Definition of TRIPLE T

and of T-algebras on an arbitrary category $\mathcal{C}$

if $\mathcal{C}$ = Set   T-algebras = exactly [illegible]
with op
rel rel
rel gen.

$\mathcal{C}$ = Top   Topological groups

$X \in$ Top $X \times X \to X$ cont. funct.

Def A TRIPLE (T, η, μ) on a category $\mathcal{C}$ is given by

$T : \mathcal{C} \to \mathcal{C}$ ENDOFUNCTOR

+ 2 natural transformation

$\eta : 1_{\mathcal{C}} \to T$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "1_{\mathcal{C}}"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{C} \end{tikzcd}$$

$\forall c \in \mathcal{C}$ $\boxed{\eta_c : c \to Tc}$

$\mu : T^2 \to T$ natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "T^2"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow \mu"] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "T"] \arrow[rr, bend right, "T^2"'] & \mathcal{C} \arrow[r, "T"] & \mathcal{C} \end{tikzcd}$$

$\forall \; x \in \mathcal{C}$

$$\boxed{\mu_X : T^2 X \to TX}$$

natural

$$\mathcal{C} \xrightarrow{T} \mathcal{C}$$

$$\boxed{\eta_X : X \to TX}$$

+ axioms

Axioms

$\forall x \in \mathcal{C}$

$$x \xrightarrow{\eta_X} TX$$

$$TX \xrightarrow{T(\eta_X)} T^2X$$

$$\boxed{C = TX}$$

$$\eta_C : C \longrightarrow TC$$

$$\eta_{TX} : TX \longrightarrow T(TX) = T^2X$$

$$\begin{tikzcd}
TX \arrow[r, "T\eta_X"] \arrow[dr, "1_{TX}"'] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "\eta_{TX}"'] \arrow[dl, "1_{TX}"] \\
& TX &
\end{tikzcd}$$

η

$(A, \cdot, e$

$x \cdot e = e$

$$\begin{tikzcd}
x \arrow[r] \arrow[dr, "1_x"'] & x \cdot e \arrow[d] \\
& x
\end{tikzcd}$$

axiom of μ multiplication

$$\begin{tikzcd}
T^3 x \arrow[r, "T\mu_x"] \arrow[d, "\mu_{Tx}"'] & T^2 x \arrow[d, "\mu_x"] \\
T x \arrow[r, "\mu_x"'] & T x
\end{tikzcd}$$

ASSOC DIAGRAM

TRIPLE

(T, η, μ)

+ 3 axioms

$\forall x$ ... the diagrams commute

$(A, \cdot)$

semigroup $\forall a, b, c \in A$

$$\boxed{\begin{gathered}(a \cdot b) \cdot c \\ = \\ a \cdot (b \cdot c)\end{gathered}}$$

$$\begin{tikzcd}
A^3 \arrow[r, "\cdot \times Id"] \arrow[d, "Id \times \cdot"'] & A^2 \arrow[d, "\cdot"] \\
A^2 \arrow[r, "\cdot"'] & A
\end{tikzcd}$$

$a\,b\,c$ $\quad$ $(a \cdot b)\,c$

Set

Example

$$\begin{tikzcd} \mathrm{Smgr} \arrow[r, shift right=1.5ex, "G"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right=1.5ex, "F"'] \end{tikzcd}$$

$F \dashv G$

$$\boxed{F(X) \overset{\mathrm{DEF}}{=} \coprod_{n \in \mathbb{N}^+} X^n}$$

$X \in \mathrm{Set}$ free semigroup $\coprod_{n \in \mathbb{N}^+} X^n = \{ X \cup X^2 \cup X^3 \cup \dots \cup X^n \}$

$= \{$ FINITE <u>words</u> of elements of X $\}$

<u>$x_1 x_2 x_3$</u> <u>$x_6 x_7$</u> $x_i x_j x_k$

$$\boxed{\mathrm{Semigr} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Set} \circlearrowleft T}$$

$$\boxed{T = GF}$$

$$\boxed{\text{FUNCTOR}}$$

$$\boxed{GFX = \coprod_n X^n}$$

$$\coprod X^n \in \mathrm{Semigr}$$

$$(x_1 x_2 x_3)\cdot(x_6 x_7) = (x_1 x_2 x_3 x_6 x_7)$$

$$\boxed{\eta_X : X \to TX = \coprod X^n}$$

$X \in \mathrm{Set}$

$\xi \in X$

$\eta_X(\xi) = (\xi)$

word with 1 elements

$$\text{Algebraic Categ} \underset{G}{\overset{F}{\rightleftarrows}} \boxed{\;\circlearrowleft\; T = GF}$$

$\mu_X : T^2X \longrightarrow TX$

$n, m, t \in \mathbb{N}^+$

$$\boxed{\mu_X = \underbrace{\coprod_m \Big(\coprod_n X^n\Big)^m}_{} \longrightarrow \coprod_t X^t}$$

"parola di parole"

$$\underbrace{(x_1 x_2 x_3)\ (x_6 x_7)\ (x_i x_j)\ (x_1 \cdots)}_{m \text{ words}} \xrightarrow{\mu_X} (x_1 x_2 x_3 x_c x_7 \cdots\cdots)$$

axioms are true

$$\begin{tikzcd} TX \arrow[r, "T\eta_X"] \arrow[dr, "1_X"'] & T^2X \arrow[d, "\mu_X"] \\ & TX \end{tikzcd}$$

$$\begin{tikzcd} \coprod_m X^m \arrow[r, "T\eta_X"] \arrow[dr, "1"'] & \coprod_m \left(\coprod_n X^n\right)^m \arrow[d, "\mu_X"] \\ & \coprod_n X^n \end{tikzcd}$$

TX

$$\begin{tikzcd} (x_1, x_2 x_3) \arrow[r, "T\eta_X"] \arrow[dr, "1"'] & (x_1)(x_2)(x_3) \arrow[d, "\mu_X"] \\ & (x_1, x_2 x_3) \end{tikzcd}$$

$x \longrightarrow (x)$

$\eta_X : X \longrightarrow TX$

$$\mathrm{Grp} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Set} \qquad T \qquad (T, \eta_X, \mu_X)$$

or

Ex 2: $(X, \leq) = \mathcal{C}$ $\boxed{\leq \text{ order}}$

$x \xrightarrow{\leq} t$? triple on $\mathcal{C}$

$\boxed{\mathcal{C} \xrightarrow{T} \mathcal{C}}$ T function that preserve order

$\boxed{x \mapsto Tx}$

1) $\boxed{\eta_x : x \longrightarrow Tx}$ $\boxed{x \leq Tx}$

2) $\boxed{\mu_x : T^2x \longrightarrow Tx}$ $T^2x \leq Tx$

$Tx \leq T^2x$

$\Longrightarrow \boxed{Tx = T^2x}$

$\boxed{T \text{ is a closure operator on } X}$

def T-algebra for a triple T on $\mathcal{C}$

$(T : \mathcal{C} \to \mathcal{C}, \eta, \mu)$ + 3 axs

is given by

$$\boxed{(X \in \mathcal{C},\ h : TX \to X)}$$

such that

axioms

$X \in \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "h"] \\ & X \end{tikzcd}
\qquad
\begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "Th"'] & TX \arrow[d, "h"] \\ TX \arrow[r, "h"'] & X \end{tikzcd}$$

Ex

$$\mathrm{Smgp} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Set} \circlearrowright T = GF \qquad (T, \eta_x, \mu_x)$$

T-algebra? $\quad X \in \mathrm{Set}$

$$\boxed{h : TX \longrightarrow X}$$

$$\boxed{h : \coprod X^n \longrightarrow X}$$

+ axioms

$$\begin{tikzcd} X^n \arrow[r, "i_n"] \arrow[rr, bend left, "h_n"] & \coprod X^n \arrow[r, "h"] & X \end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{structure}}$$

h_n is a classical operation

The 2 axioms of T. algebra

They say that $h_1 : X \to X$ is the identity

$h_2 : XXX \to X$

by ax. 2 $\to$ h_3 is obtained by h_2

h_2 its associativity

each h_n is obtained by h_2

$\mathrm{Smgp} \rightleftarrows \mathrm{Set}^{T = GF}$

then <u>T. algebras</u> are <u>exactly free groups</u>

$$\begin{tikzcd}
\mathrm{Smgp} \arrow[r, shift left] \arrow[d, phantom, "\cong" rotate=90] & \boxed{\mathrm{Set}^{T = GF}} \arrow[l, shift left] \\
T\text{. algebras} &
\end{tikzcd}$$

Ex $(X \leq)$ category

triple $T : X \to X$ $\boxed{\text{closure operator}}$

? what is a T-algebra

$x \in X$ $\boxed{h : Tx \to x}$ $\boxed{Tx \leq x}$ $x \leq Tx$

$x = Tx$

$\boxed{x \text{ is closed}}$ $\boxed{\text{T-algebras}}$