ALGEBRAIC CATEGORY

def Regular category

$$\begin{tikzcd} \cdot \arrow[r, dashed] \arrow[d, "\bar{f}"'] \arrow[dr, phantom, "pb"] & X \arrow[d, "f"] \\ \cdot \arrow[r] & Y \end{tikzcd}$$

f reg epi $\Rightarrow$ $\bar{f}$ reg epi

$$\boxed{\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads] & & Y \\ \rho & Q \arrow[ur, tail] & m \end{tikzcd}}$$

Def $\mathcal{C}$ with finite limits and finite colimits

$\mathcal{C}$ is called an <u>EXACT CAT</u>

iff

1) $\mathcal{C}$ is REGULAR

2) <u>any equivalence relation is effective</u>

Set $\quad \boxed{\text{R eq. rel}} \quad X/R$

Grp / Ab / Ring / Latt $\quad \boxed{\text{R equivalence compatible}} \quad G/R$

Top $\quad \boxed{\text{R equiv rel.}} \quad X/R$

$X \xrightarrow{q} X/R$

final topology

$\mathcal{C}$

what is a relation?

what is an $\boxed{\text{equiv. relat?}}$

what corresponds to a compatible equiv. rel?

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r] & X/R \end{tikzcd} \equiv \mathrm{Coep}(r_1, r_2)$$

$X \in \mathcal{C}$ R is an ~~equ.~~ relation on X

iff $\exists$ $\boxed{r_1, r_2 : R \rightrightarrows X}$ such that

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, "\exists! \varphi"'] & X \\ X \times X \arrow[ur, shift left, "p_1"] \arrow[ur, shift right, "p_2"'] & \end{tikzcd}$$

φ is a MONOMORPH.

$$\begin{tikzcd} R \arrow[r, tail, "\varphi"'] \arrow[rr, bend left, "r_1"] \arrow[rr, bend right, "r_2"'] & X \times X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \end{tikzcd}$$

Set

$R \rightarrowtail X \times X$

$R \subseteq X \times X$

$R \rightarrowtail X \times Y$

When R is an EQUIVALENCE ?

iff

1) $\boxed{\text{reflexive}}$

2) symmetric

3) transitive

Set

$R \to X \times X$

1) $R \longrightarrow X \times X$

reflexive

$\exists \delta$

$$\begin{tikzcd} & R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \\ X \arrow[ur, "\delta"] \arrow[urr, shift left, "1_X"] \arrow[urr, shift right, "1_X"'] & & \end{tikzcd}$$

iff

$\exists \delta : X \longrightarrow R$

s.the

$r_1 \cdot \delta = 1_X$

$r_2 \delta = 1_X$

reflect. iff

$\forall x \in X$

$(x, x) \in R$

2) R is symmetric iff $\exists \sigma : R \to R$

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d, bend right, "\sigma"'] & X \\ R \arrow[ur, shift left, "r_2"] \arrow[ur, shift right, "r_1"'] & \end{tikzcd}$$

$$r_1 \cdot \sigma = r_2$$

$$r_2 \sigma = r_1$$

So $\forall (x, y) \in R$

$$\Downarrow$$

$(y, x) \in R$

3) R is transitive

$$\begin{tikzcd} R \times_X R \arrow[r, "t_1"] \arrow[d, "t_2"'] \arrow[dr, phantom, "\text{pb}"] & R \arrow[d, "r_2"] \\ R \arrow[r, "r_1"'] & X \end{tikzcd}$$

$$\boxed{\exists\ \tau : R \times_X R \longrightarrow R \quad \text{s. th.}}$$

set

$(x\ y)$

$(\bar{y}\ z)$ $y = \bar{y}$

$R \times_X R = \left\{ (x, y, z) \;\middle|\; \begin{array}{l} (x\ y) \in R \\ (y\ z) \in R \end{array} \right\}$

set

$\forall x\, y\, z$

$(x, y) \in R$

$(y\ z) \in R$

τ

$(x\ z) \in R$

τ is such that

$$\begin{tikzcd} R\times R \times X \arrow[r, "\tau"] \arrow[d, "t_1"'] \arrow[dr, phantom, "\text{is comm}"] & R \arrow[d, "x_1"] \\ R \arrow[r, "x_1"'] & X \end{tikzcd}$$

$$\begin{tikzcd} R\times R \times X \arrow[r, "\tau"] \arrow[d, "t_2"'] \arrow[dr, phantom, "\text{is comm}"] & R \arrow[d, "x_2"] \\ R \arrow[r, "x_2"'] & X \end{tikzcd}$$

$$\begin{tikzcd} (x,y,z) \arrow[r] \arrow[d] & (x\ z) \arrow[d] \\ (x\ y) \arrow[r] & x \end{tikzcd}$$

Ex 1. $\forall \mathcal{C}$ finite limits & colimits

any kernel pair of f is an equivalence relation

$$\underbrace{R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X}_{\ker f} \xrightarrow{f} Y$$

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "pb"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Set $R = (x, \bar{x})$: $f(x) = f(\bar{x})$

$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ is an equiv. rel.

1) $\boxed{\text{reflexive}}$

$$\begin{tikzcd}
X \arrow[dr, "\delta"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\
& R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "pb"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

reflexive

$\exists\, \delta : X \to R$

by univ. property of pb $\exists\, \delta$ s.th.

$r_1 \delta = 1_X \qquad r_2 \delta = 1_X$

2) $\boxed{\text{symmetry}}$

$$\begin{tikzcd}
R \arrow[dr, "\sigma"] \arrow[drr, bend left, "r_2"] \arrow[ddr, bend right, "r_1"'] & & \\
& R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "pb"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$f r_2 = f r_1$

by univ. prop

$\exists!\, \sigma$:

$r_1 \sigma = r_2 \qquad r_2 \sigma = r_1$

3) $\boxed{\text{transitivity}}$ $\exists\ \tau : R \times_X R \to R$ with

$r_1 \tau = r_1 t_1$ $\qquad$ $r_2 \tau = r_2 t_2$

$$\begin{tikzcd}
R \times R \arrow[rr, "t_1"] \arrow[dr, "\tau"] \arrow[dd, "t_2"'] & & R \arrow[d, "r_1"] \\
& R \arrow[r, "r_1"] \arrow[d, "r_2"] & X \arrow[d, "f"] \\
R \arrow[r, "r_2"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

by uni property of p.b.

to apply the uni. property

$f r_1 t_1 \overset{?}{=} f r_2 t_2$

$\widetilde{f r_1 t_1} = \widetilde{f r_2 t_1} = \widetilde{f r_1 t_2}$

$= f r_2 t_2$

Lemma $f : X \twoheadrightarrow Y$ is a reg epi then f is the coeq. of its kernel pair

Proof

$$\begin{tikzcd}
\underbrace{R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"']}_{\text{kernel pair}} & X \arrow[r, two heads, "f"] \arrow[dr, two heads, "p"'] & Y \arrow[d, "\cong"] \\
& & \mathrm{coeq}(r_1, r_2)
\end{tikzcd}$$

Proof

f reg epi $\Vert$ $\exists\,(u,v)$, $f = Coeq(u,v)$

$\exists!\,\varphi$

$$\begin{tikzcd}
 & S \arrow[d, shift left, "v"] \arrow[d, shift right, "u"'] & & \\
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "f"] \arrow[d, "e"'] & Y = Coeq(u,v) \arrow[dl, shift left, "h"] \arrow[dl, shift right, "k"'] \\
 & Q = Coeq(r_1, r_2) \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

reg. epi

$$\begin{tikzcd}
S \arrow[dr, "\psi"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
 & R \arrow[r, "r_1"] \arrow[d, "r_2"'] & X \arrow[d, "f"] \\
 & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

R is kernel pair of f $\Vert$ $fu = fv$

Def $\quad R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X \quad$ equiv. relation

R is called **effective** iff

R is the kernel pair of the coeq of (r_1, r_2)

$$\boxed{R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X} \xrightarrow{\rho} \mathrm{Coeq}(r_1, r_2) = Q$$

$$\begin{tikzcd}
R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "\mathrm{pb}"] & X \arrow[d, "\rho"] \\
X \arrow[r, "\rho"'] & Q
\end{tikzcd}$$

def $\mathcal{C}$ is EXACT iff

1) $\mathcal{C}$ regular 

2) Any equiv. relation is effective

Counterex — an equiv. relation that is not effective

Ab$_{TF}$ torsion free ab. groups

$x^n = e \Rightarrow x = e$

$$\begin{tikzcd} P_2 \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & \mathbb{Z} \arrow[r, "q"] \arrow[dr, "\bar{\varphi}"'] & \mathbb{Z}_2 = \{0, 1\} \arrow[d, "h"] \\ & & \{0\} \end{tikzcd}$$

$\notin$ Ab$_{TF}$

Ab

$1 + 1 = 0$

Ab$_{TF}$

Ab$_{TF} \subseteq$ Ab

reflective subc.

$$
\begin{tikzcd}
R_2 \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d] & \mathbb{Z} \arrow[r, "\bar{p}"] & \{0\} \\
\mathbb{Z}\times\mathbb{Z} \arrow[ur, shift left, "p_1"] \arrow[ur, shift right, "p_2"'] & &
\end{tikzcd}
$$

kernel pair of $\bar{p}$

$$
\begin{tikzcd}
\mathbb{Z}\times\mathbb{Z} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\text{pb}"] & \mathbb{Z} \arrow[d, "\bar{p}"] \\
\mathbb{Z} \arrow[r, "\bar{p}"'] & \{0\}
\end{tikzcd}
$$

$\boxed{Ab_{TF}}$

is not an exact category

Set / Grp / Ab / Ring / Latt ....

They are EXACT categories

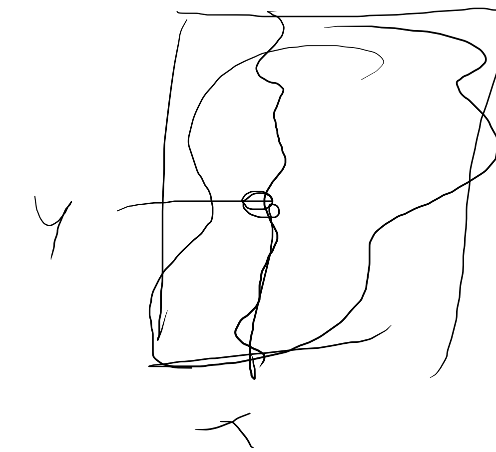
Set

equiv. rel.

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \xrightarrow{q} X/R = \mathrm{Coeq}(r_1, r_2)$$

$\cup$

Kep



Theorem: Grp / Ab ...

are exact categories

Cgf.

$R \rightrightarrows G$ equiv. relat in a categ sense.

$\boxed{R \hookrightarrow G \times G \text{ homo.}}$

$R \subseteq G \times G$ is a subgroup of the direct product

=

R is a compatible relat.

$\boxed{R \rightrightarrows G \to G/R}$

$\cancel{R \rightrightarrows G}$

equiv rel

NOT corr.

$\boxed{\text{Top}}$ is not regular

equiv. rel. are not effective

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] \arrow[d] & X \arrow[r, "p"] & X/R \\ X \times X / R \arrow[ur, shift left, "r_1"] \arrow[ur, shift right, "r_2"'] & & \end{tikzcd}$$

discrete top

quot topology

must have the subspace top.

$$\begin{tikzcd} \bullet \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "\mathrm{pb}"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "p"'] & \cdot \end{tikzcd}$$

$$x^n = 0 \Longrightarrow x = 0$$

$\boxed{\text{implication}}$

Def $\mathcal{C}$ is called algebraic category (over Set)

Grp

EXACT

$\exists\ \mathbb{Z} \in \mathrm{Grp}$

$$\mathrm{Grp} \underset{U}{\overset{F}{\rightleftarrows}}_{\perp} \mathrm{Set}$$

$\mathbb{Z} = F(1)$

$\mathbb{Z}$ is a [illegible] PROJECTIVE

and is a [illegible] GENERATOR

Def $P \in \mathcal{E}$ is a regular projective object

iff $\forall f : X \twoheadrightarrow Y$ reg epi

$\forall h : P \to Y$

$\exists \bar{h} : P \to X$ st $f\bar{h} = h$

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\bar{h}"'] \arrow[d, "h"] \\
X \arrow[r, two heads, "f"', "\text{reg epi}"'] & Y
\end{tikzcd}$$

Ex $\quad \mathbb{Z} \in \mathrm{Grp} \quad \mathbb{Z}$ is regular projective

$$
\begin{tikzcd}
 & \mathbb{Z} = F(1) \arrow[dl, dashed, "T"'] \arrow[d, "h"] \\
g \in G \arrow[r, two heads, "f"'] & H \ni \xi
\end{tikzcd}
$$

reg epi = surj. hom.

$h : \mathbb{Z} \to H$ hom. is totally defined by the image of 1 $\quad h(1) = \xi \in H$

$\boxed{\exists\, g \in G : g = f^{-1}(\xi)}$ $\quad \boxed{f \text{ is SURJECTIVE}}$ $\quad \boxed{T(1) = g}$ ← Def

$\boxed{\text{Set}}$ $\boxed{\text{Any set is a regular projective}}$

(also the empty set)

$$\begin{tikzcd} & P \ni x \arrow[dl, bend right] \arrow[d, "h"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$

surj.

$\boxed{\text{Grp} \quad \text{Ab}}$

$\mathbb{Z}$ $\quad$ $\mathbb{Z}$

The free algebra on 1 is a reg. projective

Exercise If P_i are regular proj.

Then the coproduct of P_i is reg. project.

$\mathbb{Z} \amalg \mathbb{Z}$ reg. proj.

$\mathcal{A} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathrm{Set}$

---

algebr. categ (ouSet)

$\Updownarrow$

EXACT ← regular, equiv. rel. eff.

$\exists P$ reg. proj. reg. gen.

Exercise $\mathcal{C}$ $\quad$ $P \in \mathcal{C}$

$$\boxed{P \text{ is reg. proj}} \iff \boxed{\begin{array}{c} \mathcal{C} \longrightarrow \mathrm{Set} \\ \mathcal{C}(P,-) \end{array}}$$

preserves reg. epimorphisms

$$\mathrm{Grp} \longrightarrow \mathrm{Set}$$

$$\boxed{\mathrm{Grp}(\mathbb{Z},-) \simeq U}$$