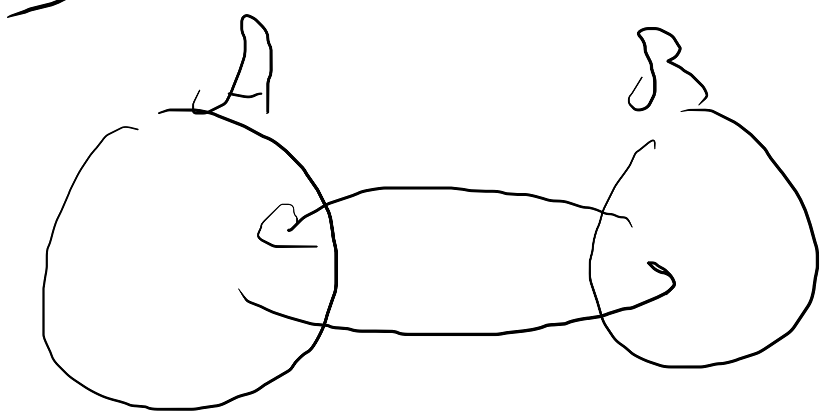
<u>Def</u> ADJOINT FUNCTORS

$$\mathcal{A} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{B}$$



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Freyd / Lawvere
Bourbaki

$\forall A \in \mathcal{A} \quad \forall B \in \mathcal{B} \quad \exists\ \varphi_{AB}$ bijection

$$\boxed{\mathcal{B}(B, GA) \xrightarrow{\varphi_{AB}} \mathcal{A}(FB, A)}$$

$+$ φ_{AB} is natural in A and B

$F \dashv G$

F is left adjoint

G right adjoint

nat. in A $\quad \forall u : A \to A'$

$$\begin{array}{ccc}
\mathcal{B}(B, GA) & \xrightarrow{\varphi_{AB}} & \mathcal{A}(FB, A) \\
{\scriptstyle \mathcal{B}(B, Gu)} \downarrow & & \downarrow {\scriptstyle \mathcal{A}(FB, u)} \\
\mathcal{B}(B, GA') & \xrightarrow[\varphi_{A'B}]{} & \mathcal{A}(FB, A')
\end{array}$$

is comm

$\forall h : B \to GA$

$$\boxed{u \cdot \varphi_{AB}(h) = \varphi_{A'B}(Gu \cdot h)}$$

Same for B

$$\boxed{v : B \to B'}$$

$$\begin{tikzcd}
\mathcal{B}(B, GA) \arrow[r, "\sim", "\varphi_{AB}"'] & \mathcal{A}(FB, A) \\
\mathcal{B}(B', GA) \arrow[u, "{\mathcal{B}(v, GA)}"] \arrow[r, "\varphi_{AB'}"'] & \mathcal{A}(FB', A) \arrow[u, "{\mathcal{A}(Fv, A)}"']
\end{tikzcd}$$

$\forall f : B' \to GA$

$$\boxed{\varphi_{AB}(f v) = \varphi_{AB'}(f)\, Fv}$$

Ex

① $\begin{tikzcd} \mathrm{Vett} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$ $\quad F(X) = V_X$

$X \in \mathrm{Set}$

② $\begin{tikzcd} \mathrm{Grp} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$

$\begin{tikzcd} \mathrm{Ab} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$

$\begin{tikzcd} \mathrm{Ring} \arrow[r, bend right, "U"'] & \mathrm{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$

$\underbrace{F}_{\text{free functor}} \dashv \underbrace{U}_{\text{underlying functor}}$

$F(X)$

$\underline{\underline{\Sigma x_i}}$ abstract words

$\underbrace{x_1 + x_2 + x_3}$ $\quad x_1 x_2 x_3$

$\underline{F(X)}$ is the polynomial ring
free ring with X variables

3) $X \in \mathbf{Top}$ $\qquad$ $S \in \mathbf{Set}$

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "U" description] & \mathbf{Set} \arrow[l, bend right=40, "D"'] \arrow[l, bend left=40, "I"] \end{tikzcd}$$

$$\boxed{\underline{\underline{D}} \dashv U \dashv \underline{I}}$$

discrete top. $\qquad$ indiscrete top

$D \dashv U$ $\qquad$ $\forall S \in \mathbf{Set}$ $\qquad$ $\forall X \in \mathbf{Top}$

$$\boxed{\mathbf{Set}(S, UX) \underset{\varphi_{SX}}{\xrightarrow{\ \sim\ }} \mathbf{Top}(DS, X)}$$

$S \xrightarrow{f \text{ function}} UX$ $\qquad$ $\underline{\underline{D}}S \xrightarrow{f} X$ continuous function

th.

$\underline{F \dashv G}$ $\qquad$ then G preserves limits

F " " colimits

$$\boxed{\begin{tikzcd} \mathrm{Grp} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right, "G"'] \end{tikzcd}}
\qquad
\boxed{\begin{tikzcd} \mathrm{Top} \arrow[r, "U" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{Set} \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"] \end{tikzcd}}$$

U preserves limits $\longrightarrow$ limits in Grp are done as in Set

U doesn't preserve colimits

for Top: $D \dashv U \dashv I$

U preserves limits & colimits

G

X is fixed

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift right, "\mathrm{Set}(X,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right, "-\times X"'] \end{tikzcd}$$

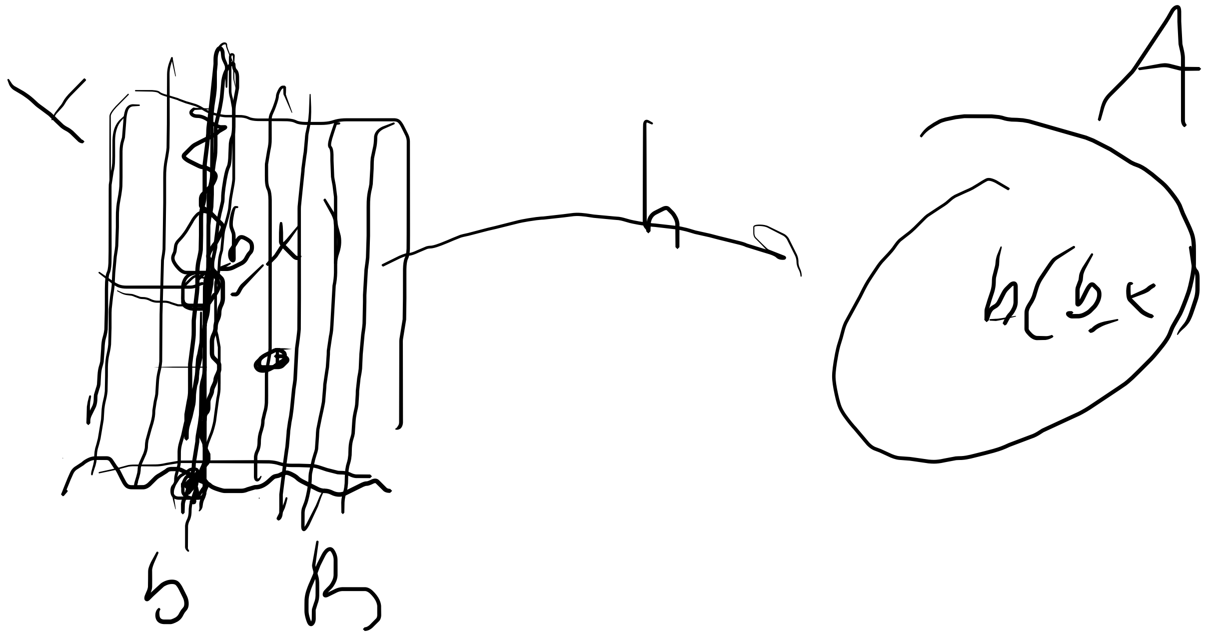
$B \times X \longleftarrow\!\shortmid\; B$

$A \longmapsto \mathrm{Set}(X, A)$

$$\boxed{-\times X \dashv \mathrm{Set}(X,-)}$$

$$\boxed{\mathrm{Set}(B, \mathrm{Set}(X, A)) \xrightarrow[\varphi_{A,B}]{\sim} \mathrm{Set}(B \times X, A)}$$

h



$\varphi^{-1}(h) : B \to \mathrm{Set}(X, A)$

$\varphi^{-1}(h)(b) : X \to A$

$\varphi^{-1}(h)(b)(x) = h(b, x)$

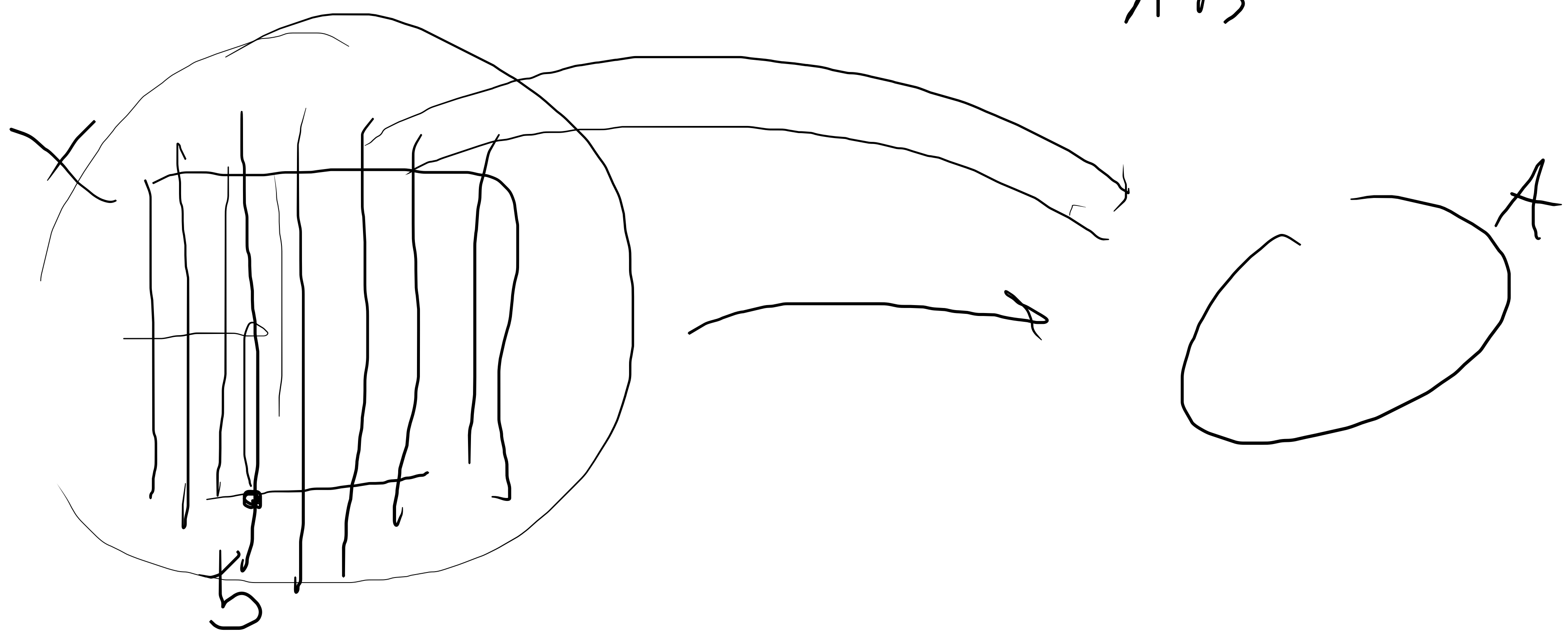
Top

$$\begin{tikzcd} \mathrm{Top}^{A} \arrow[r, bend right, "\mathrm{Top}(X,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Top}^{B} \arrow[l, bend right, "-\otimes X"'] \end{tikzcd}$$

??

$$\mathrm{Top}(B, \mathrm{Top}(X,A)) \xrightarrow[\varphi_{AB}]{\sim} \mathrm{Top}(B\otimes X, A)$$

$B \otimes X$ is the separate cont. topology

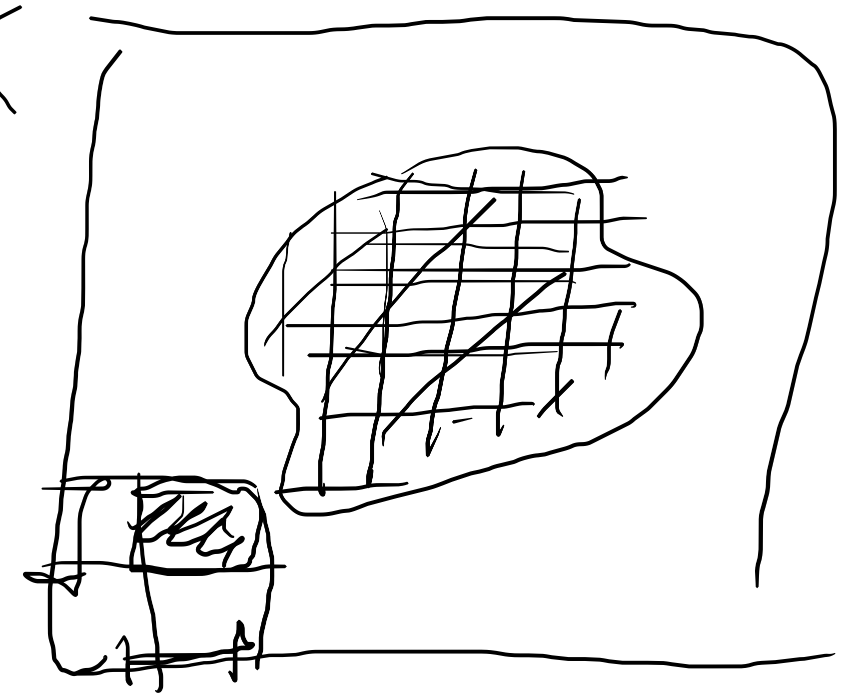


B

$$\mathrm{Top}(X,A) = \{ f: X \to A \text{ cont} \} \simeq \underline{A}^{X}$$ is a top. space

$\boxed{B \otimes X}$

X



open iff all sections are open

B

$$\begin{tikzcd} Ab \arrow[r, bend right, "Ab(X,-)"'] \arrow[r, phantom, "\perp"] & Set \arrow[l, bend right, "-\otimes X"'] \end{tikzcd}$$

tensor product

5.

$$\boxed{\begin{tikzcd}(X,\le) \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & (Y,\le) \arrow[l, bend right, "F"']\end{tikzcd}}$$

$$Y(y, Gx) \xrightarrow{\sim} X(Fy, x)$$

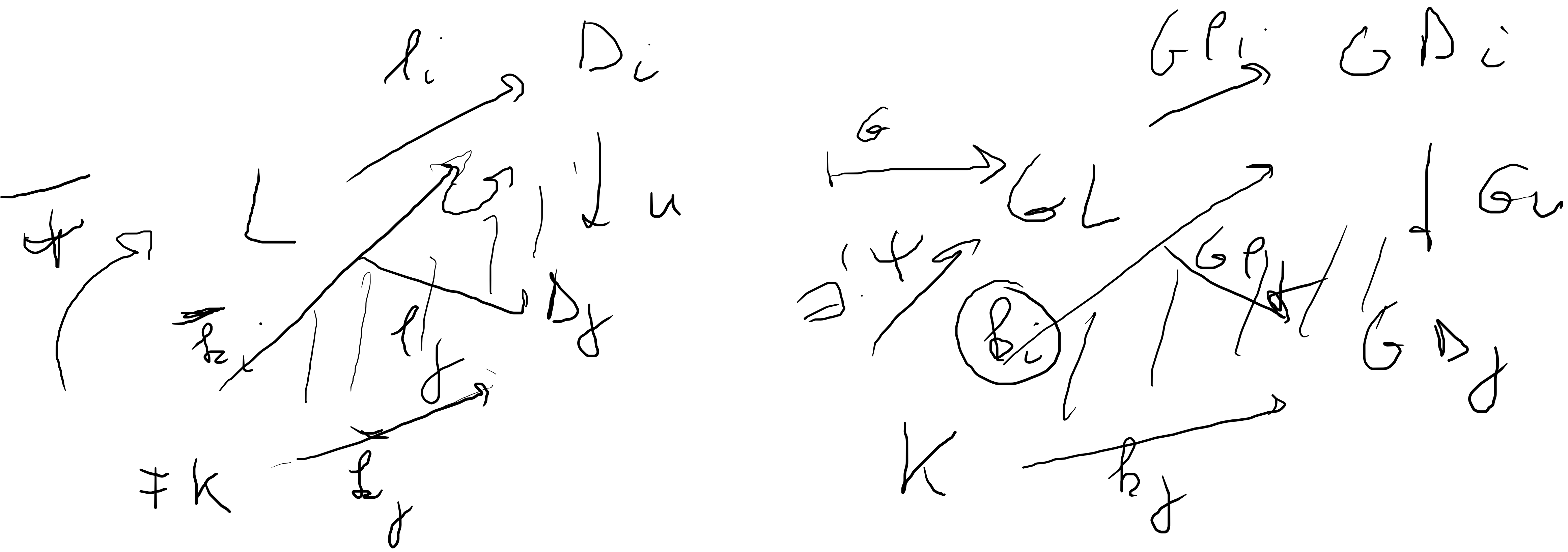
$$\boxed{y \le Gx \longleftrightarrow Fy \le x}$$

$$\left.\begin{array}{l}\text{completion}\\ \text{compactification}\\ \text{quotient field}\end{array}\right\} \text{ all adjoints}$$

theorem $F \dashv G$ $\quad A \underset{G}{\overset{F}{\rightleftarrows}} B$

F preserves colimits & $\boxed{G \text{ preserves limits}}$

Proof if $(L, \ell) = \mathrm{Lim}_A\, D \overset{\text{then}}{\Longrightarrow} (GL, G\ell_i) = \mathrm{Lim}_B\, GD$



still a cone

$\boxed{\text{UNIVERSAL}}$?

$\forall$ cone (K, k_i) $\exists!\, u : K \to GL : k_i = G\ell_i \cdot u$

from (P) $\boxed{Gu \cdot k_i = k_j}$ $\forall i$ (K, k_i) is a cone

also $(FK \xrightarrow{\bar{k}_i} D_i)$ is a cone

th $\boxed{u \cdot \bar{k}_i = \bar{k}_j}$ $\qquad D_i \xrightarrow{u} D_j$

$$
\begin{tikzcd}
\mathcal{B}(K, GD_i) \arrow[r, "\varphi_{D_i K}"] \arrow[d, "\mathcal{B}(K, Gu)"'] & \mathcal{A}(FK, D_i) \arrow[d, "\mathcal{A}(FK, u)"] \\
\mathcal{B}(K, GD_j) \arrow[r, "\varphi_{D_j K}"] & \mathcal{A}(FK, D_j)
\end{tikzcd}
$$

$k_i \mapsto \bar{k}_i$

$k_j \mapsto \bar{k}_j$

$Gu \cdot k_i = k_j$

~~we can obtain $\psi : K \to GL$ by~~

we obtain $\boxed{\overline{\psi} : FK \to L}$ from universal property of (L, ρ_i)

$(\overline{k}_i)$ is a new cone $\Rightarrow$

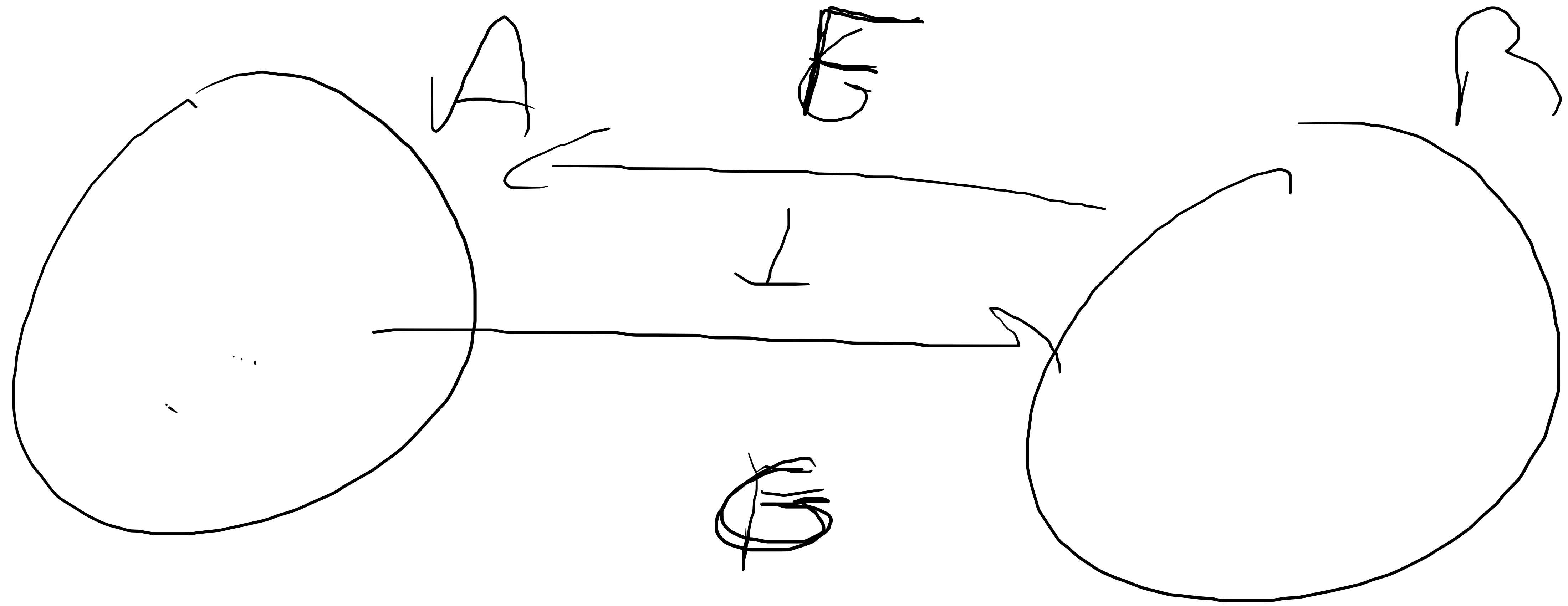
1P $\boxed{\overline{k}_i = \rho_i \cdot \overline{\psi}}$

$\boxed{\psi : K \to GL}$ is the corresponding morphism to $\overline{\psi}$ in the adjoint pair

ψ such that $\boxed{k_i = G\rho_i \cdot \psi}$

TM

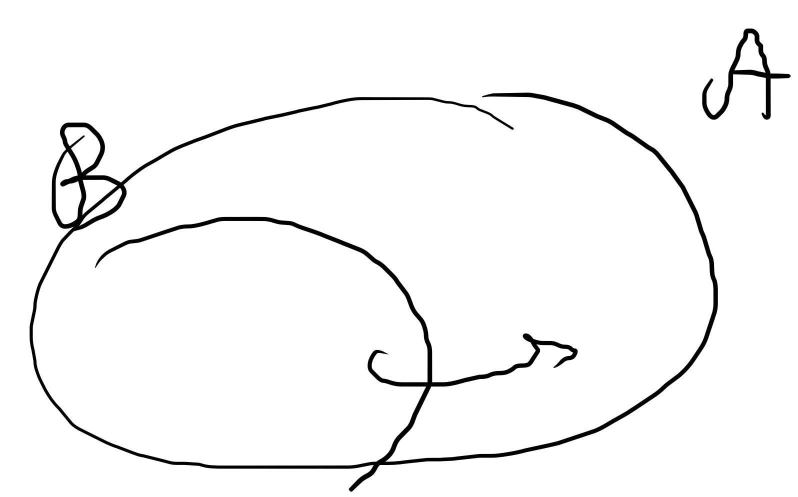
from naturality of ψ



limits go to limits

$$Ab \underset{E}{\overset{F}{\underset{\perp}{\rightleftarrows}}} Grp \qquad Ab_{TF} \underset{E}{\overset{F}{\underset{\perp}{\leftrightarrows}}} Ab$$

$$T_2Top \underset{E}{\overset{F}{\underset{\perp}{\leftrightarrows}}} Top$$



$$B \xrightarrow[E]{} A$$

$$\boxed{Ab \underset{E}{\overset{F}{\underset{\perp}{\rightleftarrows}}} Grp}$$

$(G\cdot)$

$A \subseteq B$

$$F(G) = G/\underbrace{\langle xyx^{-1}y^{-1}\rangle}$$

abelian

subgp. of commutators

REFLECTIVE SUBCAT

$$\boxed{F \dashv E}$$

$$\boxed{\text{Comp } T_2Top \underset{E}{\overset{F}{\underset{\perp}{\rightleftarrows}}} Top}$$

$X \longrightarrow \boxed{X/\sim}$

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