Def Regular category

Def EXACT category

Borceux

Handbook of

Categorical

algebra

Vol II

regular

reg epi $\circ \twoheadrightarrow$

behave well

exact

equiv relations

behave well

Def $\mathcal{C}$ with finite limits and colimits

$\mathcal{C}$ is regular IFF

$\forall\ \boxed{f : X \twoheadrightarrow Y \text{ rep epi}}$ $\quad \forall\, g : T \to Y$

the p.b. of f and g

$$\begin{tikzcd} P \arrow[r, dashed, "\bar{g}"] \arrow[d, dashed, "\bar{f}"'] \arrow[dr, phantom, "\text{pb}"] & X \arrow[d, two heads, "f"] \\ T \arrow[r, "g"'] & Y \end{tikzcd}$$

then $\bar{f}$ is a rep epi

TOP is NOT REGULAR

Examples

- Set regular
- Grp/Ab/Rng regular

Observ it suffices to have

$\mathcal{C}$ with finite limits +

$\Updownarrow$ exact coequalizers of kernel pairs

Def a kernel pair is given by the pull-back of a morphism f by itself

$$\begin{tikzcd} P \arrow[r, dashed, "q"] \arrow[d, dashed, "v"'] \arrow[dr, phantom, "p.b"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(P, q, v) is the kernel pair of f

$$\begin{tikzcd} P \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "f"] \arrow[dr, "q"'] & Y \\ & & \mathrm{Coeq}(u,v) \end{tikzcd}$$

Theorem If $\mathcal{C}$ is regular category then $\forall f : X \to Y$, there exist $q : X \to Q$, $m : Q \to Y$ s.that $f = m \cdot q$

m is a mono

q is a reg epi

The fact is UNIQUE (up to iso)

Proof

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\rho"'] & & \cdot \\ & \cdot \arrow[ur, "m"'] & \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & X \arrow[r, "f"] \arrow[dr, "q"'] & Y \\ & & Q \arrow[u, dashed, "m"'] \end{tikzcd}$$

(R, u, v) is the kernel pair of f

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\mathrm{pb}"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$Q = \mathrm{Coeq}(u, v)$

q is a reg epi

q is universal

f is s.t. $fu = fv$

$\exists!\ m : Q \to Y \ :\ \boxed{f = m \cdot q}$

m is a monomorphism

Lemma. $m : A \to B$ is a $\boxed{\text{monomorph}}$ iff its kernel pair (P, u, v) is $u = v$

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\text{p.b}"] & A \arrow[d, "m"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

Proof if m is monom $\overset{?}{\longrightarrow}$ $u = v$

we the p.b $\cancel{m}u = \cancel{m}v \longrightarrow u = v$

Conversely if $u = v$ then m is a mono

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[rr, bend left, "k"] \arrow[dr, bend right, "h"'] & P \arrow[r, "u"] \arrow[d, "u=v"'] \arrow[dr, phantom, "pb"] & A \arrow[d, "m"] \\ & A \arrow[r, "m"'] & B \end{tikzcd}$$

m is a monomorph iff

$\forall\, h, k$ $\boxed{mh = mk} \Rightarrow h = k$

$h, k : T \to A$

the pb is univ $\to \exists!\, \varphi : T \to P$ s.that

$h = u\varphi$ $\quad k = u\varphi \to h = k$

Observ if $F : \mathcal{C} \to \mathcal{D}$

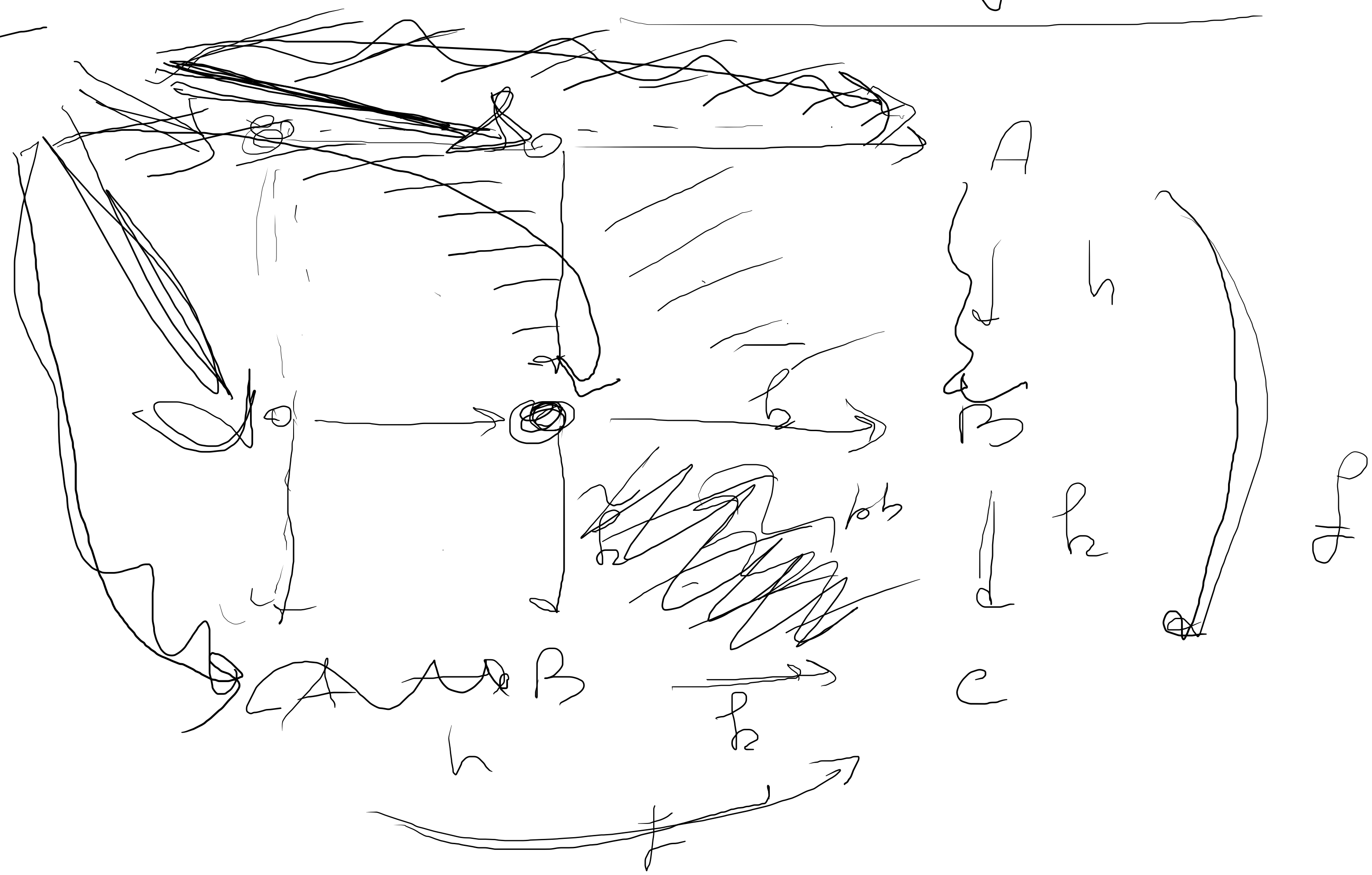
$$\begin{tikzcd} \circ \arrow[r, dashed] \arrow[d] & \circ \arrow[d, "m"] \\ \circ \arrow[r, "m"'] & \circ \end{tikzcd}$$

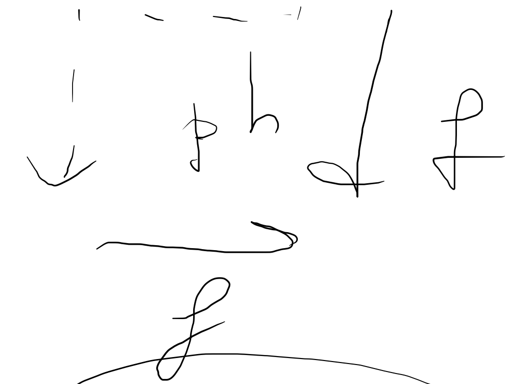
preserves limits $\to$ F preserves mono

Lemma.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend right, "f"'] & B \arrow[r, "k"] & C \end{tikzcd} \qquad f = kh$$

Construct the kernel pair of f





universal property?

$$
\begin{tikzcd}
R \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "\varphi"'] & X \arrow[r, "f"] \arrow[dr, "q"] & Y \\
S \arrow[rr, shift left, "\alpha"] \arrow[rr, shift right, "\beta"'] & & Q \arrow[u, "m"']
\end{tikzcd}
$$

$\boxed{fu = fv}$

$q = \mathrm{Coeq}(u, v)$

Hp

m is a mono ?

let (S, α, β) = kerpair of m

$$
\begin{tikzcd}
R \arrow[dr, "\varphi"] \arrow[drr, bend left, "qu"] \arrow[ddr, bend right, "qv"'] & & \\
& S \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Q \arrow[d, "m"] \\
& Q \arrow[r, "m"'] & Y
\end{tikzcd}
$$

$fu = mqu = mqv = fv$

$\exists!\ \varphi : R \to S$ s.that

$\boxed{qu = \alpha\varphi \qquad qv = \beta\varphi}$

to prove that m is a monom. we must prove that $\alpha = \beta$ by Lemma 1

we know $\alpha\varphi = \rho u$

$\beta\varphi = \rho v$

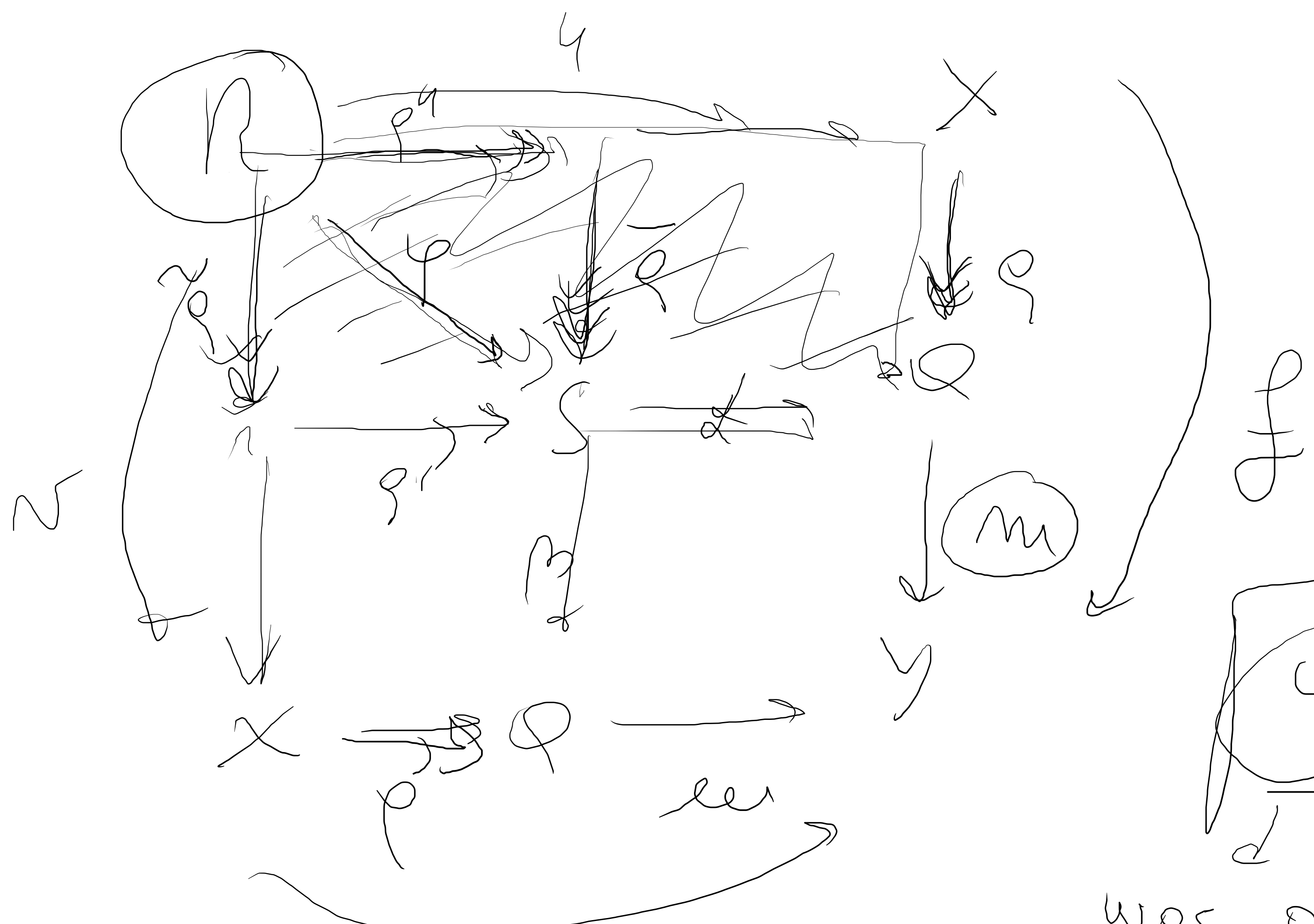
but $\boxed{\rho u = \rho v}$ by constr

$\boxed{\alpha\varphi = \beta\varphi}$

if we prove that φ is an epimorph.

then we get $\alpha = \beta$

by Lemma 2



by regularity the pb of a reg epi is a regular epi

$\varphi = \bar{\rho} \cdot \rho^4 = \rho' \cdot \rho^2$

was obtained by the uni. prop.

φ is the composite of epis $\longrightarrow$ φ is an epi $\longrightarrow$ m is iso

We have

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \\ & Q \arrow[ur, tail, "m"'] & \end{tikzcd}$$

The <u>factorization is unique</u>

<u>Proof</u> Let $\bar{m}\,\bar{p}$ be another factorization of f with $\bar{p}$ is a rp epi and $\bar{m}$ is a mono.

<u>Then</u> $\exists\ \varphi : Q \to \bar{Q}$ s.t. φ is an iso

and $\varphi p = \bar{p}$ $\quad \bar{m}\varphi = m$

$$R \overset{u}{\underset{v}{\rightrightarrows}} X \xrightarrow{f} Y$$

$\bar m \cdot \bar\rho = f$

$\bar\rho$ regepi

$\bar m$ mono

??

$\exists \varphi$ . . . . isom.

$q = \mathrm{Coeq}(u, v)$ is univ.

$\cancel{\bar m} \bar\rho u = \cancel{\bar m} \bar\rho v \longrightarrow \bar\rho u = \bar\rho v$

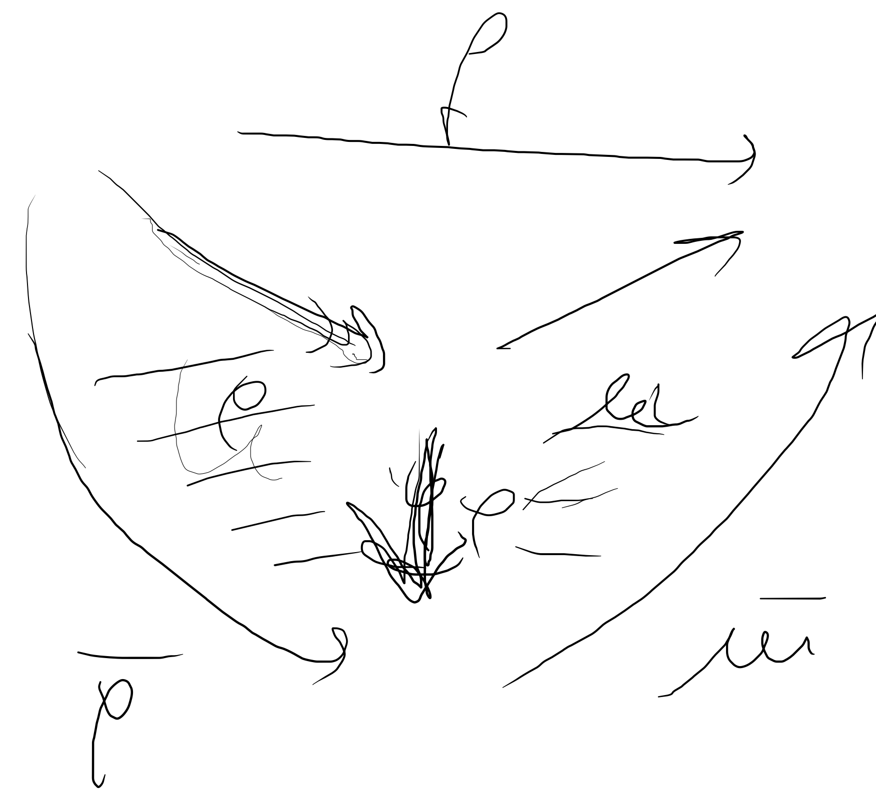
$f u = f v \longrightarrow$ by univ. of ρ

$\exists ! \varphi \quad \boxed{\varphi q = \bar\rho}$

$\bar{m}\varphi \overset{?}{=} m$ $\qquad \bar{m}$

$\bar{m}\varphi q = m q$

$\longrightarrow$ OK $\bar{m}\varphi = m$



It remains to show that φ is an iso

$\boxed{\bar{m}\varphi = m}$ mono $\longrightarrow$ φ is a mono ✓

$\boxed{\varphi q = \bar{p}}$ $\bar{q}$ regepi, q epi $\Rightarrow$ φ regepi ✓

Lemma

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[rr, bend right, "f"'] & B \arrow[r, "v"] & C \end{tikzcd}$$

IP

$f = v \cdot u$

f is a reg ep

u is an ep

TH

v is a reg ep

Proof

$$\begin{tikzcd} & & & T \\ D \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & A \arrow[r, "u"] \arrow[rr, bend right, "f"'] & B \arrow[ur, "\forall t"] \arrow[r, "v"] & C \arrow[u, "\varphi"'] \end{tikzcd}$$

IP f reg ep $\rightarrow$ $f = \mathrm{Coeq}(\alpha, \beta)$

TH $v = \mathrm{Coeq}(u\alpha, u\beta)$

Proof

$vu\alpha = vu\beta$

$\forall\, t : B \to T$ s.t. $t u \alpha = t u \beta$

?? $\exists\, \varphi : C \to T$: $\boxed{t = \varphi \cdot v}$

apply f is a coeq so you get φ s.t.

$\varphi f = t u$

$\varphi v \cancel{u} = t \cancel{u} \longrightarrow \varphi v = t$

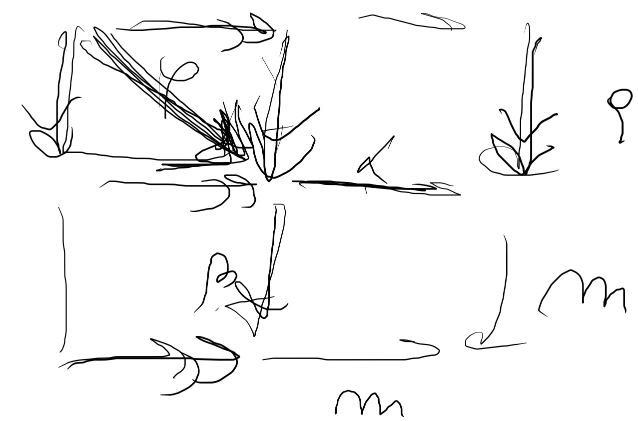
u epi

la fattorizz. è UNICA

Teor. φ regolare

$\Rightarrow$ $X \xrightarrow{f} Y$

reg epi e $\cdot$ m mono



φ EPI

Prop. $\mathcal{C}$ is a $\boxed{\text{regular}}$ cat. Then

- $$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r, "g"] & T \end{tikzcd}$$ f, g are reg epi

  then gf is a reg epi.

- If the composite $$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "t"'] & Y \arrow[r, "g"] & T \end{tikzcd}$$ t is a reg epi

  then g is a reg epi.

Grp

reg epi = surj hom = epim

Ring

$$\mathbb{Z} \xrightarrow{f} \mathbb{Q}$$

f is injective

+

monos.

$\boxed{\text{We proved } f \text{ is an EPIM}}$ not a reg epi

Ex: f is a reg epi, f is a mono → f is an isom

∀ 𝒞 category

Def regul cat $\longmapsto$ Fact. theorem

Theorem. The following are equivalent

1) $\mathcal{C}$ is a regular category

2) – any f can be factorized as $f = m \cdot e$, e: reg epi – m mono

– the factorization is stable under pb's

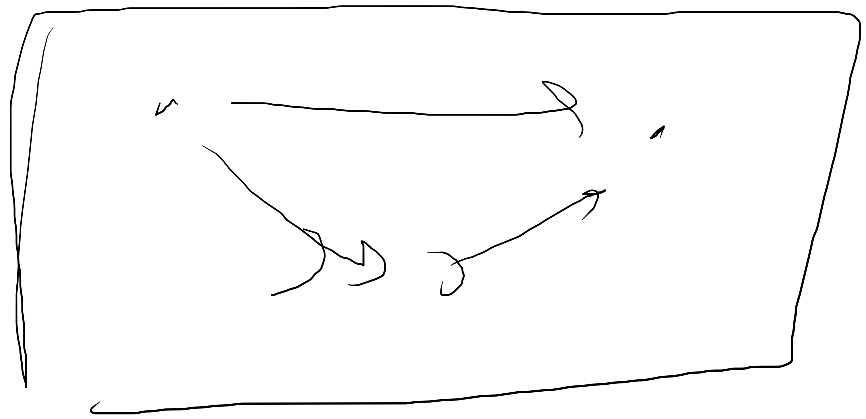
$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\
& e \arrow[ur, "m"'] &
\end{tikzcd}
$$

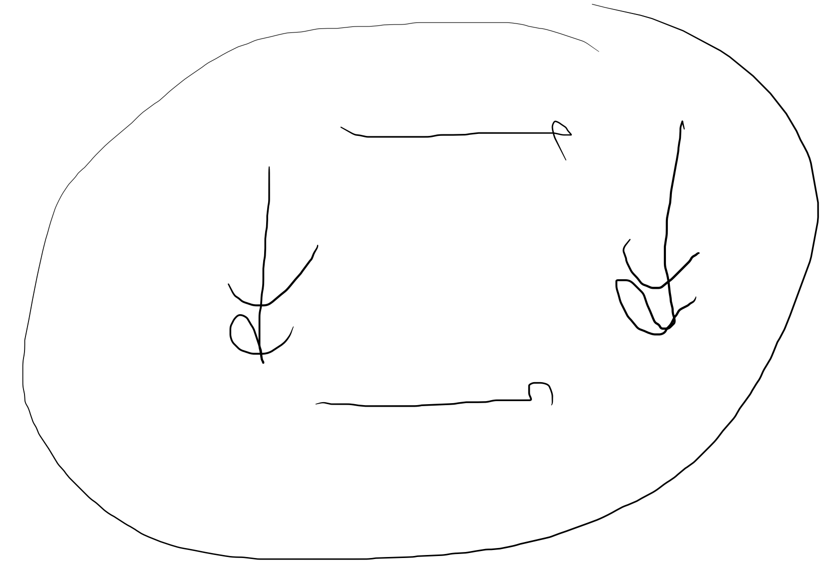
the pb of the mono is a mono

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\bar{p}"'] \arrow[dd, bend right=60, "\bar{f}"'] & \cdot \arrow[d, "p"] \arrow[dd, bend left=60, "f"] \\
\cdot \arrow[r] \arrow[d, "\bar{m}"'] & \cdot \arrow[d, "m"] \\
\cdot \arrow[r, "g"'] \arrow[ur, phantom, "pb"] & \cdot
\end{tikzcd}
$$

$\boxed{\bar{f} = \bar{m}\,\bar{p}}$

C REGULAR





Set Cpp Rec Low

rep

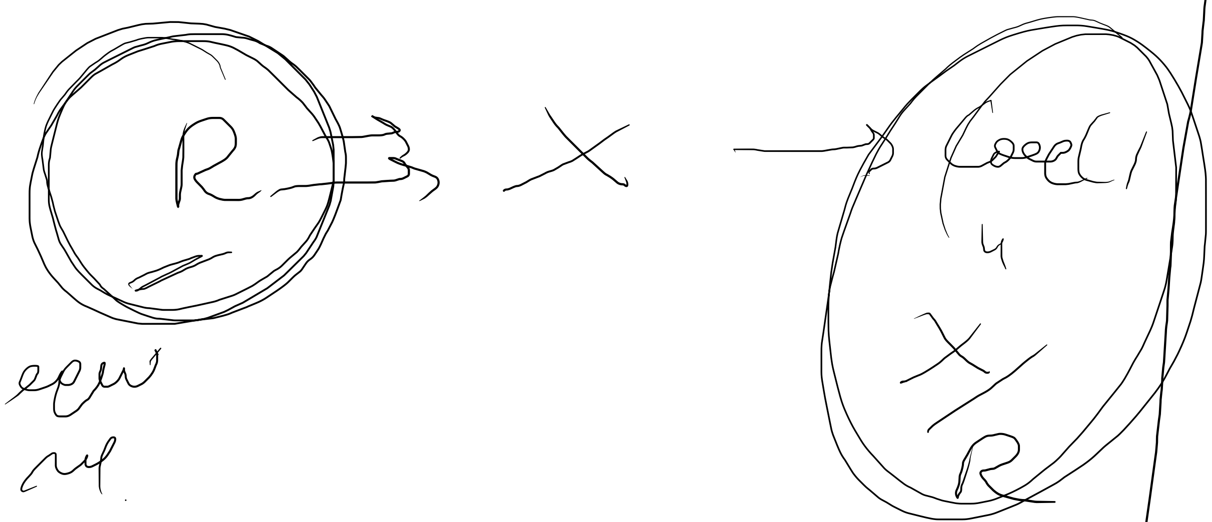
TOP

not

equal

$\boxed{\mathcal{C} \quad \text{exact}}$

equivalence relations



$\mathcal{C}$ has finite limits

$\downarrow$

we can define the notion of equiv relations

$\boxed{\text{Torsion free ab groups}}$ Ab_{TF}

Ab

$$\begin{tikzcd}
P_2 \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d] & \mathbb{Z} \arrow[r, "q"] \arrow[ddr, "\bar{p}"] & \{0,1\} = \mathbb{Z}_2 \arrow[dd, "r"] \\
\mathbb{Z}\times\mathbb{Z} \arrow[ur, shift left] \arrow[ur, shift right] & & \\
& & \{0\} = \mathrm{Coeq}(u,v)
\end{tikzcd}$$

Kernel pair of $\bar{p}$

with Ab_{TF}

$\int_e d\bar{e}$