REGULAR CATEGORIES

<u>regular</u> – <u>exact</u> categories

$xy = yx \quad \forall x, y \in X$

<u>def</u> $\mathcal{C}$ category – $\mathcal{C}$ has finite limits and colimits

$\mathcal{C}$ is called a regular category iff

$\forall f : X \to Y$ regular epi in $\mathcal{E}$

$\forall g : T \to Y$ morphism

consider the pull-back of f and g

\begin{tikzcd}
P \arrow[r, dashed, "\bar{g}"] \arrow[d, two heads, "\bar{f}"'] \arrow[dr, phantom, "pb"] & X \arrow[d, two heads, "f"] \\
T \arrow[r, "g"'] & Y
\end{tikzcd}

then $\bar{f}$ is a regular epi

$\boxed{\text{Notation } X \overset{f}{\twoheadrightarrow} Y \text{ for reg epi}}$

$\boxed{\text{Theorem.}}$ If $\mathcal{C}$ is <u>regular</u> then

$\forall f : X \to Y$ morphism in $\mathcal{C}$

$\exists\, q : X \twoheadrightarrow Q$ reg epi, $\exists\, m : Q \rightarrowtail Y$
mono

such that

$$\boxed{f = m \cdot q}$$

and the factorization is unique (up to isomorph)

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\
& Q \arrow[ur, "m"'] &
\end{tikzcd}$$

reg epi

m mono

Grp / Ab / Set / Rng

$\boxed{\text{reg epi} \equiv \text{surj homom}}$

Proof

$$X \xrightarrow{f} Y$$

Set $\supset$ define an equiv. relat on X

$$x \sim x' \iff f(x) = f(x')$$

Cat theory define R as the p.back of kernel pair of f

$$(f, f)$$

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\text{pb}"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

kernel pair

R equiv. rel. complete

$R \hookrightarrow X \times X$

$$\begin{tikzcd} R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "f"] \arrow[dr, two heads, "p"'] & Y \\ & & Q = X/R \arrow[u, "m"'] \end{tikzcd}$$

$(Q, p) = \mathrm{Coeq}(u, v)$

$[x] \mapsto f(x)$

p is universal

$fu = fv \longrightarrow$ key univ. property of p

$\exists !\, m$

$\boxed{mp = f}$

Example of reg. categories

Set

$$\begin{tikzcd} P \arrow[r, "\bar{f}'"] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "pb"] & X \ni x \arrow[d, "f"] \\ t \in T \arrow[r, "g"'] & Y \end{tikzcd}$$

f reg epi $\equiv$ surjective

$P = \{(x,t) : f(x) = g(t)\}$

Set is regular iff (f is a reg epi $\Rightarrow$ $\bar{f}$ is reg epi)

$\bar{f}$ surj : $\forall t \in T \; \exists q \in P : \bar{f}(q) = t$

$g(t) = y = f(x) \quad \exists x \in X \quad \Rightarrow \quad (t, x) \in P$

$(t, x) = q$

Grp / Ab / Rng / Latt / Bool

the same applies

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "pb"] & X \arrow[d, "f"] \\ T \arrow[r, "g"'] & Y \end{tikzcd}$$

$\Rightarrow$ $\bar{f}$ is surj. $\bar{f}$ is hom.

$\boxed{\text{reg epi} = \text{surj. hom.}}$

$$\boxed{\mathrm{Grp} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Set}}$$

U preserve limits

pb in Group or others

as in Set

(4) Every surjective homom is a reg. epi

$$\begin{tikzcd} R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, two heads, "f"] \arrow[d, "e"'] & Y \\ & Q \arrow[ur, "m"'] & \end{tikzcd}$$

kernel pair

$Q = \mathrm{Coeq}(u, v)$

$f = \mathrm{Coeq}(u, v) = X/R$

m mono

f surjective homom.

$\Longrightarrow$ m is surjective

m is surjective — m is injective

$\Longrightarrow$ m is an isom.

Set - Grp - Ab regul cates

$\boxed{\text{Top is NOT a \underline{regular cat}}}$

Counterex

F. Borceux
Handbook of Categorical algebra

$A = \{a, b, c, d\}$

$B = \{l, m, n\}$

$C = \{x, y, z\}$

$\rho \dashrightarrow^{\bar f}$ $\{a, b, c, d\} = A$

$\bar f$ is not a map

$\{a,b,c,d\} = A \xrightarrow{f} \{x, y, z\} = C$ not op.

$\{e, m, n\} = B \xrightarrow{g} \{x, y, z\} = C$

f and g are continuous

in A: $\{a, b\}$ open

in B: $\{e, m\}$ open

C has the indiscrete top.

definition of $f : A \to C$

$f(a) = x$ $\quad$ $f(b) = y = f(c)$ $\quad$ $f(d) = z$

f is cont., $\boxed{\text{is a repelr}}$ $\quad$ C has the feeial topology

$g : B \to C$

$g(e) = x$ $\quad$ $g(m) = z = g(n)$

we must construct the p.b. $(P, \bar{g}, \bar{f})$

of (f, g)

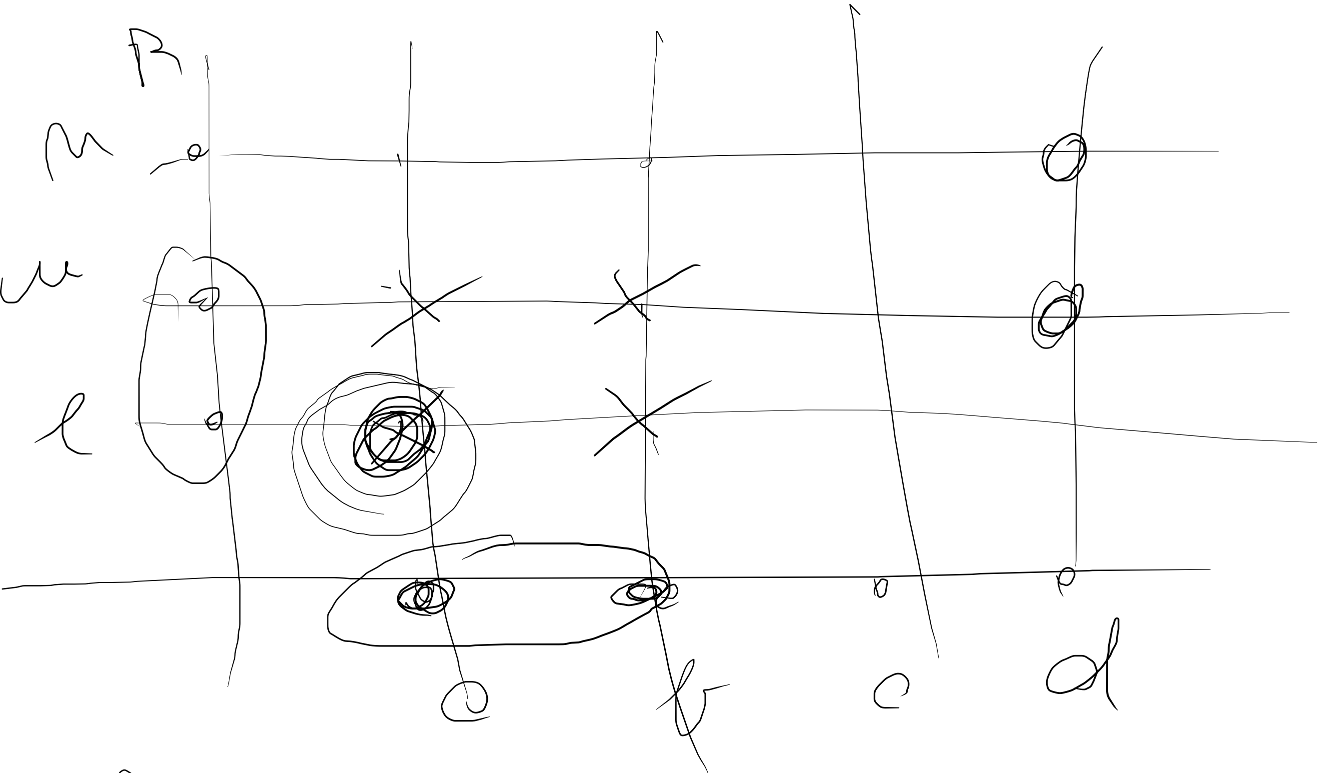
$A \times B = \{ - - - - - - \}$ 12 elements

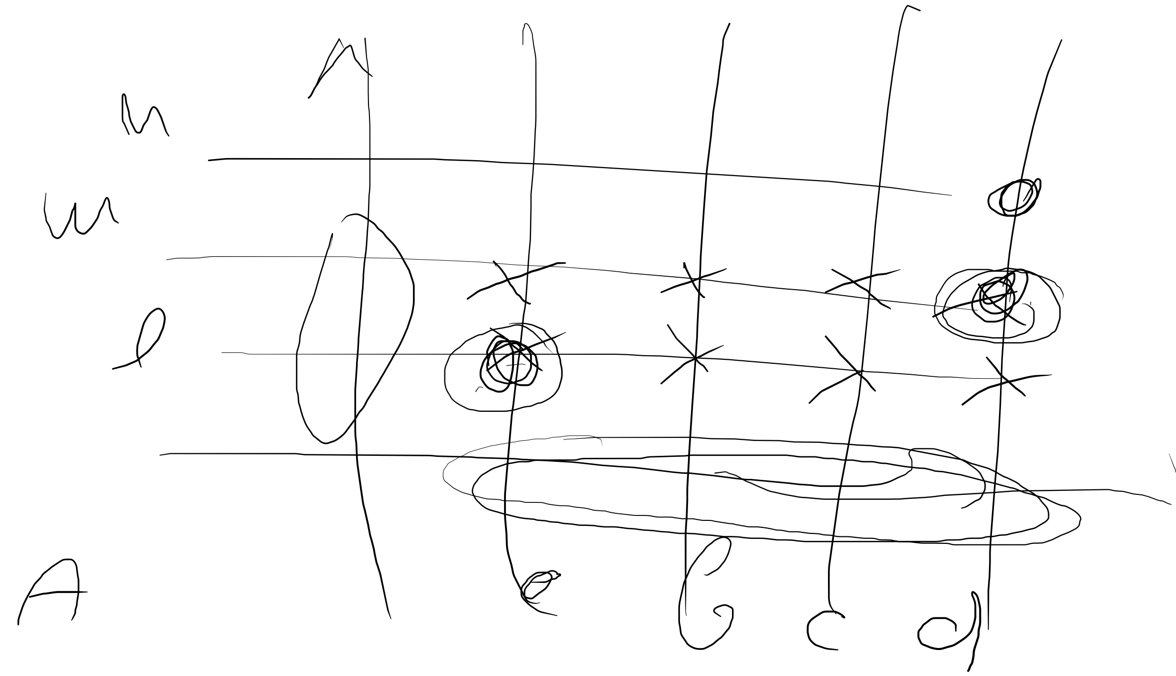
with product topology

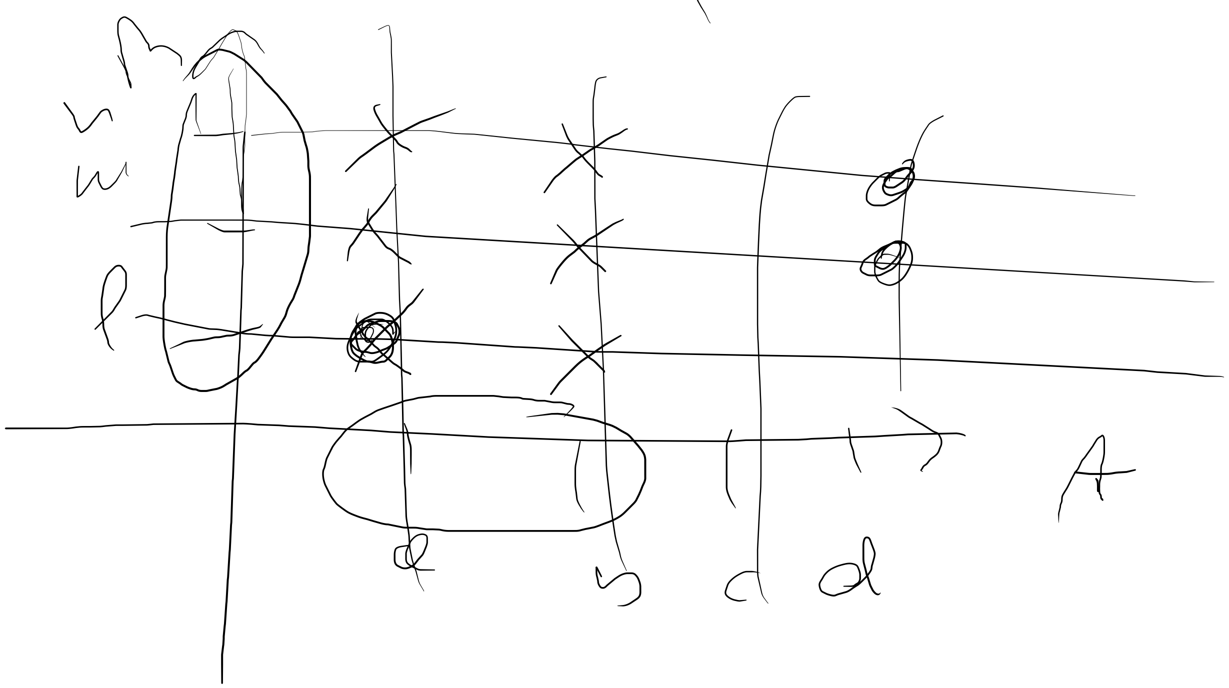
$P \subseteq A \times B$ equalizer

$\underline{(\xi, \eta)} \; | \; \underline{f(\xi) = g(\eta)}$

with the subspace topology







$$\underline{\underline{P}} = \{(\xi, \eta) : f(\xi) = f(\eta)\}$$

$$\boxed{P = \{(a, e), (d, m), (d, n)\}}$$

in $P = \{(a,e),(d,m)(d,n)\}$ the

open subsets are

$\{(a,e)\}$ $\{(a,e)(d,m)\}$

$$\begin{array}{l} P = \{(a,e)(d,m)(d\ n)\} \\ \downarrow f \\ B = \{e, m\ n\} \end{array}$$

is contin. NOT a quotient

top NOT a regular

emb.

$\bar{f}$ is not a quotient map $\Leftrightarrow$

B doesn't have the quotient top

(finest that makes $\bar{f}$ cont.)

take $\{e\} \subseteq B$

$\bar{f}^{-1}(\{e\}) = \{(a,e)\} \subseteq P$

$\{a,e\}$ open in P

$\Rightarrow$ $\{e\}$ should be open in B