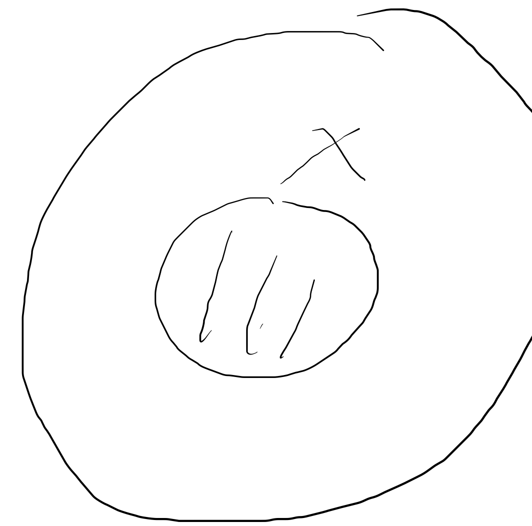
$$\text{metric } X \xrightarrow{\eta_X} RX \text{ complete}$$

$$\boxed{\mathbb{Z} \rightarrowtail \mathbb{Q}}$$

dense complete

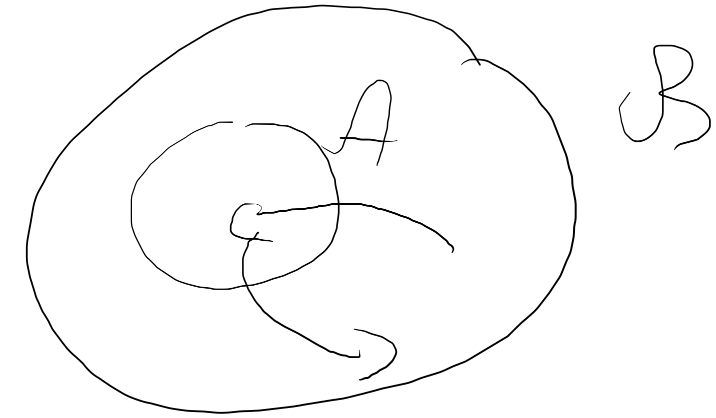
$$\begin{tikzcd} X \arrow[r, tail] \arrow[dr] & RX \arrow[d] \\ & Y \text{ complete} \end{tikzcd}$$

$$\overline{X} = RX$$



$$G \xrightarrow{\eta_G} G/[G,G]$$

$A \subseteq B$ reflective subcat



$$\begin{tikzcd} A \arrow[r, bend right, hook, "E"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "R"'] \end{tikzcd}$$

$R \dashv E$

left adjoint

Def $\mathcal{C}$ is called COMPLETE iff $\mathcal{C}$ has all limits

$\mathcal{C}$ — — COCOMPLETE iff has all colimits

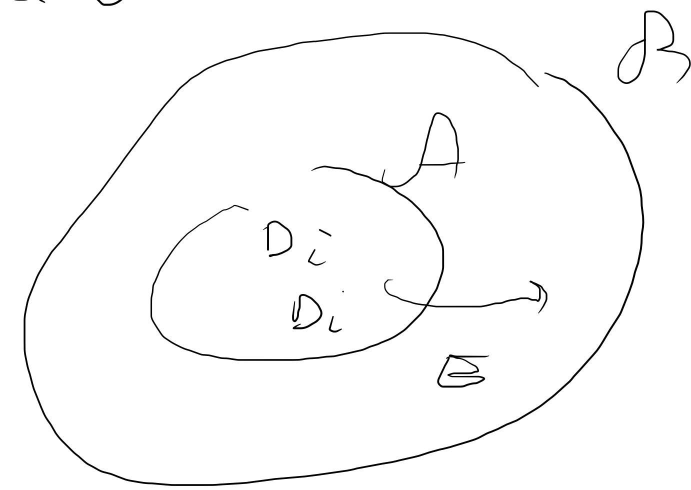
Th $A \subseteq B$ A reflective subc.

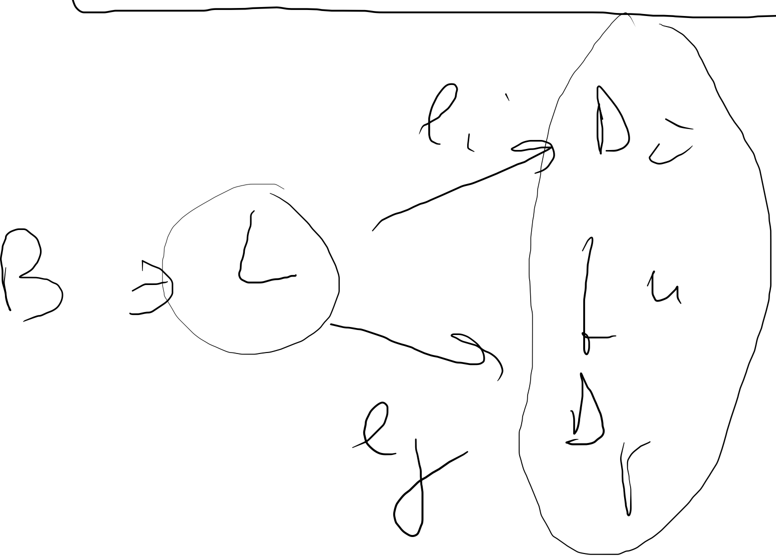
1) If B is complete then A is complete and limits in A are done as limits in B

2) If B is cocomplete then A is complete and colimits in A are the reflection of colimits in B

Proof ① $\mathcal{D}$ is a diagramme in $\mathcal{A}$

$$\boxed{(L, \ell_i) = \lim_{\mathcal{B}} \mathcal{D}}$$





~~$\exists\, \mathcal{D}$~~

~~$\exists\, D_i$~~ $\ni u$

~~$\exists\, D_j$~~

??

$$\boxed{(L, \ell_i) = \lim_{\mathcal{A}} \mathcal{D}}$$

to prove that $L \in A$

$$\begin{tikzcd}
B \ni L \arrow[r, "l_i"] \arrow[d, shift left, "\eta_L"'] & D_i \in A \\
A \ni RL \arrow[u, shift left, "\psi"'] \arrow[ur, "\varphi_i"] &
\end{tikzcd}$$

η univ $\rightarrow \exists\, \varphi_i \;\; \forall i \in I$

φ_i is a cone from RL to D

(L, l_i) is a limit $\rightarrow$

$\exists\, \psi : \varphi_i = l_i \cdot \psi$

$\underline{L \text{ is a limit}} \rightarrow \psi \cdot \eta_L = l_L$

we need prove $\boxed{\eta_L \cdot \psi \overset{?}{=} 1_{RL}}$

use η_L universal

$$\begin{tikzcd}
L \arrow[r, "\eta_L"] \arrow[dr, "1_L"'] \arrow[ddr, bend right, "\eta_L"'] & RL \arrow[d, "\psi"] \arrow[dd, bend left=60, "1_{RL}"] \\
 & L \arrow[d, "\eta_L"] \\
 & RL
\end{tikzcd}$$

$\exists!$ factors

→ consider the following diagram

by unicity of the fact. we get

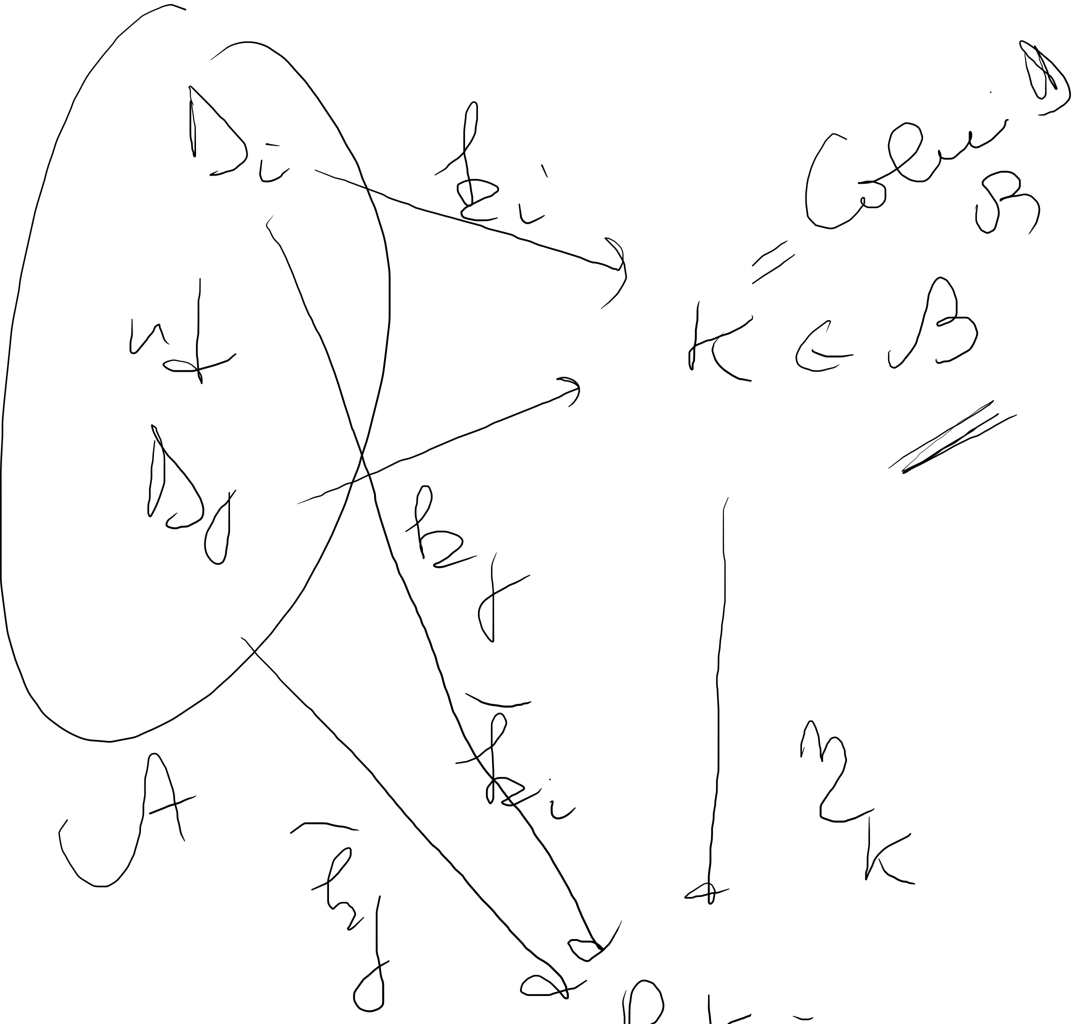
$\eta_L \cdot \psi = 1_{RL}$

Lemma. If $\eta_X : X \to RX$ has a left inverse then η_X is an isomorph.

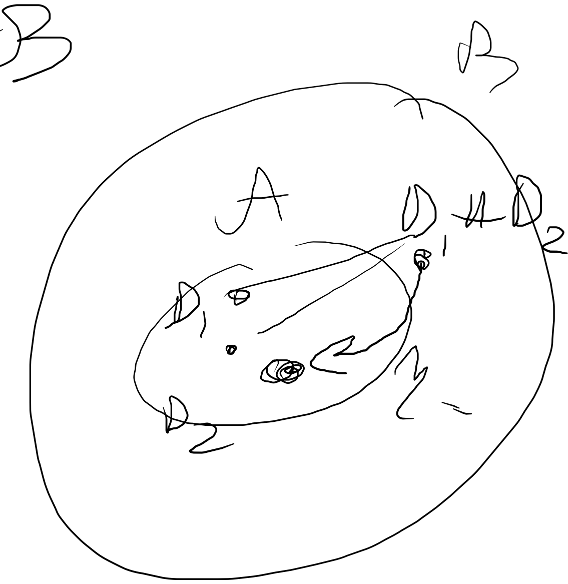
(2) cocomplete B cocompl $\Rightarrow$ A is complete

take a diagram D in A

consider the colim in B



$$(K, k_i) = \mathrm{Colim}_B \overbrace{D}^{E}$$



comp

$$\boxed{\bar{k}_i = \eta_K \cdot k_i}$$

$RK = \mathrm{Colim}_A D$ of results

$$(RK, \bar{k}_i) = \mathrm{Colim}_A D$$

Apply $\boxed{\text{left adjoint preserves colimits}}$

$$(K, k_i) = \mathrm{Colim}_{\mathcal{B}} D$$

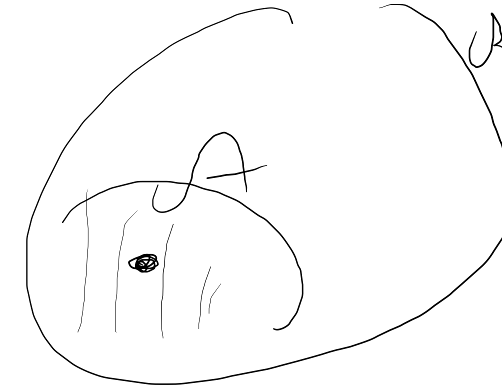
apply R left adjoint

$$(RK, Rk_i) = \mathrm{Colim}_{\mathcal{A}} RD$$

$$(RK, \eta_K \cdot k_i) = \mathrm{Colim}_{\mathcal{A}} D$$

we want this

$$\boxed{RD \cong D \iff D \in \mathcal{A}}$$



Observ. If $A \in \mathcal{A}$ then $RA \cong A$

Proof

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\eta_A"'] & A \in \mathcal{A} \\ RA \arrow[ur, "\exists ! \psi"'] & \end{tikzcd}$$

by univ. prop. of η_A and $A \in \mathcal{A}$

$\exists \psi : \psi \eta_A = 1_A$

then by previous lemma

$\Longrightarrow \eta_A$ is an iso $A \cong RA$

$$(RK, Rk_i) = \operatorname{colim}_{A} RD \simeq \operatorname{colim}_{A} D$$

$$\begin{matrix} \| \\ (RK, \eta_K \cdot k_i) \end{matrix}$$

from naturality of η

$$\begin{tikzcd} D_i \arrow[r, "k_i"] \arrow[d, "\eta_{D_i}"'] & K \arrow[d, "\eta_K"] \\ RD_i \arrow[r, "Rk_i"'] & RK \end{tikzcd}$$

$$\Longrightarrow \boxed{\eta_K \cdot k_i \simeq Rk_i}$$

Compact T_2 Spaces $\hookrightarrow$ Top

$X_i \in$ Top $\qquad$ $X_i \in$ CompT_2

$\Rightarrow$ $\underbrace{\prod X_i}_{\text{top. product}}$ is compact & T_2

$Ab_{TF} \subseteq Ab$

$\rho_2 \Rightarrow$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "q"] \arrow[dr, "\bar{\rho}"'] & \{0,1\} \arrow[d, "\eta"] \\ & \{0\} \end{tikzcd}$$

not torsion free

quotient in Ab

quotient $\bar{\rho}$ in Ab_{TF}

Example of adjoints

Limits and colimits are adjoints

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[r, shift right, "\pi"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "\Delta"'] \end{tikzcd}$$

$(X,Y) \mapsto$ $\qquad\qquad$ $\mapsfrom A$

$(X,Y) \in \mathcal{C}\times\mathcal{C}$

$\pi(X,Y) = X\times Y$ categorical product

$\Delta(A) = (A,A)$

then $\boxed{\Delta \dashv \pi}$

adjoint law —

$$\mathcal{C}\times\mathcal{C}\left(\Delta A, (X,Y)\right) \xrightarrow[\varphi_{A,(X,Y)}]{\sim} \mathcal{C}(A, X\times Y)$$

$$\overset{h}{\curvearrowright}$$

$$= \mathcal{C}\times\mathcal{C}\left((A,A)\ (X,Y)\right)$$

$$\Delta \dashv \Pi$$

$$(h_X, h_Y) \in \mathcal{C}\times\mathcal{C}\left(\Delta A, (X,Y)\right)$$

$$\begin{tikzcd}
& X \\
X\times Y \arrow[ur, "p_X"] \arrow[dr, "p_Y"'] & \\
& Y \\
A \arrow[uu, "h"] \arrow[uuur, "h_X"] \arrow[ur, "h_Y"']
\end{tikzcd}$$

$$(h_X, h_Y) \longleftrightarrow h$$

univ. property of the product

Lim $\mathbb{D}$ is a right adjoint

we must think $\mathbb{D}$ as functors from an index category I to $\mathcal{C}$

$$\bullet \rightrightarrows \bullet$$

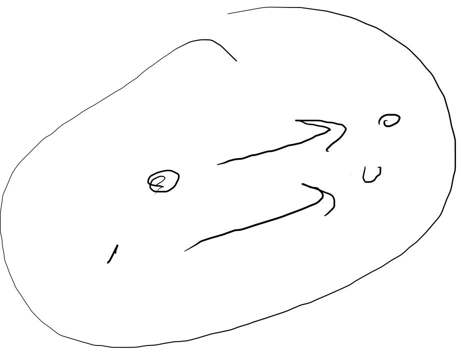
for equaliser

$\underline{I}$ is the index category

$$\bullet \rightrightarrows \bullet \;=\; \underline{I}$$

$I \xrightarrow{D} \mathcal{C}$

any functor is a Diagram in $\mathcal{C}$



$c_0 \rightrightarrows c_1$

$\mathcal{C}^I$

objs functors $I \xrightarrow{D} \mathcal{C}$

morph. nat. transf.

$\mathcal{C}^I \xrightarrow[\lim_{\mathcal{C}}]{} \mathcal{C}$ is a functor

$\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^I$

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{I}$$

$x \rightsquigarrow$ diagramma dal diagramma in x avente solo l'identità.

$I = \bullet \rightrightarrows \bullet$ $\qquad$ $\Delta x = x \overset{1_x}{\underset{1_x}{\rightrightarrows}} x$

$$\mathcal{C}^{I} \underset{\lim_{\mathcal{C}}}{\overset{\Delta}{\leftrightarrows}} \mathcal{C} \qquad \perp$$

$$\Delta \dashv \mathrm{Lim}_{\mathcal{C}}$$

limits are right adjoints to Δ

colimits are left adjoints to Δ

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathcal{C}\times\mathcal{C} \arrow[l, bend right=60, "\sqcup"'] \arrow[l, bend left=60, "\Pi"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta"'] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathcal{C}^{I} \arrow[l, bend right=60, "\mathrm{Colim}"'] \arrow[l, bend left=60, "\mathrm{Lim}"]
\end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend right, "E"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "R"'] \end{tikzcd}$$

<u>reflective sub</u>

categorical language

<u>cat.</u> <u>functor</u> <u>nat. trans</u>

limits

adjoint functor

What is an algebraic category?

algebras (universal algebra theory)

$(X, \boxed{\text{operations}}, \boxed{\text{axioms are equations}})$

set

$\forall x, y \quad x \cdot y = y \cdot x$

$X \times X \to X \qquad X \to X$

$X \times X \times X \to X \qquad \{0\} \to X$

ex of torsion free

$x \neq e$ $\quad \boxed{x^n = e \Longrightarrow n = 0}$

$x \in G \quad n \in \mathbb{N}$

multiplicative axiom

$$\boxed{\{x, y, t\} \quad x(yt) = (xy)t}$$

associat. $\quad (G, \cdot)$

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\cdot \times G"] \arrow[d, "G \times \cdot"'] & G \times G \arrow[d, "\cdot"] \\
G \times G \arrow[r, "\cdot"] & G
\end{tikzcd}$$

What is a categ description of an "algebraic" category

Group

$\mathbb{Z}$

different approaches

- algebra = exact + a projective generating obj

- T-algebras for a triple T