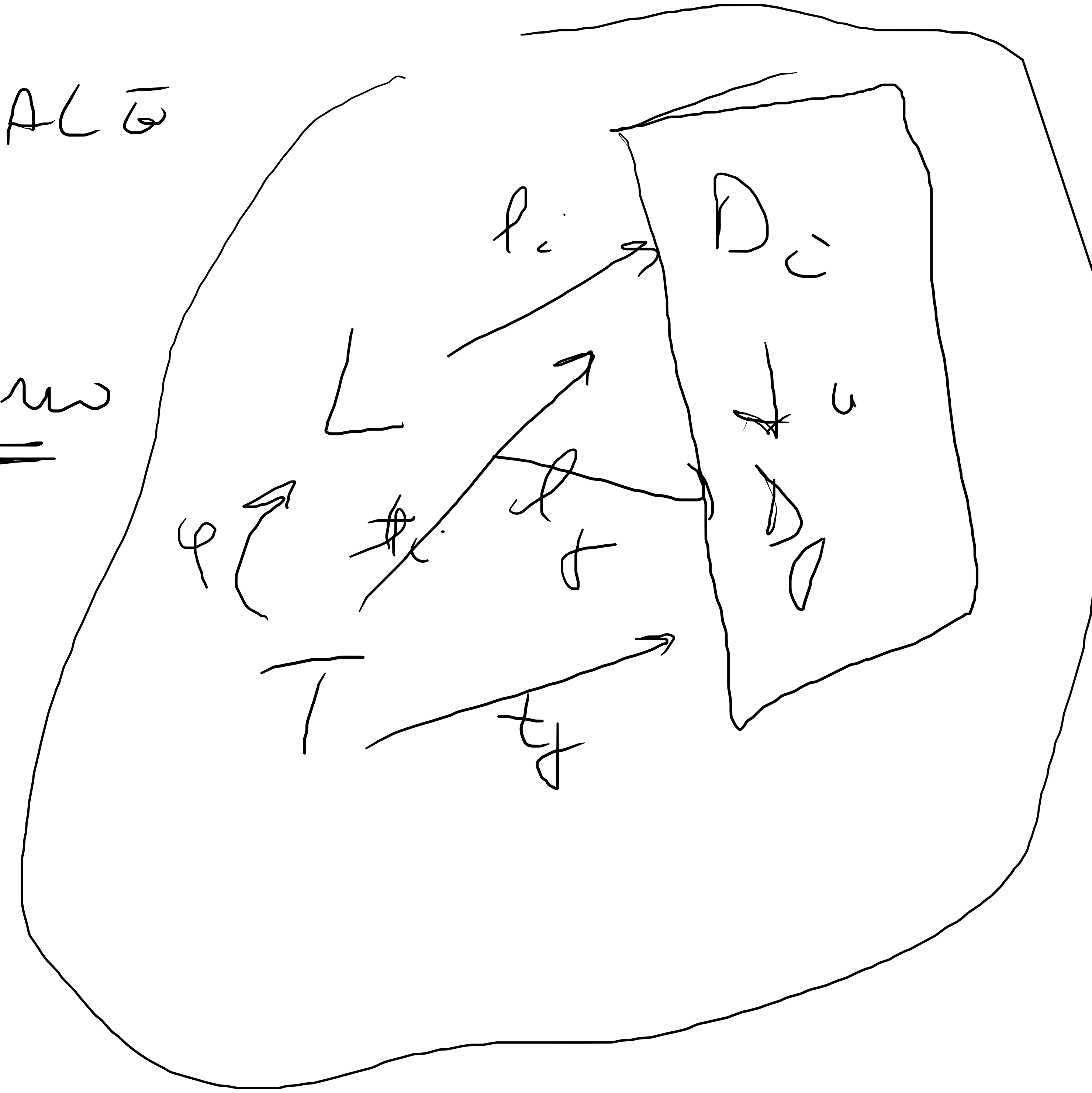
$u \cdot l_i = l_j$

② (L, ρ_i) è UNIVERSALE

$\forall$ $(T, t_i : T \to D_i)$ altro cono

$\exists!$ $\varphi : T \to L$ t.c.

$t_i = \rho_i \cdot \varphi$ $\quad \forall i$



$\underline{Ex}$ 1)

$$
\begin{tikzcd}
 & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \\
L \arrow[ur, "\ell_x"] \arrow[urr, "\ell_y"'] & &
\end{tikzcd}
$$

$\underline{\text{cono}}$ $\forall$ morph.

$$\boxed{\begin{aligned} u\ell_x &= \ell_y \\ v\ell_x &= \ell_y \end{aligned}}$$

$$\Updownarrow$$

$$\boxed{u\ell_x = v\ell_x}$$

(L, ℓ_x, ℓ_y)

$\boxed{\text{essere cono} \iff \text{equalizza } (u,v)}$ ℓ_x equaliz.

idea

equaliz = sottostruttura

Ex   Prodotti X_1 , X_2

$$\begin{tikzcd}
X_1 \times X_2 \arrow[r, "p_1"] \arrow[rd, "p_2"'] & X_1 \\
\forall T \arrow[u, "\exists!\varphi"] \arrow[ur, "t_1"] \arrow[r, "t_2"'] & X_2
\end{tikzcd}$$

Diagramma discreto
non ho morfismi.

$\prod X_i \quad i \in I$ vale anche per $I = \emptyset$

(ogg. terminale)

Ex PRODOTTI FIBRATI

pull-back

$$
\begin{tikzcd}
 & T \arrow[d, "u"] \\
X \arrow[r, "v"'] & Y
\end{tikzcd}
$$

diagramma

$$
\begin{tikzcd}
L \arrow[rr, bend left, "l_T"] \arrow[drr, "l_Y"] \arrow[dr, bend right, "l_X"'] & & T \arrow[d, "u"] \\
 & X \arrow[r, "v"'] & Y
\end{tikzcd}
$$

(L, l_X, l_Y, l_T)

cono $\iff$ $\boxed{\begin{aligned} u l_T &= l_Y \\ v l_X &= l_Y \end{aligned}}$

$\boxed{u l_T = v l_X}$ $\longleftrightarrow$

$$\begin{tikzcd}
A \arrow[dr, "\varphi"] \arrow[drr, bend left, "\alpha_T"] \arrow[ddr, bend right, "\alpha_X"'] & & \\
& L \arrow[r, "l_T"] \arrow[d, "l_X"'] & T \arrow[d, "u"] \\
& X \arrow[r, "v"'] & Y
\end{tikzcd}$$

$\boxed{\text{CONE}}$ i.e. "common way to close the square"

$u \, l_T = v \, l_X$

+ univ. property

$\forall$ cone $\equiv$ $\forall$ other way to close the square

$\boxed{(A, \alpha_T, \alpha_X \mid u \cdot \alpha_T = v \cdot \alpha_X)}$ $\rightarrow$ $\exists! \, \varphi : A \to L$

$\forall \quad \alpha_T = l_T \, \varphi$

$\alpha_X = l_X \cdot \varphi$

$$
\begin{tikzcd}
S \cap T \arrow[r] \arrow[d] \arrow[dr, phantom, "pb"] & S \arrow[d, "d"] \\
T \arrow[r] & X
\end{tikzcd}
$$

Set

$$
\begin{tikzcd}
f^{-1}(S) \arrow[r, "f_1"] \arrow[d, "d"] \arrow[dr, phantom, "pb"] & S \arrow[d, "d"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}
$$

Exercise in $\mathcal{C}$

(1)
$$
\begin{tikzcd}
P \arrow[r, "p_X"] \arrow[d, "p_T"] & X \arrow[d, "d"] \\
T \arrow[r, "t"] & Y
\end{tikzcd}
$$

iff $\boxed{m \text{ is a monomorphism}}$

m H then p_T is a monomorphism

$$\begin{tikzcd} H \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & P \arrow[r, "p_X"] \arrow[d, "p_T"'] & X \arrow[d, "m"] \\ & T \arrow[r, "t"] & Y \end{tikzcd}$$

Wts p_T is a monic

$\Longleftrightarrow$ $\forall\, h, k$ if

$p_T h = p_T k \implies h = k$

$p_X h \quad p_X k$ they make m equal

$\cancel{m}\, p_X h = \cancel{m}\, p_X k \implies p_X h = p_X k$

$\implies$ univ. prop. of the p.b. $\xRightarrow{\text{hyp.}}$ $\boxed{h = k}$

Ex

in $\mathcal{C}$

$$\begin{tikzcd}
T \arrow[r, "h"] \arrow[rr, bend left] \arrow[dr, bend right, "k"'] & X \arrow[r, dashed, "1_X"] \arrow[d, "1_X"'] \arrow[dr, phantom, "pb"] & X \arrow[d, "m"] \\
 & X \arrow[r, "m"'] & Y
\end{tikzcd}$$

m is a monomorphism

iff

the pb is given

by $X \xrightarrow{1_X} X$, $X \xrightarrow{1_X} X$

Proof

$$\forall (T, h, k) \Longrightarrow mh = mk$$

$$\xrightarrow[m \text{ mono}]{} h = k$$

Ex

$$\underbrace{\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}}_{(3)}$$

- (1), (2) are pb

$\downarrow$

(3) is a pb

- (2) (3) are pb $\rightarrow$ (1) is a pb

$\boxed{\text{Set}}$

$$\begin{tikzcd}
P \arrow[r, "p_X"] \arrow[d, "p_T"'] & X \arrow[d, "f"] \\
T \arrow[r, "g"'] & Y
\end{tikzcd}$$

$$P = \{ (x, t) \mid x \in X,\ t \in T,\ f(x) = g(t) \}$$

$$\begin{tikzcd}
P \arrow[dr, "e"] \arrow[rrr, bend left, "\pi_X e"] \arrow[ddr, bend right, "\pi_T e"'] & & & \\
& X \times T \arrow[rr, "\pi_X"] \arrow[d, "\pi_T"'] & & X \arrow[d, "f"] \\
& T \arrow[rr, "g"'] & & Y
\end{tikzcd}$$

$$e = Eq(f \circ \pi_X, g \circ \pi_T)$$

pb is obtained by a product and an equalize

Propos. **Limits are unique**

(up to isomorph.)

Proof 1) $(L, \ell_i) = \lim_{\mathcal{C}} \Delta$

Let us suppose that (T, t_i) is another limit

(univ. cone)

$$\begin{tikzcd}
L \arrow[r, "l_i"] \arrow[dr, "l_j"] \arrow[dd, dashed, "\varphi"'] & D_i \arrow[d, "f_u"] \\
 & D_j \\
T \arrow[uur, "t_i"] \arrow[ur, "t_j"'] &
\end{tikzcd}$$

by univ. property of T

L is another cone

$\Downarrow$ univ. prop.

$\exists!\ \varphi: L \to T \quad t_i \varphi = l_i$

Viceversa L is universal and T is another cone

$\exists!\ \psi: T \to L$ s. $l_i \psi = t_i$

$$\begin{tikzcd}
L \arrow[d, "\varphi"] \arrow[r, "l_i"] \arrow[dd, bend right=60, "1_L"'] & D_i \arrow[d, "L"] \\
T \arrow[ur, "l_i"] \arrow[r] & D_j \\
L \arrow[u, "\psi"] \arrow[uur] \arrow[ur, "l_j"'] &
\end{tikzcd}$$

The fow is unique

also

$l_i \psi \varphi = l_i$

$\Rightarrow \boxed{\psi \varphi = 1_L}$

$t_i \varphi = l_i$

$$\begin{tikzcd}
L \arrow[r, "l_i"] & D_i \\
2nd \ T \arrow[ur, "t_i"'] &
\end{tikzcd}$$

similarly $\varphi \cdot \psi = 1_T$

φ is an iso; $\psi = \varphi^{-1}$

DUAL SITUATION

COLIMIT

$$\begin{tikzcd}
D_i \arrow[d, "u"'] \arrow[rr, dashed] \arrow[dr, "k_i"] & & C & \\
D_j \arrow[r, "k_j"'] \arrow[urr, dashed] & K \arrow[ur, "\exists!\varphi"'] \arrow[r, phantom, "\|"] & \mathrm{colim} &
\end{tikzcd}$$

$(K, k_i : D_i \to K)$

this a cone $\quad \forall u \quad k_j u = k_i$

+ UNIV. PROPERTY

Ex . COPRODUCT } disjoint sums

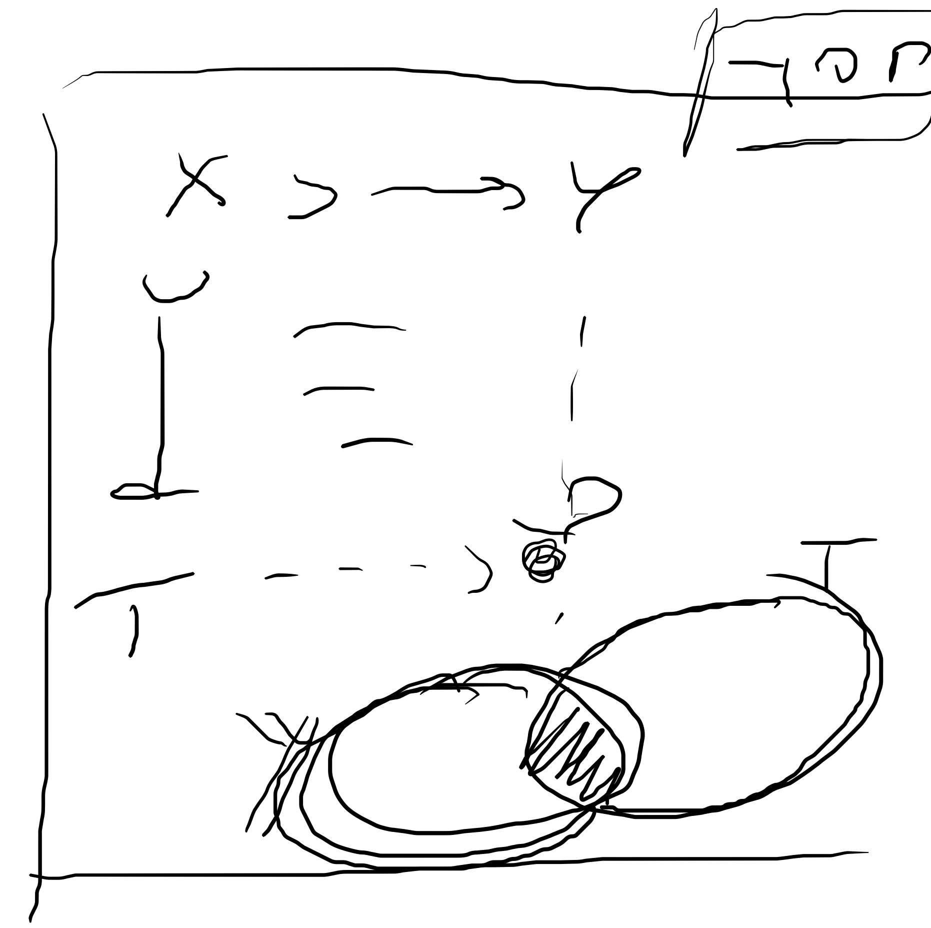
. COEQ $\cdot \overset{u}{\underset{v}{\rightrightarrows}} \cdot$ } quotient

. push out

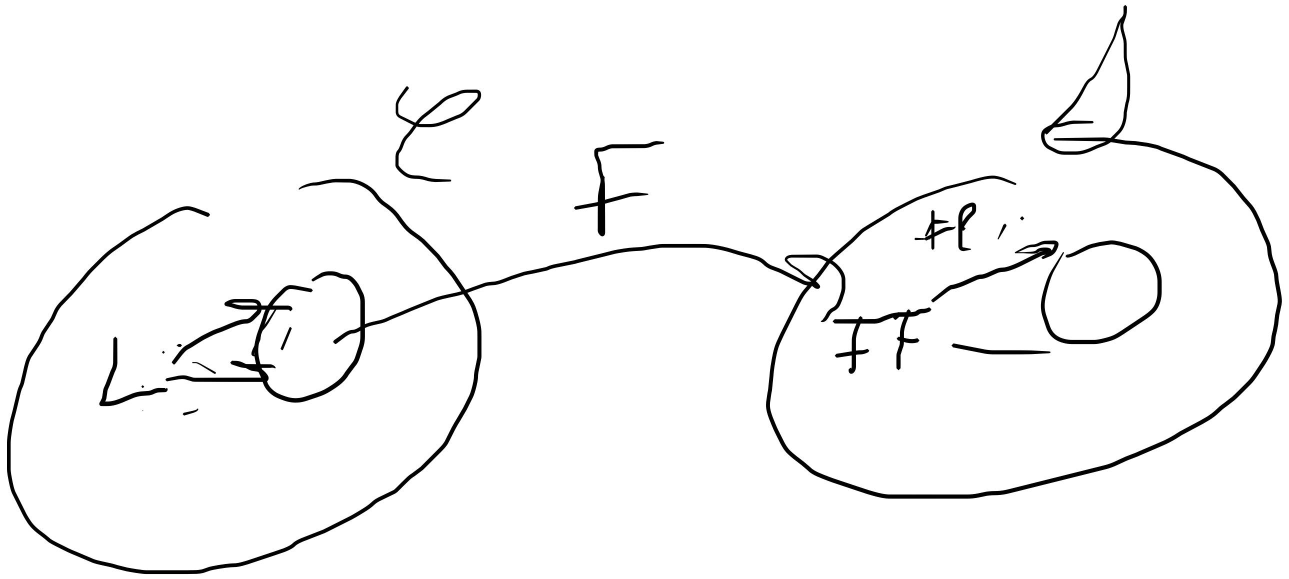
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"'] & Y \arrow[d, dashed, "h"] \\ T \arrow[r, dashed, "k"'] & H \end{tikzcd}$$

of (u, v) is given by

(H, h, k) : $hu = kv$ + univ. property



Def $F : \mathcal{C} \to \mathcal{D}$ $\boxed{F \text{ is preserving}}$ **LIMITS**



$\forall$ $\mathbb{D}$ diagram in $\mathcal{C}$

if $(L, e_i : L \to D_i) = \mathrm{Lim}_{\mathcal{C}}\, \mathbb{D}$

$$\boxed{\underline{\text{then}}\ (FL, Fe_i : FL \to FD_i) = \mathrm{Lim}_{\mathcal{D}}\, F(\mathbb{D})}$$

UNIV

$$L \rightrightarrows \begin{matrix} D_i \\ \downarrow d \\ D_j \end{matrix} \quad \xrightarrow{F} \quad FL \rightrightarrows \begin{matrix} FD_i \\ \downarrow Fd \\ FD_j \end{matrix}$$

still a cone

$\boxed{\text{Adjoint functors}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "G"'] \end{tikzcd}$$

Propos^n. Any representable functor preserves limits

$$\mathcal{C} \xrightarrow{F} Set$$

$\underline{X \in \mathcal{C}}$

$F = \mathcal{C}(X, -)$

Grp

$X = \mathbb{Z}$

$Grp \xrightarrow{F} Set$

$Grp(\mathbb{Z}, -) = F$

$G \rightsquigarrow Grp(\mathbb{Z}, G)$

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, "\mathrm{Grp}(\mathbb{Z},-)", bend left] \arrow[r, "U"', bend right] \arrow[r, phantom, "\cong"] & \mathrm{Set} \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[d, "u"] \\ H \end{tikzcd} \longmapsto \begin{tikzcd} \mathrm{Grp}(\mathbb{Z}, G) \ni t \arrow[d, "\mathrm{Grp}(\mathbb{Z},u)"] \\ \mathrm{Grp}(\mathbb{Z}, H) \end{tikzcd}$$

Grp

Set

$\forall t \quad t : \mathbb{Z} \to G$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "t"] \arrow[rr, bend right, "ut"'] & G \arrow[r, "u"] & H \end{tikzcd}$$

$\mathrm{Grp}(\mathbb{Z}, u)(t) = u \cdot t$

$\mathrm{Grp}(\mathbb{Z}, -)$

$\cong$

$U(-)$

$\underline{\mathrm{Alg}}$ is a category of "algebras"

$$\mathrm{Alg} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Set} \qquad F(X) \text{ free algebra}$$

$\underline{\underline{F(1)}}$ is playing the role of $\mathbb{Z} \in \mathrm{Grp}$

$$\underline{\mathrm{Alg}(\underline{F(1)}, A)} \simeq |A| \simeq U(A)$$

Prop. $\mathcal{C} \longrightarrow Set$, $F = \mathcal{C}(X, -)$ rep. functor.

F preserves limits

Proof

$$L = \lim_{\mathcal{C}} D \rightsquigarrow \mathcal{C}(X, L)$$

$$\begin{tikzcd}
 & & D_i \arrow[d, "f_u"] \\
X \arrow[r, "\varphi_\xi"] \arrow[urr, bend left, "s_i(\xi)"] \arrow[drr, bend right, "s_j(\xi)"'] & L \arrow[ur, "\ell_i"] \arrow[r, "\ell_j"] & D_j
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathcal{C}(X, D_i) \arrow[d, "\mathcal{C}(X,u)"] \\
S \ni \xi \arrow[r, "\varphi"] \arrow[urr, bend left, "s_i"] \arrow[drr, bend right, "s_j"'] & \mathcal{C}(X, L) \arrow[ur, "\mathcal{C}(X, \ell_i)"] \arrow[r, "\mathcal{C}(X, \ell_j)"] & \mathcal{C}(X, D_j)
\end{tikzcd}$$

$\forall (S, s_i : S \to \mathcal{C}(X, D_i))$ cone

$\exists!\ \varphi : S \to \mathcal{C}(X, L)$: $s_i = \mathcal{C}(X, \ell_i) \cdot \varphi \quad \forall i$

$\boxed{\xi \in S}$

$$\boxed{\varphi : S \longrightarrow \mathcal{E}(X, L)}$$

def of φ $\quad \forall \xi \in S \quad \varphi(\xi) = \varphi_\xi$

---

all representative functions preserve limits

Contrav. representable fts

$$\mathrm{Top} \xrightarrow{\mathrm{Top}(-,2)} \mathrm{Set} \qquad 2 = \{0, 1\}$$

open

$$\begin{tikzcd} X \arrow[d, "f"'] & & \mathrm{Top}(X, 2) \\ Y & \longmapsto & \mathrm{Top}(Y, 2) \arrow[u, "\mathrm{Top}(f,2)"'] \end{tikzcd}$$

$F : \boxed{\mathcal{C} \to \Delta}$

contravar functor (and reverses limits)

becomes

$\boxed{\text{covariant}}$ as

a functor from

$\boxed{\mathcal{C}^{o} \longrightarrow \Delta}$

$\mathcal{C}^{o} \longrightarrow \Delta$

$\mathcal{C}^{o}(X, Y) = \mathcal{C}(Y, X)$

$\mathcal{C} \longrightarrow \Delta^{o}$

$\boxed{\text{limit in } \mathcal{C}^{o} = \text{colimit in } \mathcal{C}}$