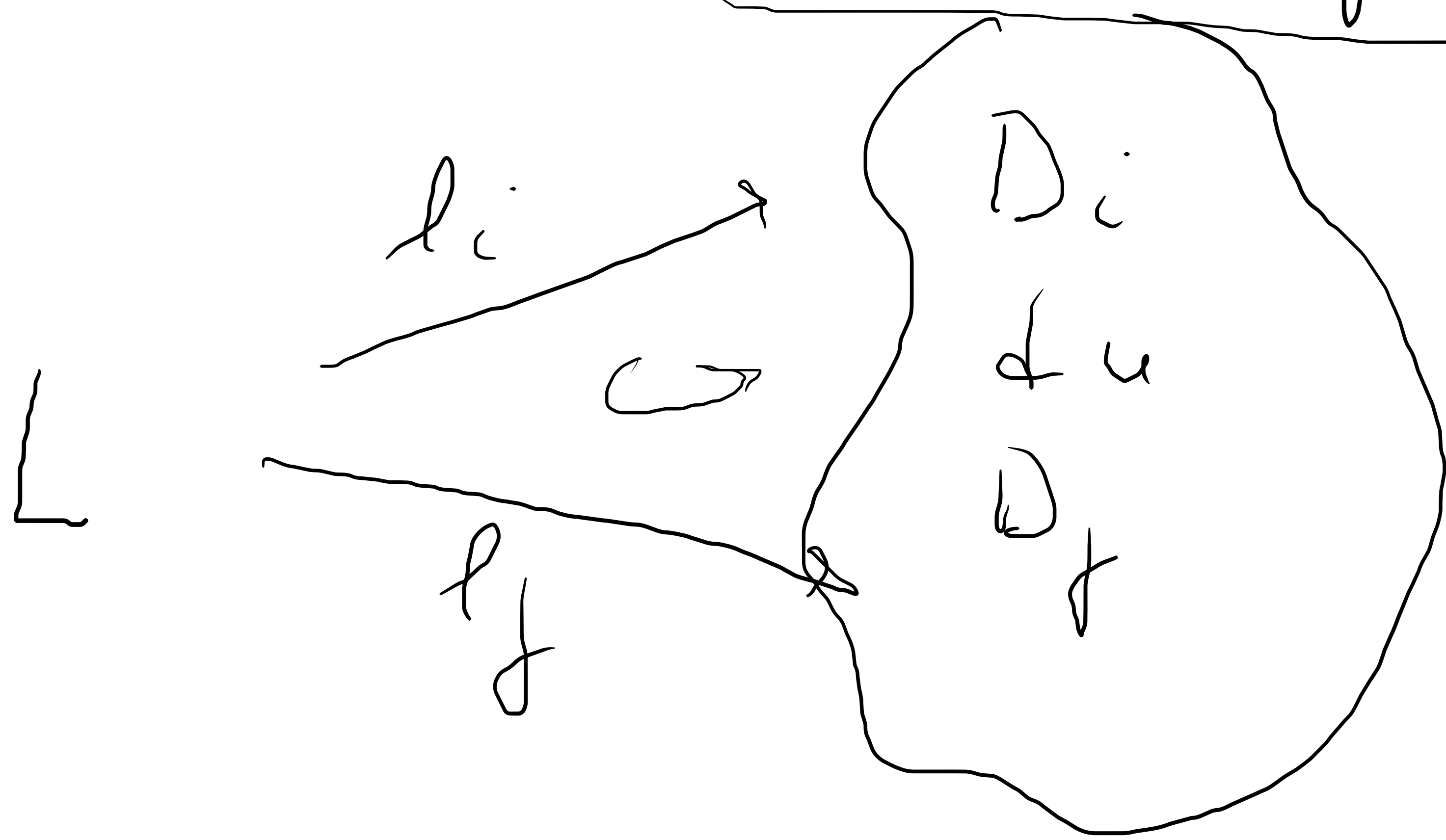
$\mathcal{C}$

$\boxed{\text{limit for a diagram}}$

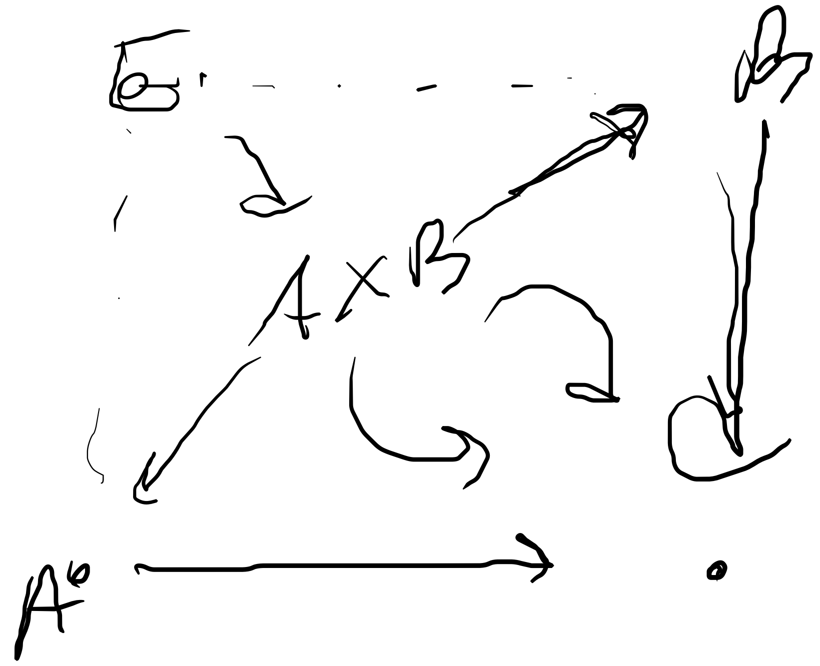


$\underline{\text{cone}}$: $u\, l_i = l_j$

$\forall u$ morph in D

+ univ property

pull-back

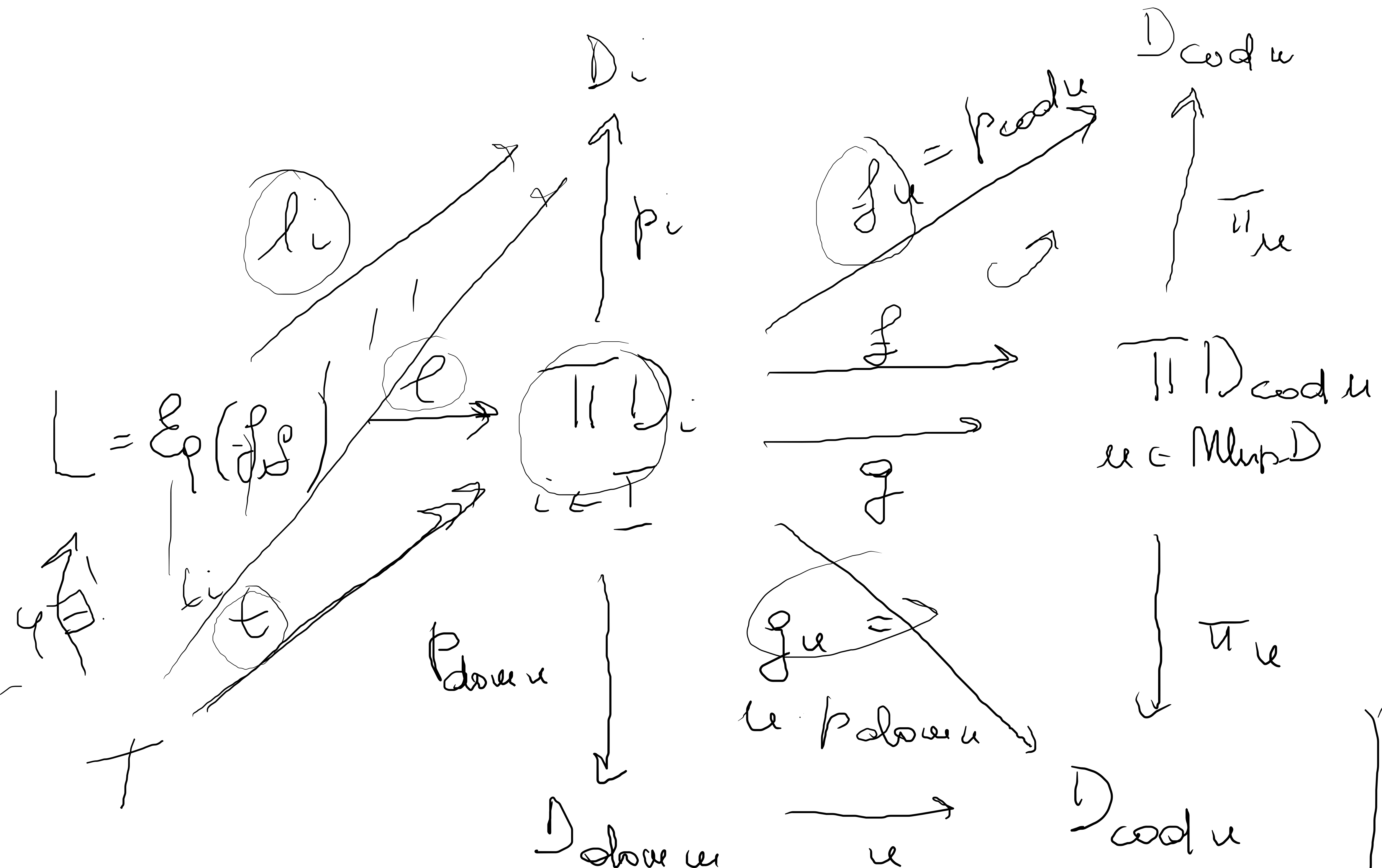


Theorem. If $\mathcal{C}$ has all products and equalizers $\Rightarrow$ $\mathcal{C}$ has all limits

---

$\boxed{\text{DUAL}}$ $\mathcal{C}$ has coproduct & coequalizer $\longrightarrow$ $\mathcal{C}$ has all colimits in $\mathcal{C}$ we have a diagram $\mathbb{D}$ $(D_i)_{i \in I}$

$\lim \mathbb{D}$ $(L, \ell_i) = \lim_{\mathcal{C}} \mathbb{D}$???



$\boxed{f_u \overset{def}{=} p_{cod\,u}}$

$\boxed{g_u \overset{def}{=} u \cdot p_{dom\,u}}$

$u : D_{dom\,u} \to D_{cod\,u}$

$u : D_i \to D_j$

$l_i = p_i \cdot l$

$$\boxed{(L, l_i) = \mathrm{Lim}_e D}$$

$f t = p t \quad \longleftrightarrow$

the t_i $\quad t_i = p_i \cdot t$ form a cone T to D

$(L, \ell_i) = \lim_{C} D$

we must prove it is universal

$\forall$ cone (t_i) $\exists!\ \varphi : T \to L \ldots$

$$\begin{tikzcd} & D_i \\ T \arrow[ur, "t_i"] \arrow[r, "t"'] & \prod D_i \arrow[u, "p_i"'] \end{tikzcd}$$

Dcolim

u

cone

Dcool u

tcool u

tcool u

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "f_i"'] & \prod_{i \in I} A_i \arrow[d, "p_i"] \\ & A_i \end{tikzcd}$$

$$\boxed{f \longleftrightarrow (f_i)_{i \in I}}$$

$$f \longmapsto p_i f = f_i$$

$$\boxed{p_i f = f_i \quad f} \longleftarrow f_i$$

by the universal property of products

universal prop. of (L, e_i)

$\forall (T, t_i : T \to D_i)$ cone $\to$

$t_i \longleftrightarrow t : T \to \pi D_i$

$\boxed{\text{being } t_i \text{ a c.} \Rightarrow ft = gt}$

univ. property of the equal. (L, e)

$\Rightarrow \exists! \varphi : T \to L : t = e\varphi$ $\Rightarrow$ compose with p_i

$p_i t = p_i e \varphi \longleftrightarrow \boxed{t_i = e_i \varphi}$ univ. prop. for the cone

universal
prop. of (L, e_i)

$\forall\ (T,\ t_i : T \to D_i)$ cone $\longrightarrow$

$t_i \longleftrightarrow t : T \to \pi D_i$

$\boxed{\text{being } t_i \circ \ldots \cancel{\Rightarrow} \ f t = g t}$

univ. property of the equaliz. (L, e)

$\Rightarrow \exists ! \ \varphi : T \to L \ : \ t = e \cdot \varphi \longrightarrow$ compose with p_i

$p_i t = p_i e \varphi \longleftrightarrow \boxed{t_i = e_i \varphi}$

univ. prop. for the cone

$$\boxed{ft = gt} \quad \text{for an arbitrary } t : T \to \prod D_i$$

$$\begin{tikzcd}
 & & & D_{\mathrm{cod}\,u} \\
T \arrow[r, "t"] & \prod D_i \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \prod D_{\mathrm{cod}\,u} \arrow[ur, "\pi_u"] &
\end{tikzcd}$$

Def

$t_i : T \to D_i$

$t_i = p_i \cdot t$

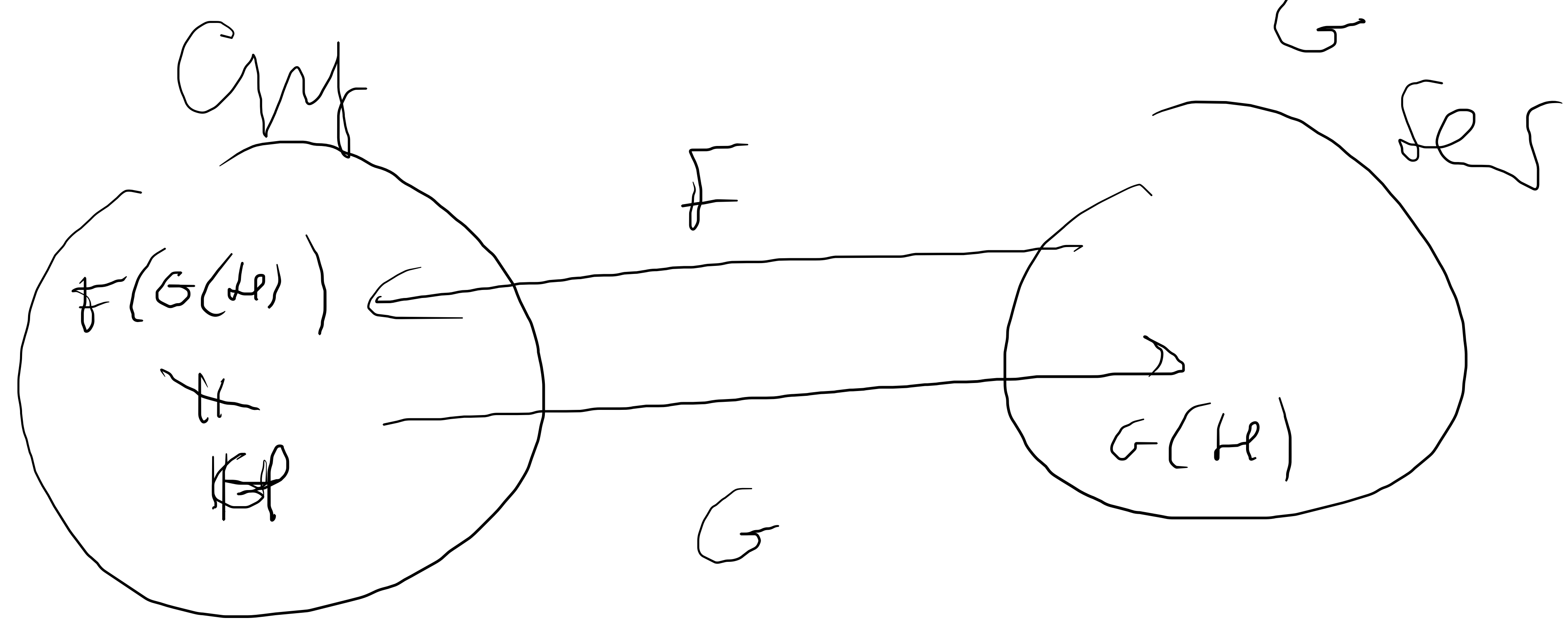
$$ft = gt \iff \pi_u f t = \pi_u g t \quad \forall \pi_u$$

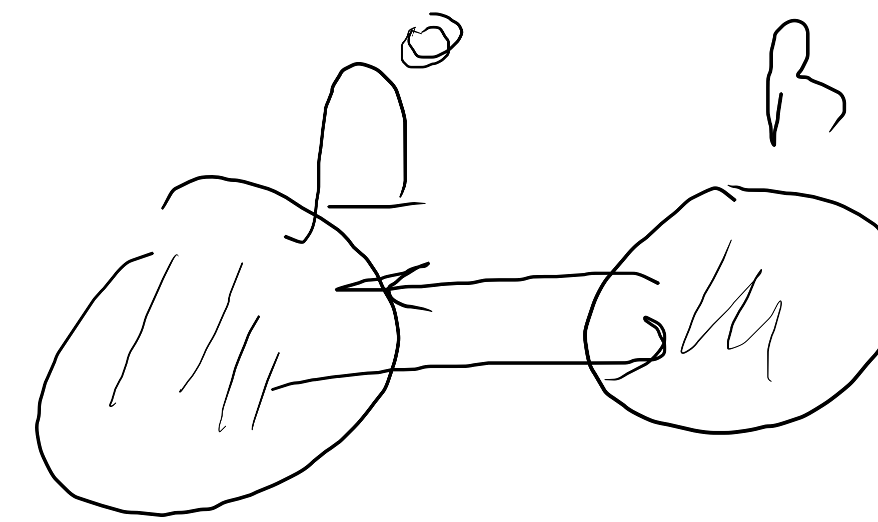
$$\iff f_u t = g_u t \quad \forall u \iff$$

$$\iff p_{\mathrm{cod}\,u} \cdot t = u \cdot p_{\mathrm{dom}\,u} \cdot t \iff \boxed{t_{\mathrm{cod}\,u} = u\, t_{\mathrm{dom}\,u}} \quad \forall u$$

Adjoint functors

$$A \underset{G}{\overset{F}{\leftrightarrows}} B$$





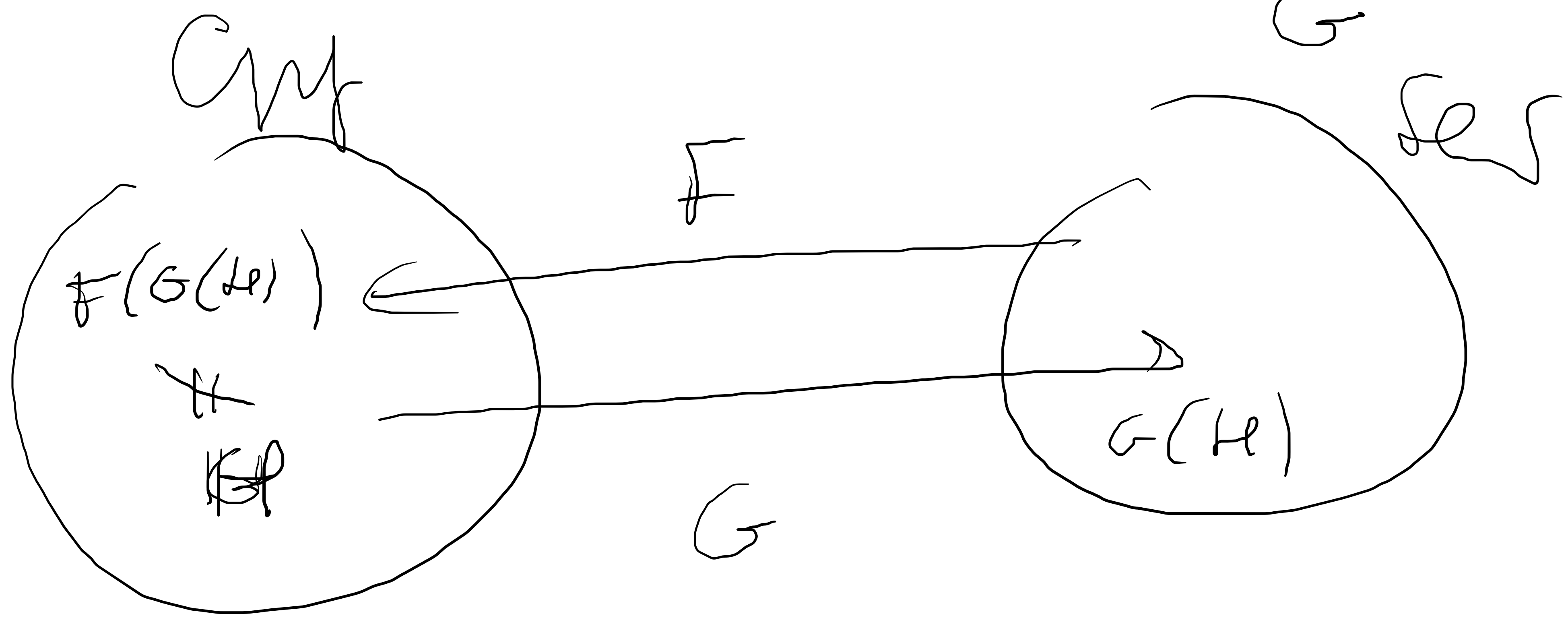
$$A \mapsto G(A) \longrightarrow F(G(A)) \simeq A$$

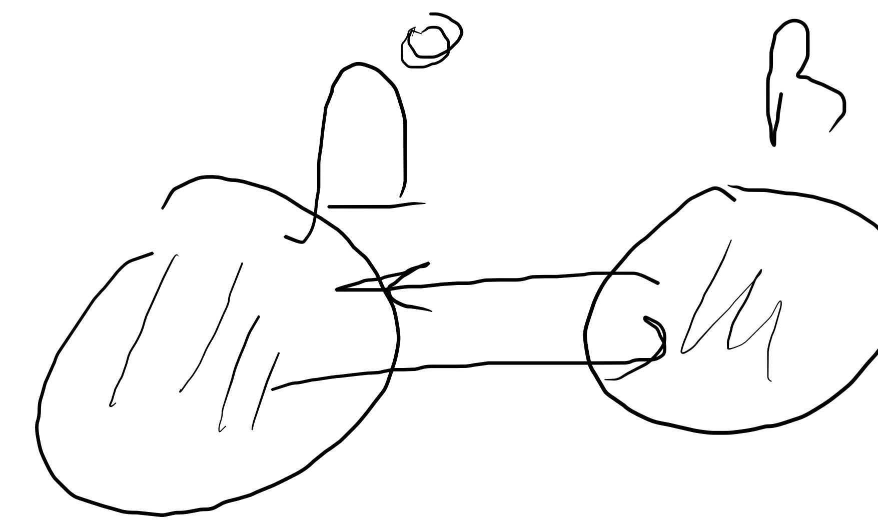
$$B \longrightarrow F(B) \longrightarrow G(F(B)) \simeq B$$

will be a categ ISOMORF

Adjoint functors

$$\begin{tikzcd} A \arrow[r, shift right, "G"'] & B \arrow[l, shift right, "F"'] \end{tikzcd}$$





$A \mapsto G(A) \longrightarrow F(G(A)) \simeq A$

$B \longrightarrow F(B) \longrightarrow G(F(B)) \simeq B$

will be a categ ISOMORF

$$\begin{tikzcd} \mathrm{Vect}_K \arrow[r, "G"'] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

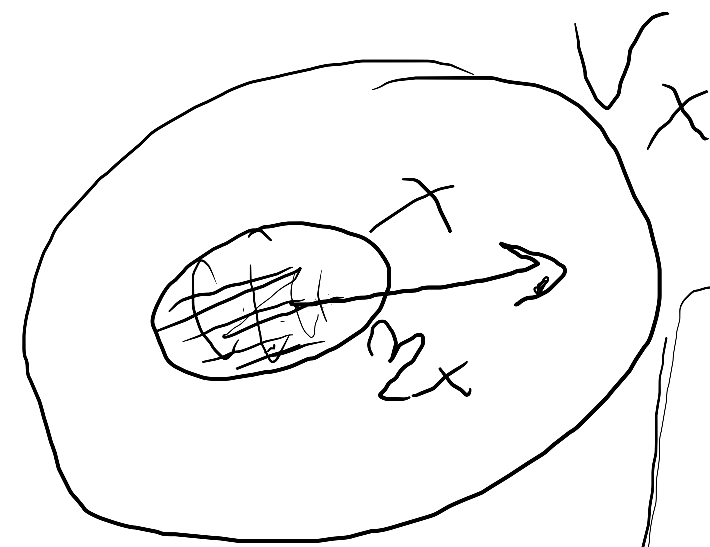
$F(X) = V_X$ vector space with X as a base

$\forall X \in \mathrm{Set}$

$$X \xrightarrow{\boxed{\eta_X}} F(X) = V_X$$

$$X \xrightarrow{\eta_X} F(X) = GF(X)$$

NOT an ISO



$\lambda_i \in K$

$$\boxed{\begin{array}{l} \sum \lambda_i x_i \\ x_i \in X \end{array}}$$

η_X will be an UNIVERSAL ARROW

$$\begin{tikzcd} \mathrm{Vect}_K \arrow[r, "G"'] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

$F(X) = V_X$ vector space with X as a base

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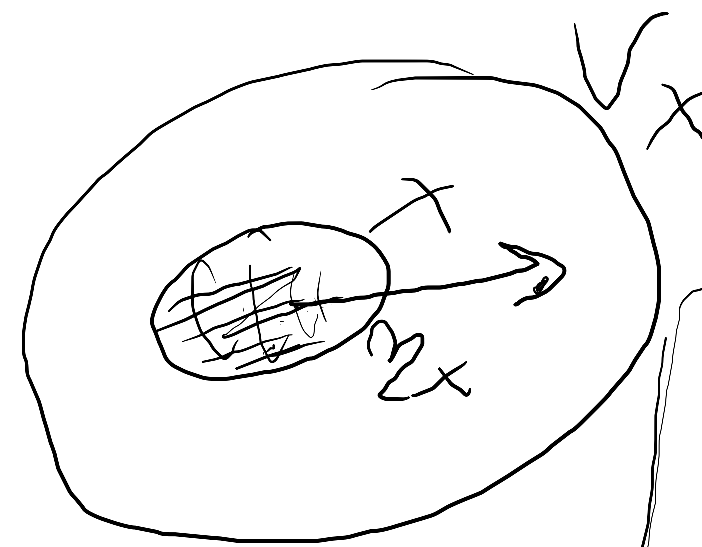
NOT an iso



$\lambda_i \in K$

$\sum \lambda_i x_i$

$x_i \in X$

η_X will be an $\boxed{\text{UNIVERSAL}}$ ARROW

$$\begin{tikzcd} \mathrm{Vect}_K \arrow[r, bend right, "G"'] & \mathrm{Set} \arrow[l, bend right, hook', "F"'] \end{tikzcd}$$

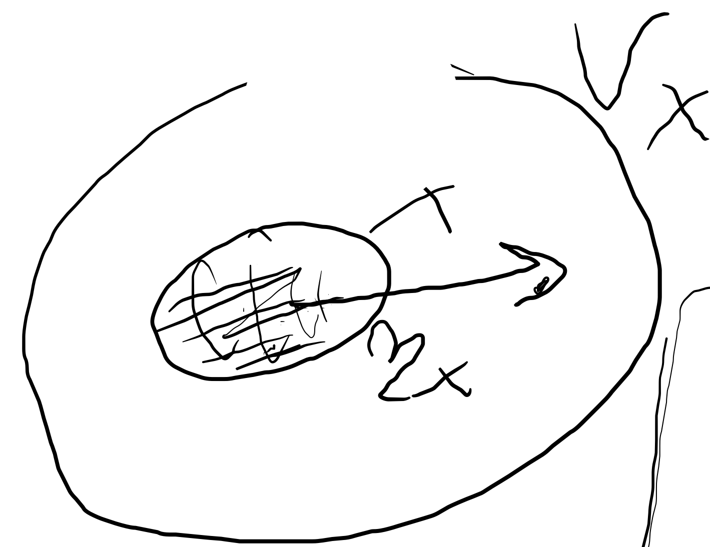
$F(X) = V_X$ vector space with X as a base

$\forall\, \underline{X \in \mathrm{Set}}$

$X \xrightarrow{\boxed{\eta_X}} F(X) = V_X$

$X \xrightarrow{\eta_X} F(X) = \underline{GF(X)}$

NOT an ISO



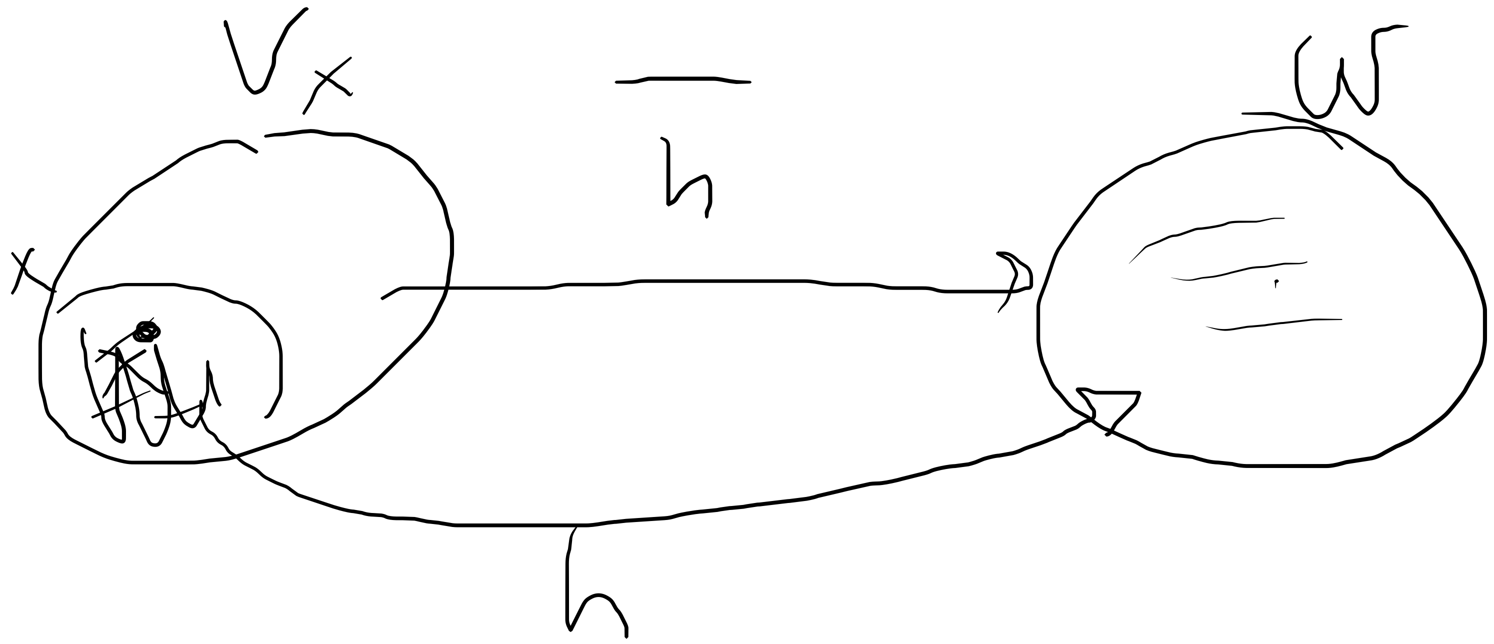
$\lambda_i \in K$

$$\boxed{\begin{array}{l} \sum \lambda_i x_i \\ x_i \in X \end{array}}$$

η_X will be an $\boxed{\text{UNIVERSAL}}$ ARROW

$$\mathrm{Vect} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Set}$$

X base $\qquad F(X) = \underline{V_X}$ vector space



$\forall W \in \mathrm{Vect}.$

$\forall h : X \to W$ in Set

$\exists!\, \bar{h} : V_X \to W$ in Vect.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, bend right, "\forall h"'] & V_X \arrow[d, "\exists!\, \bar{h}"] \\
& W
\end{tikzcd}$$

$\bar{h} \circ \eta_X = h$

$\bar{h}$ is an extension of h

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr] & GFX \arrow[d, dashed, "G\bar{h}"] \\ & GW \end{tikzcd}$$

$\forall h$

$$\begin{tikzcd} FX \arrow[d] \\ W \end{tikzcd}$$

$\exists! \, \bar{h}$ linear funct.

$$\boxed{\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "F"] & \mathrm{Vect} \arrow[l, shift left, "G"] \end{tikzcd}}$$

$h \longmapsto h$ | conversely $h \longrightarrow G\bar{h} \cdot \eta_X = h$

$$\boxed{A \underset{G}{\overset{F}{\leftrightarrows}} B \quad \forall A \ \forall B}$$

$$\mathrm{Vect} \underset{G}{\overset{F}{\leftrightarrows}} \mathrm{Set} \quad \forall W \ \forall X$$

$\forall X$

$\forall W$

$\forall B \ \forall A$ we have

$$\boxed{\mathrm{Set}(X, GW) \xrightarrow{\sim} \mathrm{Vect}(FX, W)}$$

$$\boxed{\mathcal{B}(B, GA) \xrightarrow[\varphi_{AB}]{\sim} \mathcal{A}(FB, A)}$$

Def

F is LEFT adjoint to G (RIGHT)

$F \dashv G$

iff

$\forall A, B$

$\exists\, \varphi_{AB}$ bijection

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "G"'] & \mathcal{B} \arrow[l, shift right, "F"'] \end{tikzcd}$$

$A \in \mathcal{A}$, $B \in \mathcal{B}$

def $F \dashv G$

$\boxed{\begin{array}{l} G \text{ right adj.} \\ F \text{ left adj} \end{array}}$

iff $\forall A \in \mathcal{A}$ $\forall B \in \mathcal{B}$

$\exists$ bijection $\boxed{\mathcal{B}(B, GA) \xrightarrow[\varphi_{A,B}]{\sim} \mathcal{A}(FB, A)}$ Set

f

φ_{AB} is bijection

and it is $\boxed{\text{natural in } A \text{ and in } B}$

$$
\begin{tikzcd}
\mathcal{B}(B, GA) \arrow[r, squiggly, "\varphi_{A,B}"] \arrow[d, "\mathcal{B}(B, Gu)"'] & \mathcal{A}(FB, A) \arrow[d, "\mathcal{A}(FB, u)"] \\
\mathcal{B}(B, GA') \arrow[r, squiggly, "\varphi_{A'B}"] & \mathcal{A}(FB, A')
\end{tikzcd}
$$

$\underline{\text{natural}}$ in $\mathcal{A}$

$\forall \; A \xrightarrow{u} A'$ in $\mathcal{A}$

$$B \xrightarrow{f} GA \;\overset{\varphi}{\rightsquigarrow}\; \varphi_{AB}(f) : FB \to A$$

$$Gu \cdot f \qquad FB \to A \xrightarrow{u} A'$$

$$\varphi_{A'B}(Gu \cdot f) = u \cdot \varphi_{AB}(f)$$

$$\boxed{\varphi_{A'B}(Gu \cdot f) = u \cdot \varphi_{AB}(f)}$$

$$\boxed{\varphi(Gu \cdot f) = u \cdot \varphi(f)}$$

natural in B

$$\boxed{\forall v : B \to B'}$$

$$\begin{array}{ccc} \mathcal{B}(B, GA) & \xrightarrow{\varphi_{AB}} & \mathcal{A}(FB, A) \\ \uparrow {\scriptstyle \mathcal{B}(v, GA)} & \circlearrowleft & \uparrow {\scriptstyle \mathcal{A}(Fv, A)} \\ \mathcal{B}(B', GA) & \xrightarrow[\varphi_{AB'}]{} & \mathcal{A}(FB', A) \end{array}$$

must comm.

$B \xrightarrow{v} B' \xrightarrow{h} GA$ $\rightsquigarrow$ $FB \xrightarrow{Fv} FB' \xrightarrow{\varphi(h)} A$

$h v$

$B' \xrightarrow[h]{} GA$ $\rightsquigarrow$ $\underline{\varphi(h) : FB' \to A}$

$$\boxed{\varphi(h) \cdot F(v) = \varphi(h \cdot v)}$$

$$\begin{tikzcd} \mathrm{Alg} \arrow[r, shift right=1.5, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right=1.5, "F"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, shift right=1.5, "G"'] & \mathrm{Set} \arrow[l, shift right=1.5, "F"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "U"' description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Set} \arrow[l, bend right=40, "D"'] \arrow[l, bend left=50, "I"] \end{tikzcd}$$

th

any right adjoint preserves limits