DEF

$\mathcal{A} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{B}$ $F \dashv G$ G is a right adjoint, F — left —

$\forall A \in \mathcal{A}$ $\forall B \in \mathcal{B}$

$\exists$ $\varphi_{A,B} : \mathcal{B}(B, GA) \xrightarrow{\sim} \mathcal{A}(FB, A)$

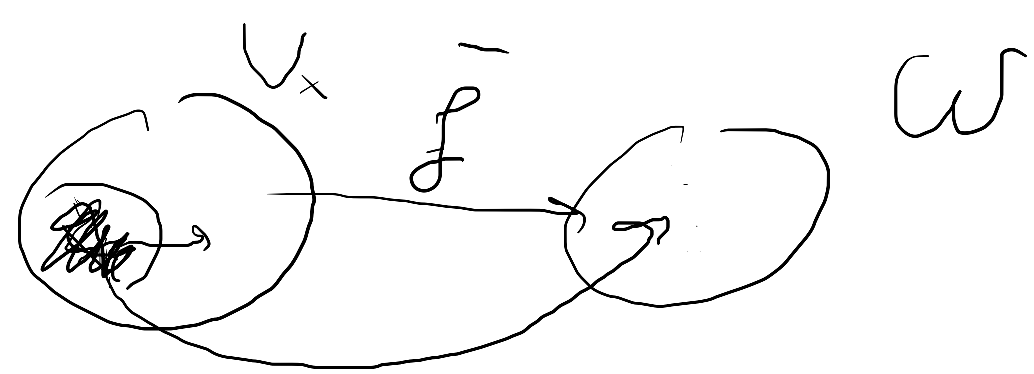
φ bijection

φ is nat in A & in B

$\mathrm{Vect} \underset{G}{\overset{F}{\leftrightarrows}} \mathrm{Set} \ni X$ $F(X) = V_X$

$W \in \mathrm{Vect}$

$X \xrightarrow{\iota_X} GV_X$, Gf, GW, f



V_X f

$\bar{f}$ linear ext.

W $\bar{f}$ $\exists!$

Theorem. TFAE

$$A \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

① $F \dashv G$ ... φ

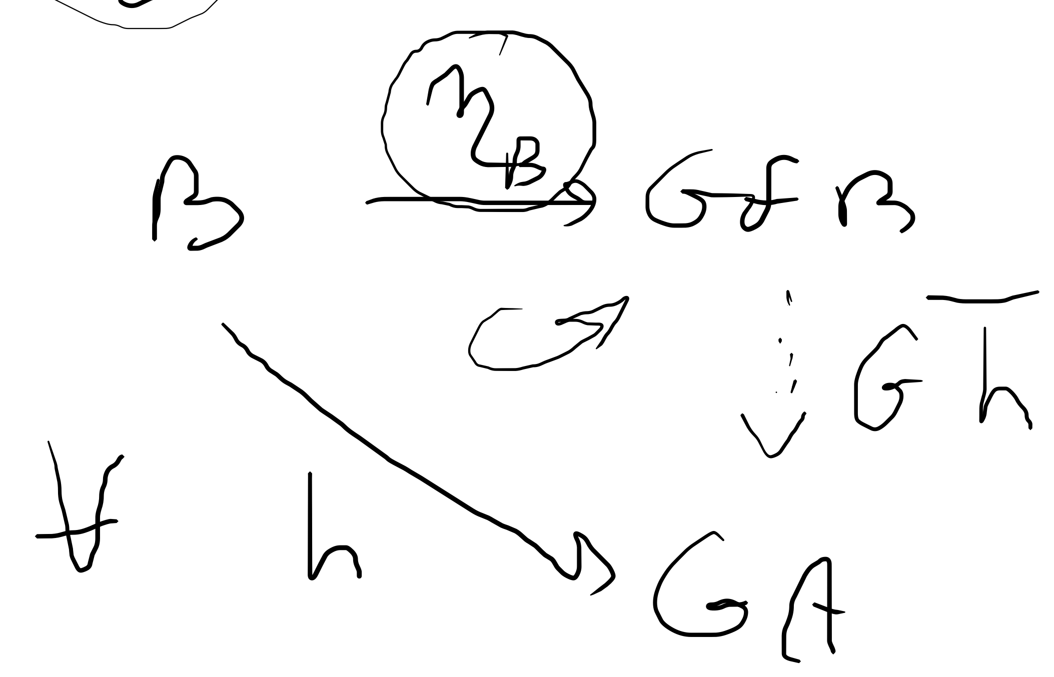
② $\forall B \quad B \in \mathcal{B} \quad \exists$ $\boxed{\eta_B : B \longrightarrow GFB}$ UNIT

s.t. η is natural and universal.

η natural: $\forall\ B \xrightarrow{u} B'$ then must commute

$$\begin{array}{ccc} B & \xrightarrow{\eta_B} & GFB \\ u \downarrow & \circlearrowright & \downarrow GFu \\ B' & \xrightarrow[\eta_{B'}]{} & GFB' \end{array}$$

η is UNIVERSAL



in $\mathcal{B}$

$\forall A \in \mathcal{A}$

$\forall h : B \to GA$

$\exists! \ \bar{h} : FB \to A$

s.th.

$$G\bar{h} \cdot \eta_B = h$$

$$FB \xrightarrow{\ \bar{h}\ \exists!\ } A$$

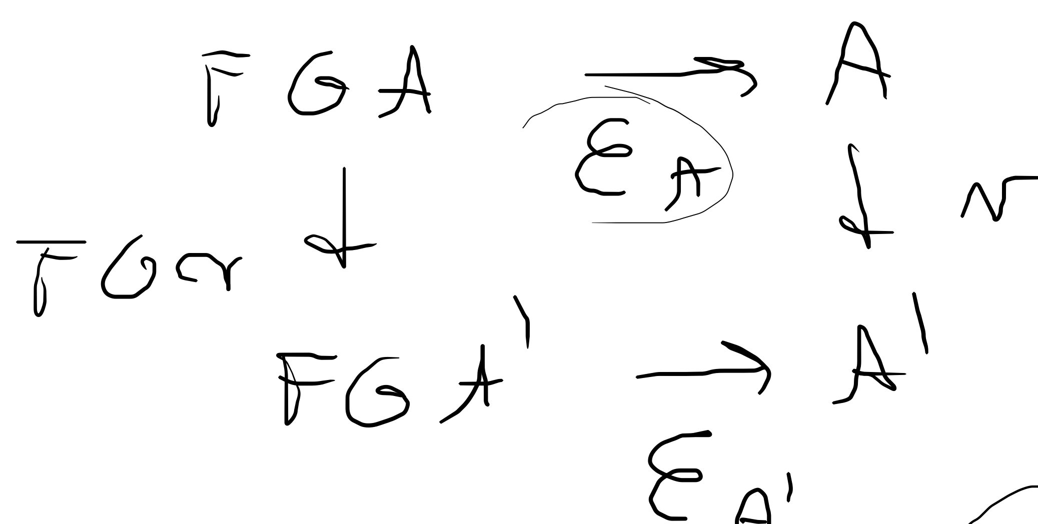
in $\mathcal{A}$

(3) dual of 2

$\forall A \in \mathcal{A}$ $\exists$ $\varepsilon_A : FGA \to A$ **COUNIT** of the adjunction

natural & universal

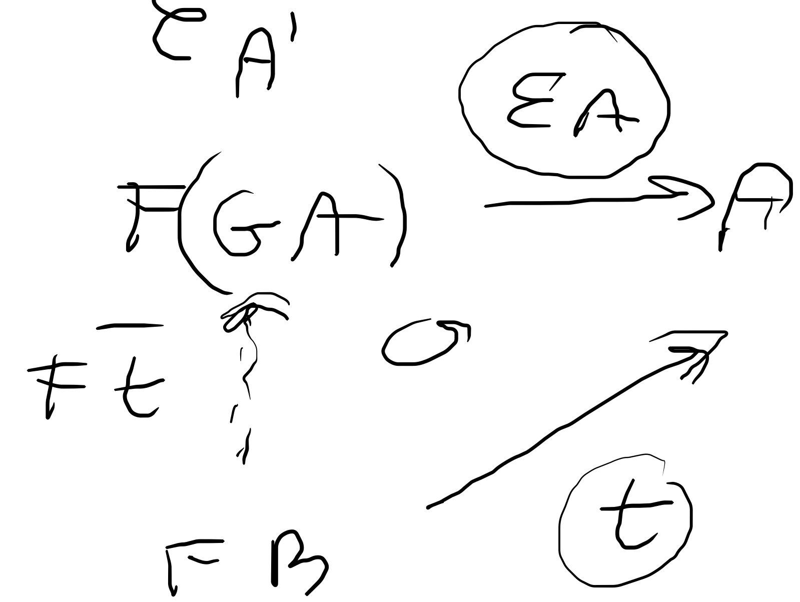
$\forall A \xrightarrow{\alpha} A'$



must commute

universal

$\forall B$ $\forall t : FB \to A$



in A

$\exists! \, \bar{t}$

$B \to GA$

in B

s.t.

$t = \varepsilon_A \cdot F\bar{t}$

4) $\exists$ η and ε natural transformations

$\forall B$ $\quad \eta_B : B \to GFB$ $\qquad \forall A$ $\quad \varepsilon_A : FGA \to A$

s.t. the following triangular identities hold

$$\begin{array}{ccc} GA & \xrightarrow{\eta_{GA}} & GFGA \\ & {\scriptstyle 1_{GA}} \searrow & \downarrow {\scriptstyle G(\varepsilon_A)} \\ & & GA \end{array} \qquad\qquad \begin{array}{ccc} FB & \xrightarrow{F(\eta_B)} & FGFB \\ & {\scriptstyle 1_{FB}} \searrow & \downarrow {\scriptstyle \varepsilon_{FB}} \\ & & FB \end{array}$$

$$\boxed{G(\varepsilon_A) \cdot \eta_{GA} = 1_{GA}} \qquad\qquad \boxed{\varepsilon_{FB} \cdot F(\eta_B) = 1_{FB}}$$

TFAE

(1) $F \dashv G$ (4) φ bij. nat. in A, B

(2) $\eta_B : B \to GFB$ UNIT nat univ

(3) $\varepsilon_A : FGA \to A$ COUNIT nat univ

(4) η_B, ε_A nat.

+ triang. identities

Proof 1) $\Rightarrow$ 2)

Hyp: $\exists\, \varphi_{AB}$ nat & bij $\quad \varphi_{AB} : \mathcal{B}(B, GA) \xrightarrow{\sim} \mathcal{A}(FB, A)$

Th: $\forall B \quad \eta_B : B \to GFB$ nat + univ

Proof $\quad A = FB \quad \mathcal{B}(B, GFB) \xrightarrow{\varphi_{FB,B}} \mathcal{A}(FB, FB)$, $\quad \eta_B \mapsto 1_{FB}$

$$\eta_B \underset{\text{DEF}}{=} \varphi^{-1}_{FB,B}(1_{FB})$$

η is nat

η is univ.

η_B NATURAL we must prove $\forall\, u : B \to B'$

$$\begin{array}{ccc} B & \xrightarrow{\eta_B} & GFB \\ u \downarrow & \circlearrowright & \downarrow GF(u) \\ B' & \xrightarrow[\eta_{B'}]{} & GFB' \end{array}$$

$$\boxed{GF(u) \cdot \eta_B = \eta_{B'} \cdot u}$$

we know that φ is natural

$$\begin{array}{ccc}
\mathcal{B}(B, GFB) & \xrightarrow[\varphi_{FB,B}]{\sim} & \mathcal{A}(FB, FB) \\
\eta_B & \mapsto & 1_{FB} \\
\downarrow \mathcal{B}(B, GFu) & & \downarrow \mathcal{A}(FB, Fu) \\
\mathcal{B}(B, GFB') & \xrightarrow[\varphi_{FB',B}]{\sim} & \mathcal{A}(FB, FB') \\
\uparrow \mathcal{B}(u, GFB') & & \uparrow \mathcal{A}(Fu, FB') \\
\mathcal{B}(B', GFB') & \xrightarrow[\varphi_{FB',B'}]{\sim} & \mathcal{A}(FB', FB') \\
\eta_{B'} & \mapsto & 1_{FB'}
\end{array}$$

? η_B is natural

$B \xrightarrow{u} B'$

in Set

then

$$\begin{array}{ccc}
B & \xrightarrow{\eta_B} & GFB \\
u \downarrow & & \downarrow GFu \\
B' & \xrightarrow{\eta_{B'}} & GFB'
\end{array}$$

$$\begin{array}{ccc}
\eta_B & \longmapsto & 1_{FB} \\
\downarrow & & \downarrow \\
GF(u)\cdot\eta_B & \overset{\varphi}{\longleftrightarrow} & Fu \\
\| & & \uparrow \\
\eta_{B'}\cdot u & & \\
\uparrow & & \\
\eta_{B'} & \longmapsto & 1_{FB'}
\end{array}$$

η_B is UNIVERSAL

thesis ??

$$B \xrightarrow{\eta_B} GFB \qquad \exists!\, G\bar{h} \qquad FB$$

$$\forall h : B \to GA \qquad \exists! \ \bar{h} : FB \to A \ : \ \boxed{G\bar{h} \cdot \eta_B = h} \ \checkmark$$

⊃ define $\boxed{\bar{h} = \varphi_{A,B}(h)}$

true

$$h \in \mathcal{B}(B, GA) \xrightarrow{\varphi_{A,B}} \mathcal{A}(FB, A) \ni \bar{h}$$

$$\mathcal{B}(B, G\bar{h}) \uparrow \qquad \uparrow \mathcal{A}(FB, \bar{h})$$

$$\eta_B \in \mathcal{B}(B, GFB) \xrightarrow{\varphi_{FB,A}} \mathcal{A}(FB, FB) \ni 1_{FB}$$

$$h \longleftrightarrow \bar{h}$$

$$\boxed{G\bar{h} \cdot \eta_B} \uparrow \qquad \uparrow$$

$$\eta_B \longleftrightarrow 1_{FB}$$

$$\boxed{G\bar{h} \cdot \eta_B = h} \ !$$

1 ⇒ 2 OK

$\varphi \Rightarrow \eta$ univ.

2 ⇒ 1

η not univ. ⇒ φ not biject. — in A, in B

proof — we must define φ by having η

?? $\varphi_{AB} : \mathcal{B}(B, GA) \longrightarrow \mathcal{A}(FB, A)$ — $h \mapsto t$

$\varphi(h) \underset{DEF}{=} \bar{h}$

we know — TRUE

$B \xrightarrow{\eta_B} GFB$, Fh: $B \to GA$, $\circlearrowleft$ $G\bar{h}$, Gt; $\exists!\, t \downarrow (\bar{h})$ $FB \to A$

define φ^{-1} ??

$\varphi^{-1}_{AB}(t) \underset{DEF}{=} G(t) \cdot \eta_B$

φ^{-1} is inverse

we want to prove that φ is nat. in A (H1)

we know that η is nat & univ. (1P)

$$
\begin{array}{ccc}
\mathcal{B}(B, GA) & \xrightarrow{\varphi_{AB}} & \mathcal{A}(FB, A) \\
t & \mapsto & \bar{t} \\
\downarrow \mathcal{B}(B, Gu) & & \downarrow \mathcal{A}(FB, u) \\
\mathcal{B}(B, GA') & \xrightarrow{\varphi_{AB'}} & \mathcal{A}(FB, A')
\end{array}
\qquad
\begin{array}{c}
A \\
\downarrow u \\
A'
\end{array}
$$

?

$$
\begin{array}{ccc}
B & \xrightarrow{\eta_B} & GFB \\
 & & \downarrow G\bar{t} \\
 & \xrightarrow{t} & GA \\
 & & \downarrow Gu \\
 & & GA'
\end{array}
= G(u\bar{t})
\qquad
\begin{array}{c}
FB \\
\downarrow \bar{t} \\
A \\
\downarrow u \\
A'
\end{array}
\quad u \cdot \bar{t}
$$

$Gu \cdot t$

$$\varphi_{AB'}(Gu \cdot t) = \, ?$$

$$= u \cdot \bar{t}$$

$1 \longleftrightarrow 3 \qquad \varphi \longleftrightarrow \varepsilon$ COUNIT

$\varepsilon_A : FGA \to A \quad \forall A$

$\underline{1} \Rightarrow 3$

$$\mathcal{B}(GA, GA) \xrightarrow{\varphi_{A,GA}} \mathcal{A}(FGA, A)$$

$$1_{GA} \mapsto \varepsilon_A$$

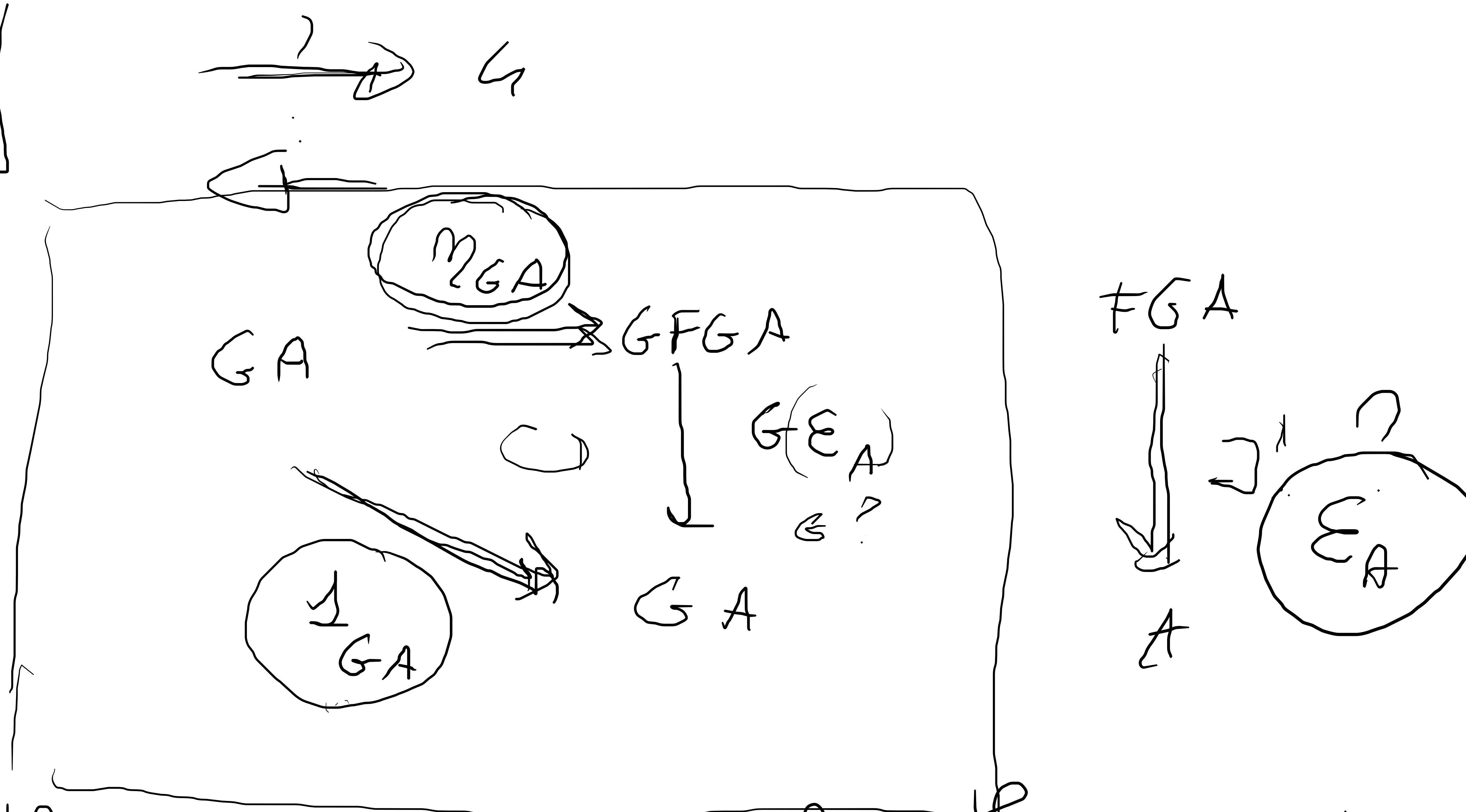
$B = GA$

def $\boxed{\varepsilon_A := \varphi_{A,GA}(1_{GA})}$

$3 \Rightarrow 1$

ε

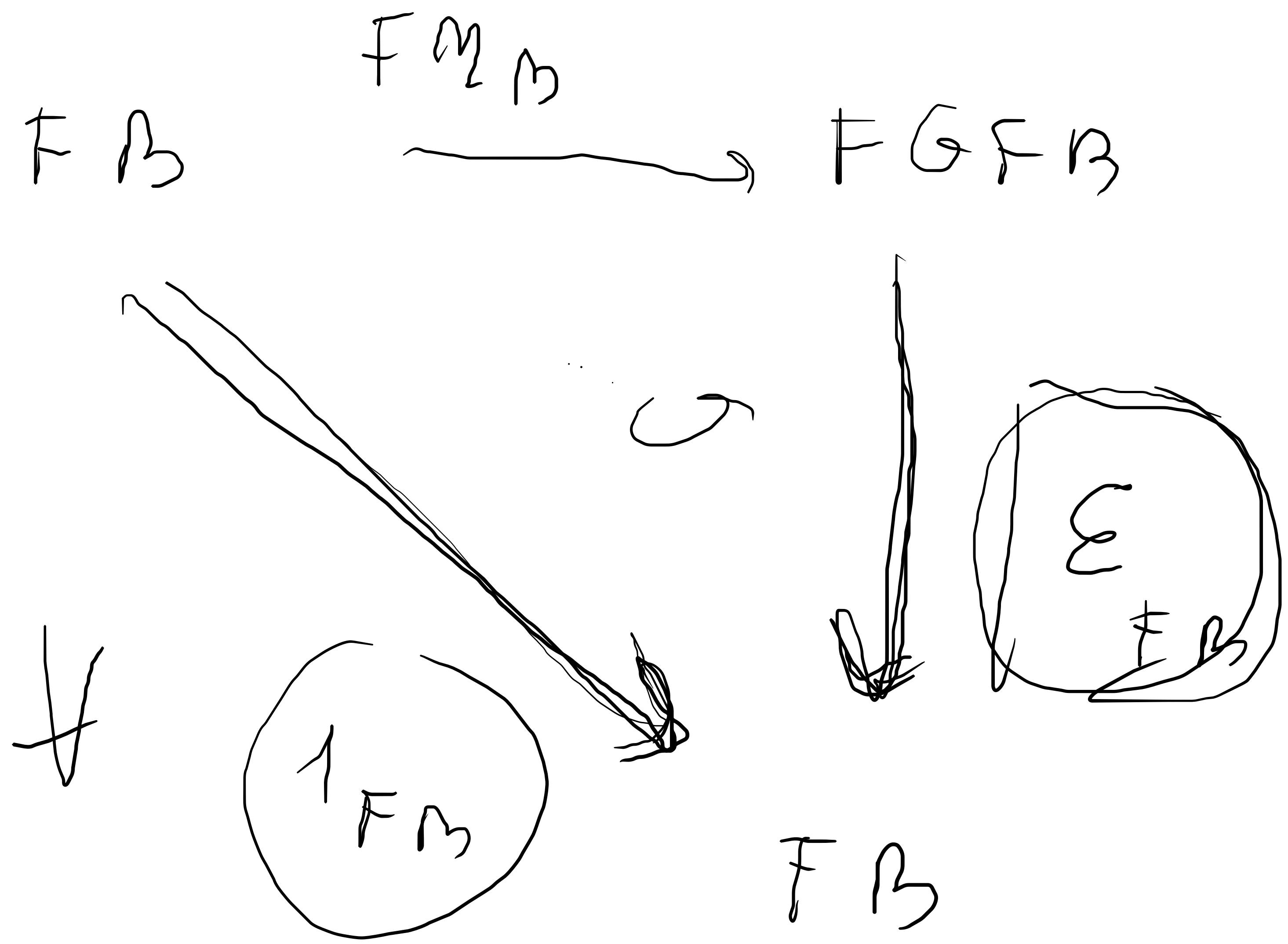
$1 \Leftrightarrow 2 \Leftrightarrow 3$



<u>identities</u>

η univ. → ∀ arrow take 1_{GA}

$1_{GA} \Leftrightarrow \varepsilon_A$ they correspond each other via φ by def of ε_A



ε is univ.

$\forall\, 1_{FB}$ $\left[\ \exists!\ B \xrightarrow{\eta_B} GFB\ \right]$

$1_{FB} \overset{\varphi}{\longleftrightarrow} \eta_B$

by def of η

Conversely $\quad \Leftarrow \quad \Rightarrow \quad 1 + 2 + 3$

η ε natural $\mapsto \varphi$ $\qquad$ must prove that φ is a bijection $\quad \varphi^{-1}$

theorem: the adjoint (if it exists) is unique (up to isomorphism)

hyp: $A \underset{G}{\overset{F,\ \bar F}{\rightleftarrows}} B$ — G has 2 left adjoints $F : B \to A$, $\bar F : B \to A$

$F \dashv G \qquad \bar F \dashv G$

Th: F is isomorphic to $\bar F$ (not iso)

$\forall B \quad FB \xrightarrow[h_B]{\sim} \bar F B$ — h_B is an iso

$\forall\, B \xrightarrow{u} B'$,

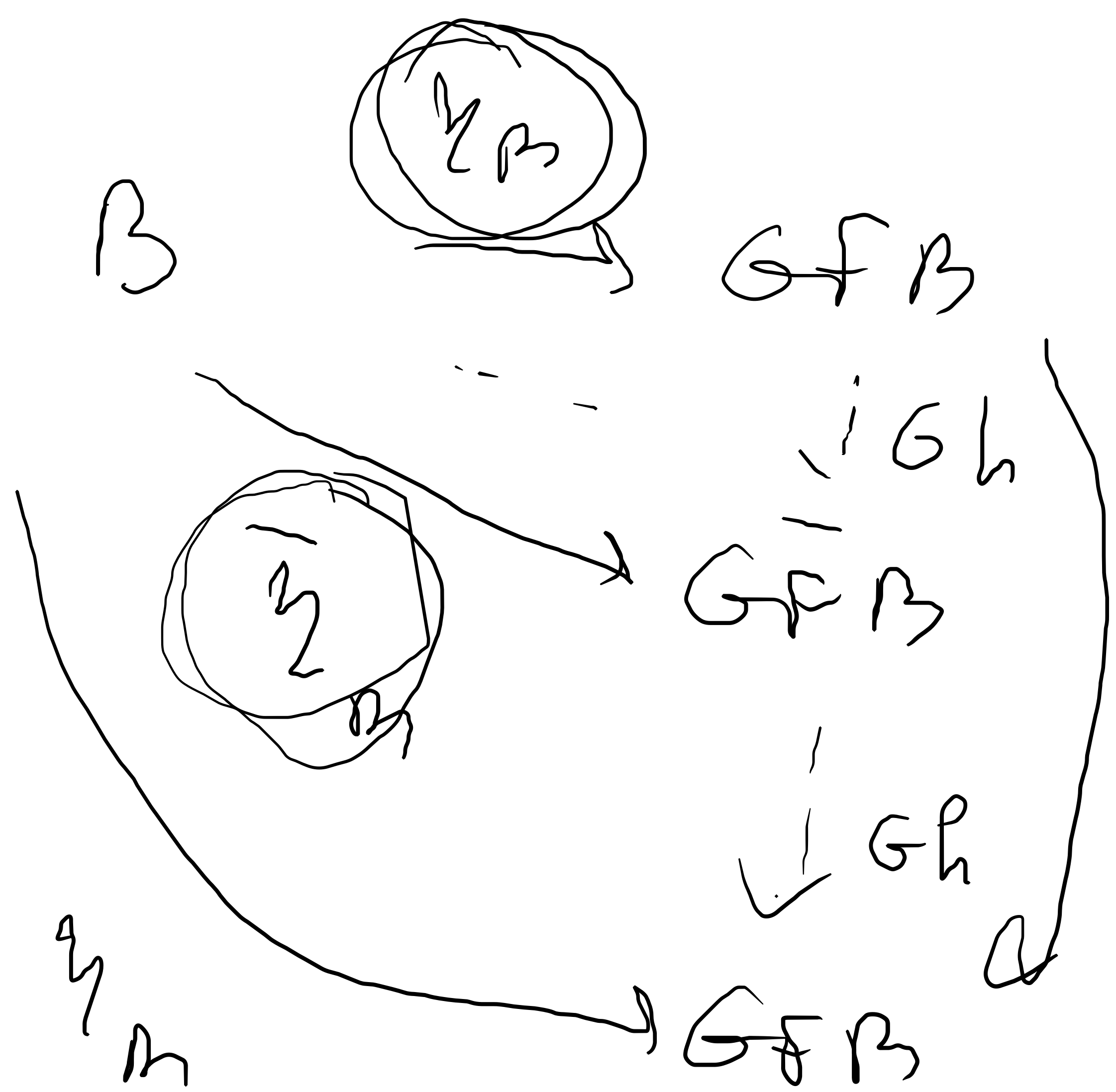
$$\begin{array}{ccc} FB & \xrightarrow{h_B} & \bar F B \\ Fu \downarrow & & \downarrow \bar F u \\ FB' & \xrightarrow{h_{B'}} & \bar F B' \end{array}$$

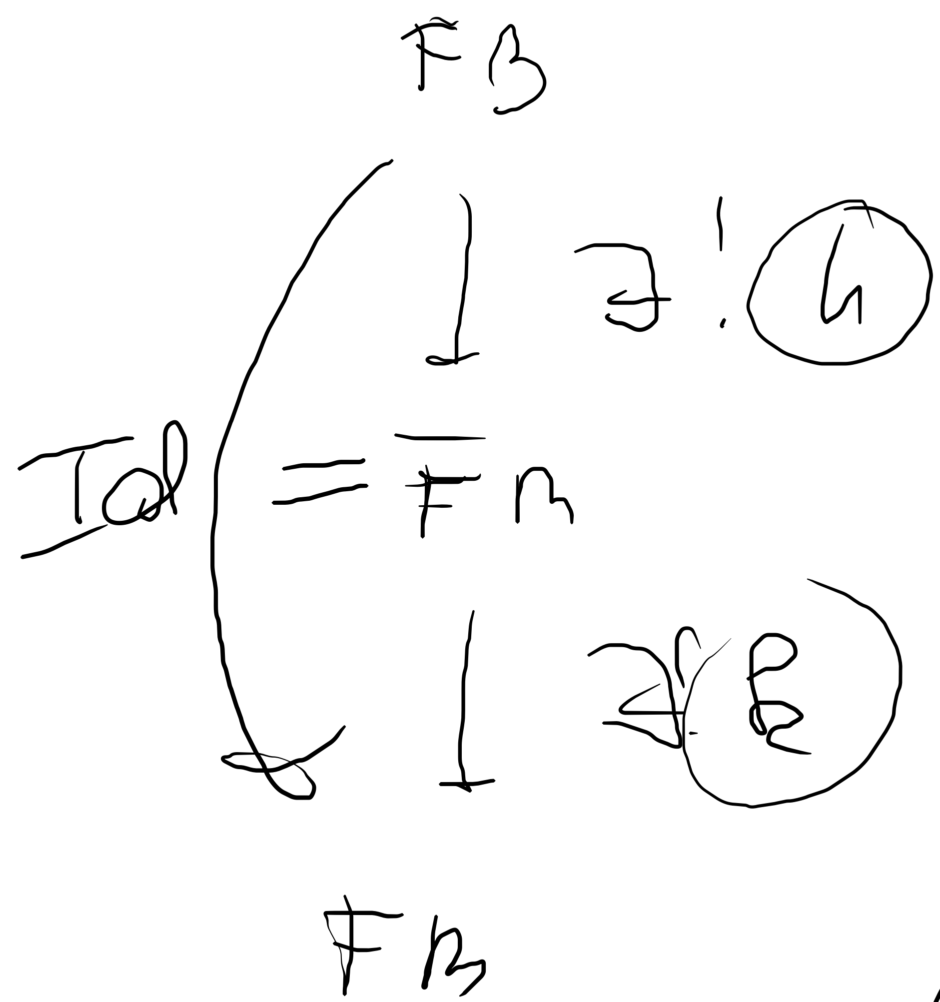
//

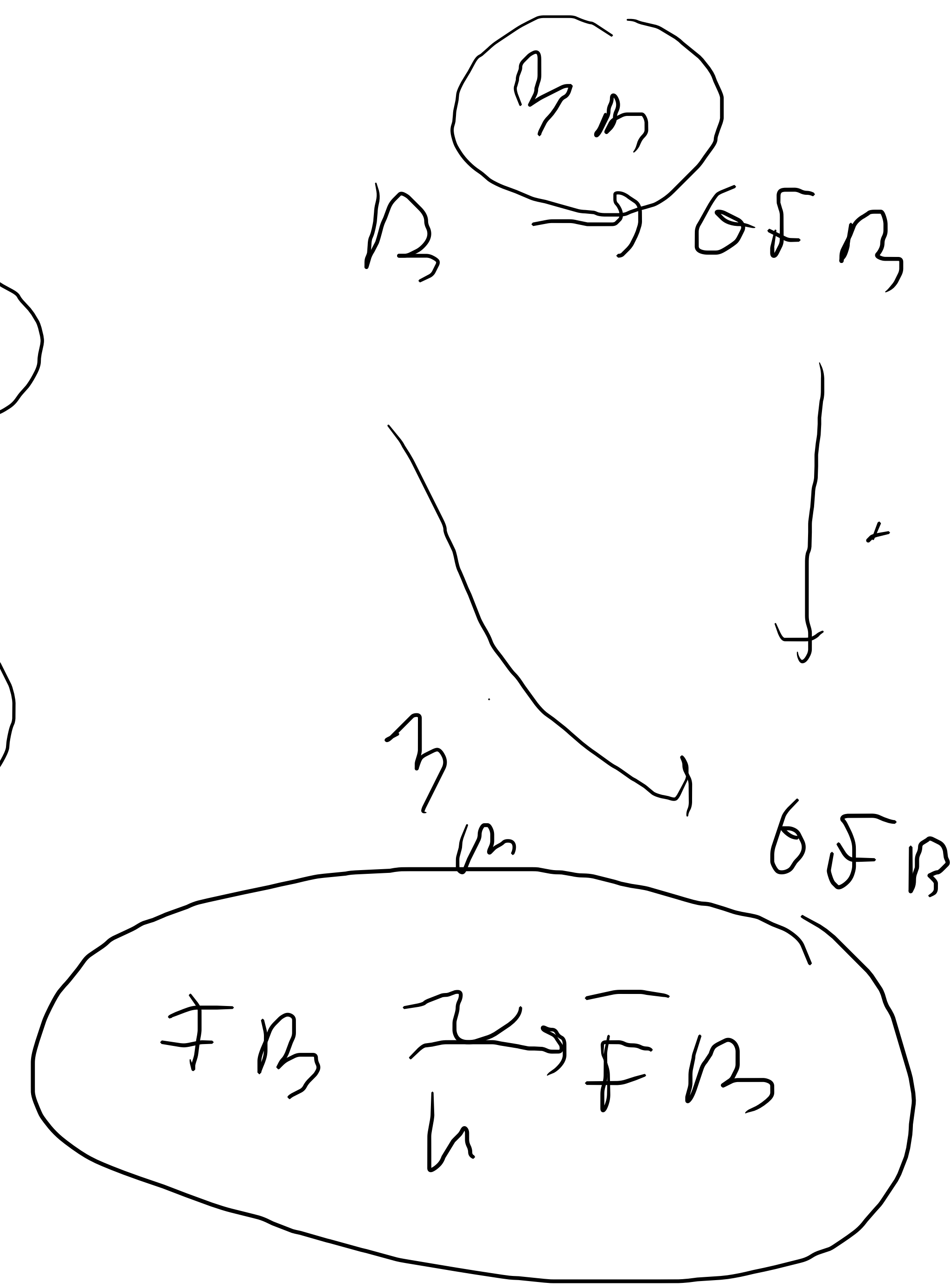
Proof

$\eta_B : B \to GFB$ $\qquad F \dashv G \quad \eta$

$\bar{\eta}_B : B \to G\bar{F}B$ $\qquad \bar{F} \dashv G \quad \bar{\eta}$







verify $h_B : FB \xrightarrow{\sim} \bar{F}B$

it is a NATURAL isom.

① $\varphi \leftrightarrow \eta$ ✓

$\leftrightarrow \varepsilon$

② the adjoints are unique

ex subcategories

$Ab \underset{\bar{F}}{\overset{F}{\rightleftarrows}} Grp$. $\perp$

$Ab \subseteq Grp$

right adj is an inclusion

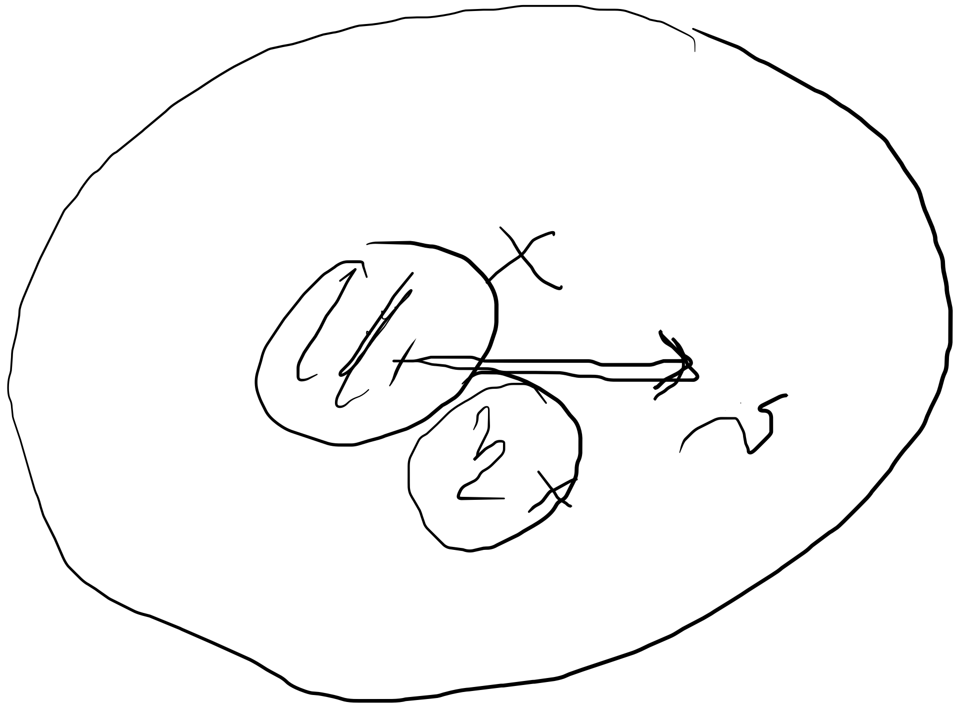
φ nat. bij $\longleftrightarrow$ η nat univ. (UNIT)

φ nat bij $\longleftrightarrow$ (ε) nat univ. COUNIT

$\eta, \varepsilon \longleftrightarrow \varphi, \varphi^{-1}$

nat.

(Δ) $\Longrightarrow$ φ is bijective



V_K

$\sum \lambda_i x_i \quad \lambda_i \in K$

$x_i \in X$

$v = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \ldots$

$x = 1 \cdot x$