$$\begin{tikzcd} A \arrow[r, shift right=2, "G"'] & B \arrow[l, shift right=2, "F"'] \arrow[l, shift right=6, "\overline{F}"'] \end{tikzcd}$$

$F \overset{\eta}{\dashv} G \qquad \overline{F} \overset{\overline{\eta}}{\dashv} G$

$\Rightarrow \boxed{F \overset{\alpha}{\simeq} \overline{F}}$

α is a $\boxed{\text{natural iso}}$

$$\begin{tikzcd} B \arrow[r, bend left, "F"] \arrow[r, bend right, "\overline{F}"'] \arrow[r, phantom, "\Downarrow \cong \alpha"] & A \end{tikzcd}$$

$\forall B \in \mathcal{B} \quad FB \quad \overline{F}B$ in A

$\exists \alpha_B : FB \xrightarrow{\sim} \overline{F}B$

α_B is an iso in A $\forall B$

$\forall B \xrightarrow{u} \overline{B}$ in $\mathcal{B}$

$$\begin{tikzcd} FB \arrow[r, "\alpha_B"] \arrow[d, "Fu"'] & \overline{F}B \arrow[d, "\overline{F}u"] \\ F\overline{B} \arrow[r, "\alpha_{\overline{B}}"'] & \overline{F}\overline{B} \end{tikzcd}$$

$$A \underset{G}{\overset{F}{\leftrightarrows}} B \qquad \underline{\text{Set}}$$

ν is unique

- **<u>Charact Th</u>**

$$\underline{\varphi} \longleftrightarrow \boxed{\eta_B : B \to GFB}$$

UNIT

natural & <u>universal</u>

- G preserves limits

  F ____ colimits

<u>Ex</u>:

$$\text{Grp} \underset{G}{\overset{F}{\leftrightarrows}} \text{Set}$$

$$\begin{array}{c} \xleftarrow{\ \delta\ } \\ \text{Top} \quad \xrightarrow{\qquad} \quad \text{Set} \\ \xleftarrow[\ \cdot\ \nabla\ ]{} \end{array}$$

def Reflective subcategories $\boxed{Ab \subseteq Grp}$

$\mathcal{A} \subseteq \mathcal{B}$ subcat such that

1) $\mathcal{A}$ is a full subcat -

$\forall x, y \in \mathcal{A}$ $\boxed{\mathcal{A}(x,y) = \mathcal{B}(x,y)}$

2) $\mathcal{A}$ is $\boxed{\text{closed w.r. to isomorphisms}}$

if $A \in \mathcal{A}$ $B \in \mathcal{B}$ and

$\exists\, \alpha : A \xrightarrow{\sim} B$ isom

$\Longrightarrow B \in \mathcal{A}$

3) there exists a left adjoint $R : B \to A$ to the inclusion $E : A \to B$

$$\begin{tikzcd} A \arrow[r, bend right, "E"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "R"'] \end{tikzcd}$$

def Coreflective subcategory $A \subseteq B$ iff

① ②

③ $\exists\, S : B \to A \quad E \dashv S$

Exemples

1) $\boxed{Ab \subseteq Grp}$ $(G, \cdot)$

$$\begin{tikzcd} Ab \arrow[r, shift right, "E"'] \arrow[r, phantom, "\perp"] & Grp \arrow[l, shift right, "R"'] \end{tikzcd} \quad G$$

$\boxed{\forall\, G \in Grp \quad R(G) \in Ab}$

Foncteur

$\boxed{q = \eta_G}$

$G \longrightarrow G/[G,G]$

$[G,G]$ commutateur

$x, y \in G$

$\langle x y x^{-1} y^{-1} \rangle$

$\boxed{\text{normal}}$

$\boxed{\eta_G : G \to ER(G)}$

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "\forall f"'] & G/[G,G] \arrow[d, "\bar{f}"] \\ & \cancel{\exists} A \end{tikzcd}$$

$A \in \mathcal{A}b$

~~univ. property~~

$\exists!$ $G/[G,G] \xrightarrow{\bar{f}} A$ $\cancel{\in \mathcal{A}}$

η is universal

B

$$\boxed{\bar{f} \cdot \eta_G = f}$$

$$\boxed{\overline{f}([x]) \underset{DEF}{=} f(x)} \quad \forall x \in G$$

$$\overline{f} \circ \eta_G = f \qquad \text{true} \qquad \eta_G = ?$$

\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dr, "f"'] & {[x]} \arrow[d, "\overline{f}"] \\
 & f(x) = \overline{f}([x])
\end{tikzcd}

verificato

$\overline{f}$ is a homom.

MAINLY

it is well defined

$$\begin{tikzcd} G \arrow[r, "\pi_G"] \arrow[dr, "f"'] & G/[G,G] \arrow[d, dashed, "\exists\, \bar{f}"] \\ & A \end{tikzcd}$$

$f[x] = [y]$ then $\bar{f}[x] = \bar{f}[y]$

$\bar{f}[x] = f(x) \overset{?}{=} f(y) = \bar{f}[y]$

$[x] = [y] \longleftrightarrow x^{-1}y \in [G,G]$

$x^{-1}y = a^{-1}b^{-1}ab$

$x^{-1}y = a^{-1}b^{-1}ab$ não és o comutor

$$\underline{f(x^{-1}y)} = f(a^{-1}b^{-1}ab) = f(a^{-1})\,f(b^{-1})\,f(a)\,f(b)$$

$$= f(a)^{-1}\,f(b)^{-1}\,f(a)\,f(b)$$

A $\underline{\text{abelian}}$ $\Rightarrow$

$$= f(a)^{-1} f(a)\, f(b^{-1})\, f(b) = e_A$$

$$f(x^{-1}y) = e \longleftrightarrow \boxed{f(x) = f(y)}$$

$$f(x^{-1})\,f(y)$$

$G \xrightarrow{\eta_G} G/[G,G]$ best way to make G abelian

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, bend right, "E"'] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, bend right, "R"'] \end{tikzcd}$$

$\boxed{R \dashv E \iff \eta \text{ univ}}$

② $\boxed{\mathrm{Ab}_{TF} \subseteq \mathrm{Ab} \subseteq \mathrm{Grp}}$

all reflective subcats

$(G,\cdot)$ is torsion free iff $\forall$ $\boxed{x^n = e \Rightarrow x = e}$

$\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 = \{0,1\}$ not torsion free

$1+1 = 2 \underset{\mathbb{Z}_2}{=} 0$

$\mathbb{Z}_2 \to \{0\}$ torsion free

$$\boxed{\mathrm{Ab}_{TF} \subseteq \mathrm{Ab}}$$

reflect. sub.

$$\boxed{\eta_A : A \longrightarrow A/\sim}$$

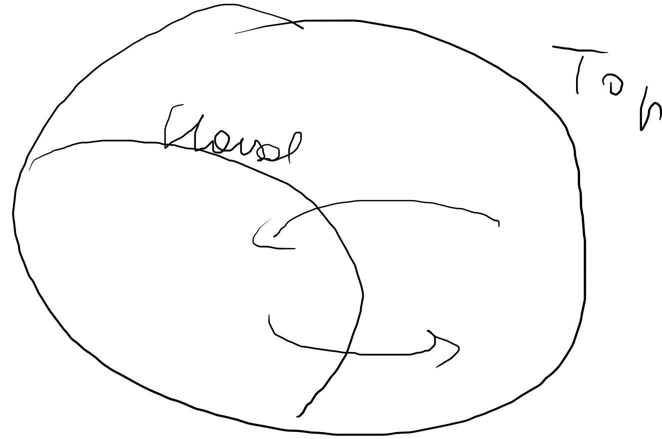
$$\mathrm{Ab}_{TF} \underset{G}{\overset{R}{\leftrightarrows}} \mathrm{Ab}$$

$RA = R/_{\sim}$ $\qquad$ $R \dashv G \iff \eta : R \to R/_{\sim}$

The unit is a $\boxed{\text{regular EP}}$

3

③ $\mathrm{Haus} \subseteq \mathrm{Top}$



$\forall X \in \mathrm{Top} \qquad X \xrightarrow{\eta_X} \boxed{X/\sim}$

equiv rel

\+ quotient topology

$x \sim y$

η_X natural map $\Rightarrow$ $\mathrm{Haus} \underset{E}{\overset{R}{\rightleftarrows}} \mathrm{Top}$ with $\perp$

def epireflective subcats are
$A \subseteq B$ reflective subc. when
η_B is a regular epi.

---

(4)

\begin{tikzcd}
\text{Complete metric spc} \arrow[r, bend right, "E"'] \arrow[r, phantom, "\perp"] & \text{Metric} \arrow[l, bend right, "R"']
\end{tikzcd}

$X \in$ Metic.
completen of $X = RX$

d compactification of a space

R Stone-Čech compactification

$$\begin{tikzcd} \mathrm{Comp\,Haus} \arrow[r, shift right, "E"'] \arrow[r, phantom, "\perp"] & \mathrm{Top}_{3\frac{1}{2}} \arrow[l, shift right, "R"'] \arrow[r, shift right, "E"'] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, shift right, "R"'] \end{tikzcd}$$

$\mathrm{Top}_{3\frac{1}{2}}$ = Tychonoff

$$\begin{tikzcd} \mathrm{Comp\,Haus} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, shift right, "F"'] \end{tikzcd}$$

$\eta_X : X \longrightarrow \overline{X}$

$$\mathrm{Comp\,Haus} \xrightarrow[G]{} \mathrm{Top}$$

$\boxed{\text{G preserves limits}}$

$(X_i)_{i \in I}$ $\quad X_i$ Top. Haus. compact.

$\prod X_i$ Top $\in$ Comp. Haus.

5

Domain of integrity $(D, +, \cdot)$

Q_D field of quotients

$\mathbb{Z} \hookrightarrow \mathbb{Q}$

$\frac{n}{m}$, $m \neq 0$

$$\boxed{D \xrightarrow{\eta_D} Q_D}$$

$$\begin{tikzcd} \text{Fields} \arrow[r, "\perp", "E"'] & \text{Dom}_I \arrow[l, bend right, "F"'] \end{tikzcd}$$

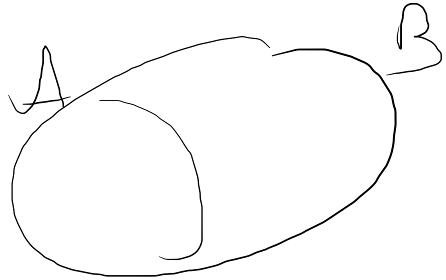
I

only injective homom.

$$
\begin{tikzcd}
D \arrow[r, hook, "\eta_D"] \arrow[dr, "\forall h"'] & R(D) = Q_D \arrow[d, "\exists!\, \bar{h}"] \\
& K
\end{tikzcd}
\qquad \mathbb{Z} \to \mathbb{Q}
$$

<u>injective</u>

field.



$A \subseteq B$ reflective subcategories

$$\begin{tikzcd} A \arrow[r, shift right, "E"'] \arrow[r, phantom, "\perp"] & B \arrow[l, shift right, "R"'] \end{tikzcd}$$

__Theorem.__ Let $A \subseteq B$ be a __reflective subcategory__

then

(1) if B has limits then

A has limits and

they are done as in B

② If $\mathcal{B}$ has colimits then $\mathcal{A}$ has colimits, and they are the reflection of the colimit in $\mathcal{B}$

$$\boxed{Ab_{TF} \subseteq Ab}$$

products in Ab

$$\begin{tikzcd}
Ab_{TF} \ni \boxed{\rho_2 \rightrightarrows (\mathbb{Z}, +)} \arrow[r, "q"] \arrow[dr, "\bar{q}"'] & \mathbb{Z}_2 = \{0,1\} \arrow[d] \\
& \{0\}
\end{tikzcd}$$

Proof (1P) $A \subseteq B$

$$\begin{tikzcd} A \arrow[r, bend right, "E"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "R"'] \end{tikzcd}$$

A reflective subcat.

If $(L, \ell_i) = \lim_B D$

D is in A

Then $\boxed{\lim_B D = \lim_A D}$

D is a diagram in A

This means

$\boxed{L \in A}$

Let $\mathcal{D}$ be a diagram in $\mathcal{A}$

$D_i \longrightarrow D_j$

let $(B, \ell_i) = \operatorname{Lim}_{\mathcal{B}} \mathcal{D}$

??

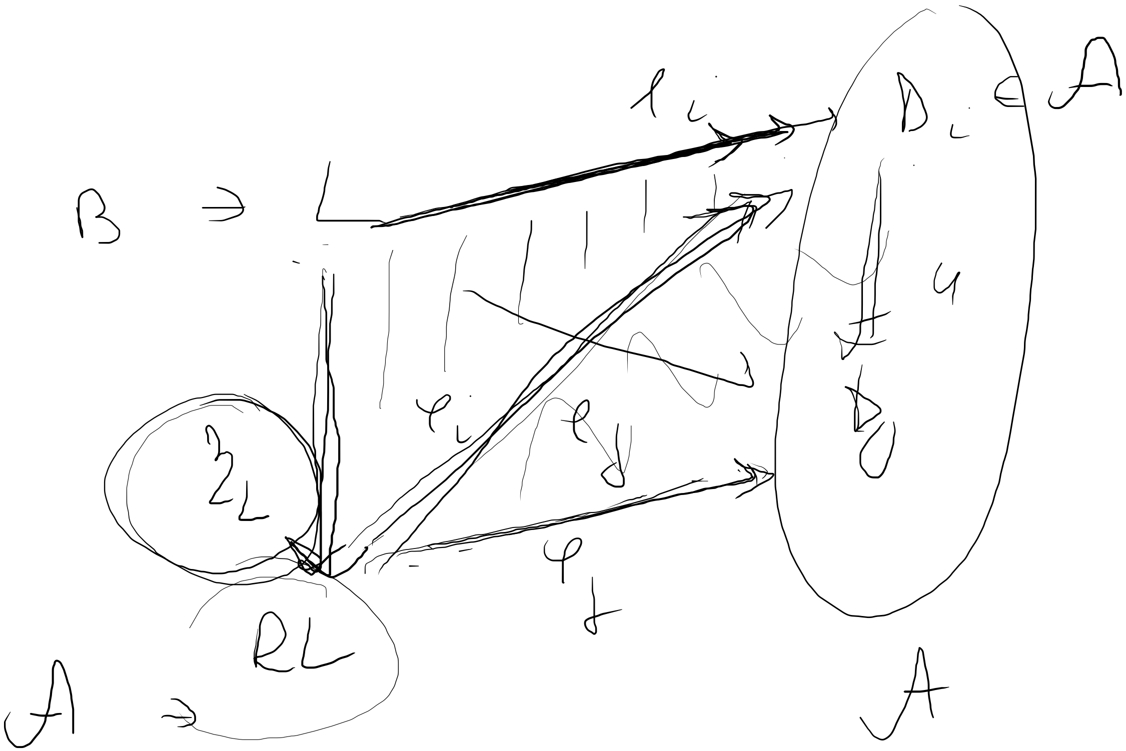
$$\begin{tikzcd}
 & D_i \arrow[dd, "u"] \\
B \ni L \arrow[ur, "\ell_i"] \arrow[dr, "\ell_j"'] & \\
 & D_j
\end{tikzcd}$$

$\mathcal{A}$

$\underline{\text{Th}}$ $L \in \mathcal{A}$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook] & \mathcal{B} \arrow[l, bend right, "R"']
\end{tikzcd}$$

$\eta_B : B \to RB$



η_L is universal

$D_i \in A$

$\exists!\ \varphi_i : RL \longrightarrow D_i$ s.that $e_i = \varphi_i \cdot \eta_L$

$\forall i$

φ_i form a cone from RL to D

$u \varphi_i = \varphi_j$
$\forall u$

$\boxed{\varphi_j \cdot \eta_L = \ell_j}$ $\boxed{\varphi_i \cdot \eta_L = \ell_i}$

ℓ_i is a cone $\quad u\,\ell_i = \ell_j \quad$ $\forall$ cones

Yes! $\boxed{\varphi_i \text{ is a cone}}$

?

$\boxed{u\,\varphi_i = \varphi_j}$

Proven universal property of η_L is the unique

arrow s.t. $\boxed{\varphi_j \cdot \eta_L = \ell_j}$

$\exists u\, \varphi_i \cdot \eta_L =$

$u\,\ell_i = \ell_j$

univ. cone

$$\begin{tikzcd}
L \arrow[r, "\ell_i"] \arrow[d, "\eta"] \arrow[dd, bend right=60, "Id"'] & D_i \\
RL \arrow[ur, "\varphi_i"] \arrow[d, "\psi"] & \\
L \arrow[uur, bend right, "\ell_i"', "\text{cone}"] &
\end{tikzcd}$$

by univ. property of (L, ℓ_i) cone, $\exists$

$\psi : RL \to L$ such:

$$\boxed{\ell_i \psi = \varphi_i}$$

$$L \xrightarrow{\eta} RL \xrightarrow{\psi} L$$

$$\boxed{\psi \eta = 1_L}$$

by the univ. property of L

LEMMA:

$\eta_{L} \downarrow$

RL

$\boxed{\psi \cdot \eta_L = 1_L}$ true

η_L has a left inverse

$\Longrightarrow$ η_L will be an iso.

You must prove $\boxed{\eta_L \cdot \psi = 1_{RL}}$

$\boxed{\text{use the universality of } \eta}$ ....