

Definition of TRIPLE T
 on an arbitrary
 category \mathcal{C}
 and of T -algebras

if $\mathcal{C} = \text{Set}$ T -algebras \equiv exactal
 with a P
 map
 map

$\mathcal{C} = \text{Top}$

Topological groups

$X \in \text{Top}$

$$X \times X \xrightarrow{\circ} X$$

and func

Def A TRIPLO (T, η , μ) on a category \mathcal{C} is given by

$T: \mathcal{C} \rightarrow \mathcal{C}$ ENDO FUNCTOR

+ 2 natural transformations

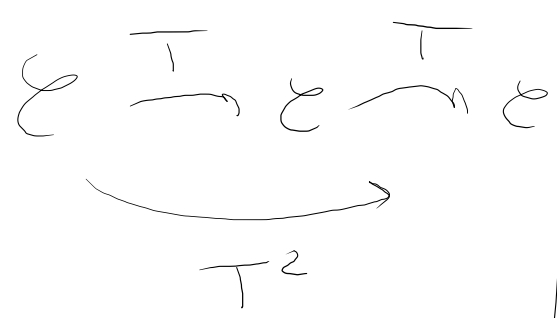
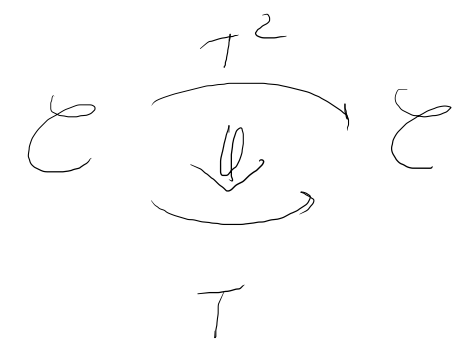
$$\eta: 1_{\mathcal{C}} \rightarrow T \quad \mathcal{C} \begin{array}{c} \xrightarrow{1_{\mathcal{C}}} \\ \Downarrow \\ \xrightarrow{T} \end{array} \mathcal{C}$$

$\forall c \in \mathcal{C}$

$$\boxed{\eta_c: c \rightarrow Tc}$$

$$\mu: T^2 \rightarrow T$$

natural transformation



$$\forall x \in C$$

$$\mu_x: T^2_x \rightarrow T_x$$

natural

$$\mu_x: X \rightarrow T_x X$$

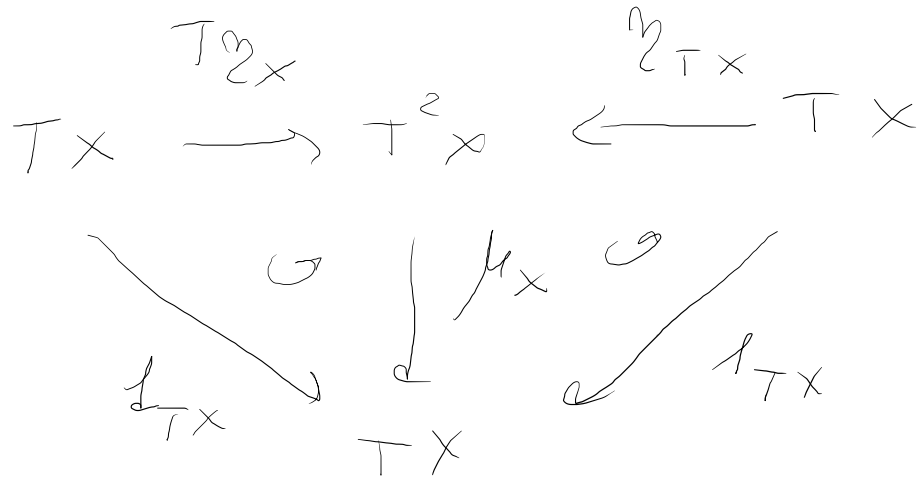
functor

AXIOMS

$\forall x \in \mathcal{C}$

$$x \xrightarrow{\eta_x} TX$$

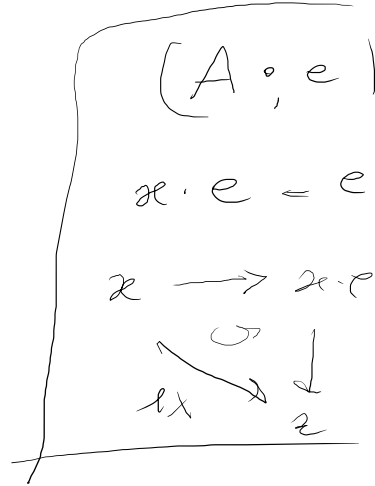
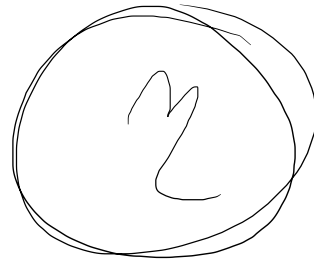
$$TX \xrightarrow{T(\eta_x)} T^2X$$



$$C = TX$$

$$\eta_C : C \rightarrow TC$$

$$\eta_{TX} : TX \rightarrow T(TX) = T^2X$$



axiom of μ

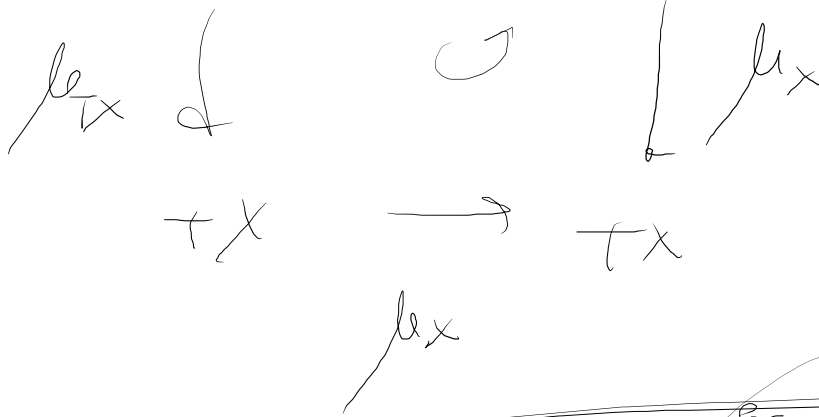
multiplicative

TRIPLO

$$T^3 \xrightarrow{T\mu_x} T^2$$

ASSOC

$$(T, \mu, \mu)$$



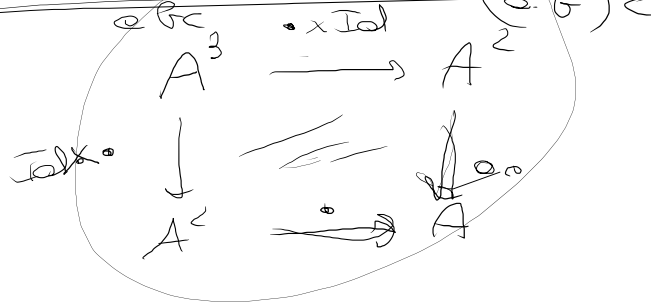
DIAGRAM

+ 3 axioms

Use ... the diagrams commute

$(A \cdot)$
 $\text{few } \mu$
 ~~μ~~

$(\mu \cdot \mu) \cdot c$
 μ
 $e(b \cdot c)$



Set

Example

Smgr $\xrightleftharpoons[\perp]{F}$ Set

$F \rightarrow G$

$$F(x) = \bigcup_{m \in \mathbb{N}^+} X^m$$

$X \subseteq \text{Set}$

free semigroup

$$\bigcup_{m \in \mathbb{N}^+} X^m$$

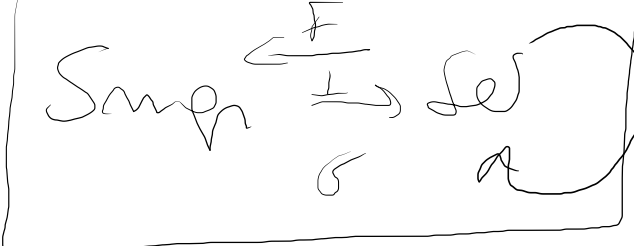
$$= \left\{ X \cup X^2 \cup X^3 \cup \dots \cup X^m \right\}$$

= $\left\{ \overset{\text{FINITE}}{\text{words}} \text{ of elements of } X \right\}$

$x_1 x_2 x_3$

x_0

$x_1 x_2 x_3$



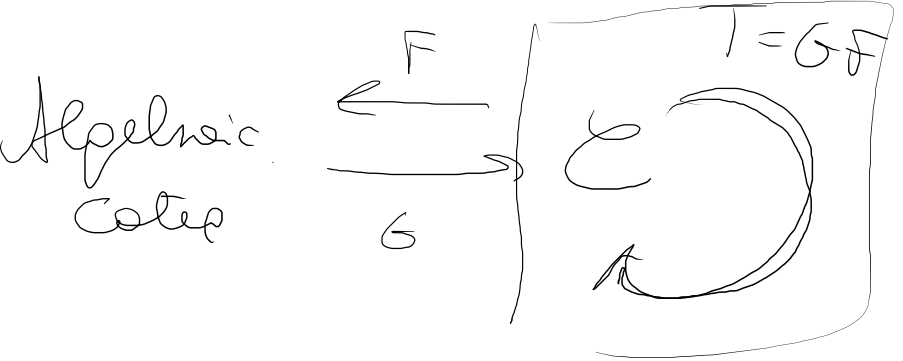
$$T = GF$$

FUNCTOR

$$GF X = \coprod_n X^n$$

$\coprod_n X^n \in \text{Simplicial}$

$$(x_1, x_2, x_3) \cdot (x_0, x_7) = (x_1, x_2, x_3, x_0, x_7)$$



$$M_X = X \rightarrow TX = \coprod_n X^n$$

$x \in \text{set}$

$\xi \in X$

$$M_X(\xi) = (\xi)$$

world with 1 element

$$\mu_x : T^2 X \longrightarrow TX$$

$$\mu_x = \underbrace{\text{fl}}_m \left(\underbrace{\frac{\text{fl} X^m}{m}}_m \right) \longrightarrow \frac{\text{fl} X^t}{t}$$

$$m, n, t \in \mathcal{N}^+$$

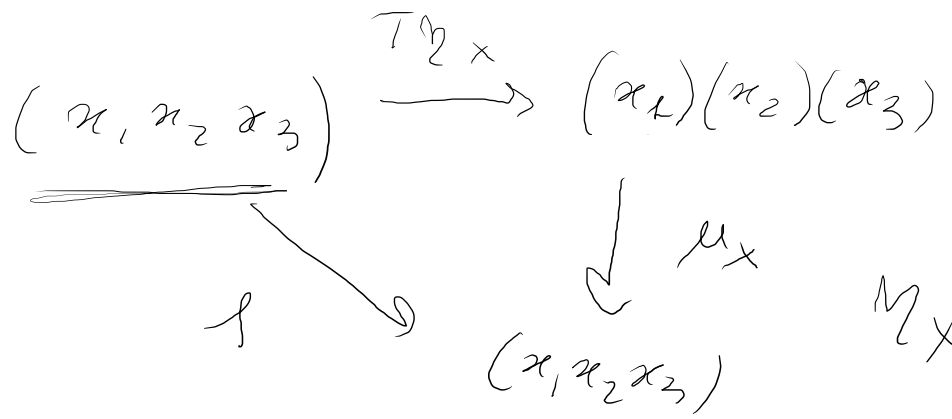
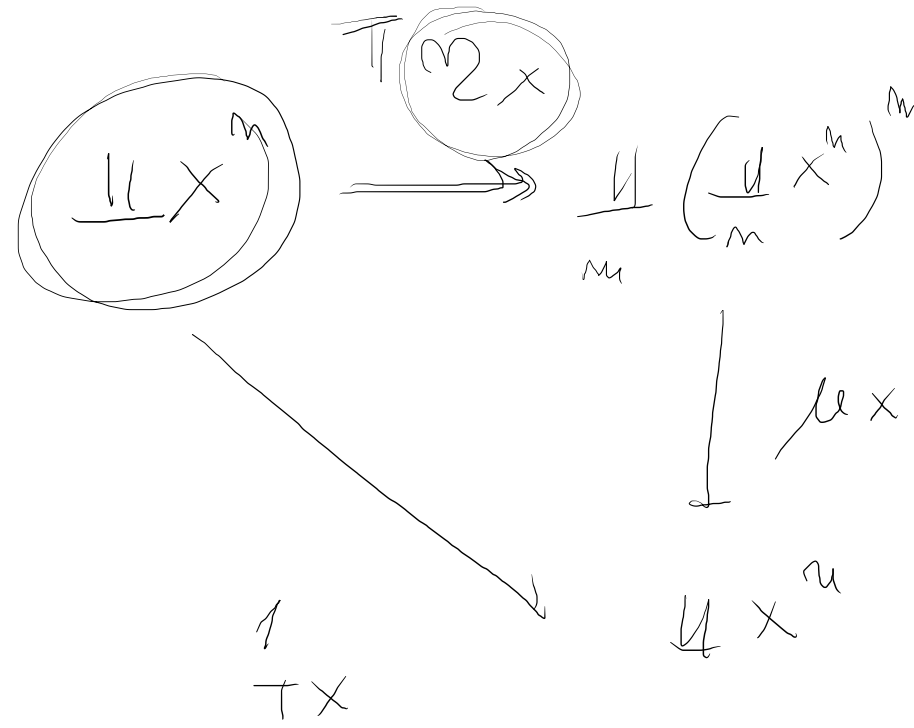
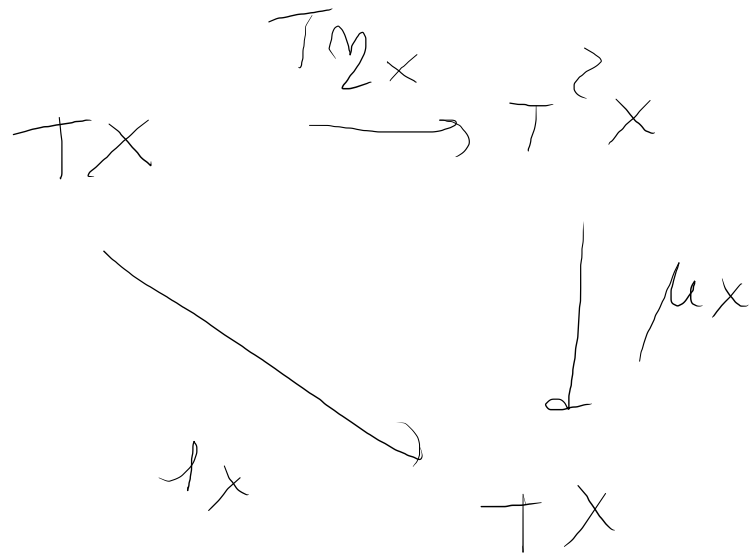
~ parts of parts ~

$$\underbrace{(\alpha_1 \alpha_2 \alpha_3) (\alpha_0 \alpha_2) (\alpha_i \alpha_j) (\alpha_i \dots)}_m$$

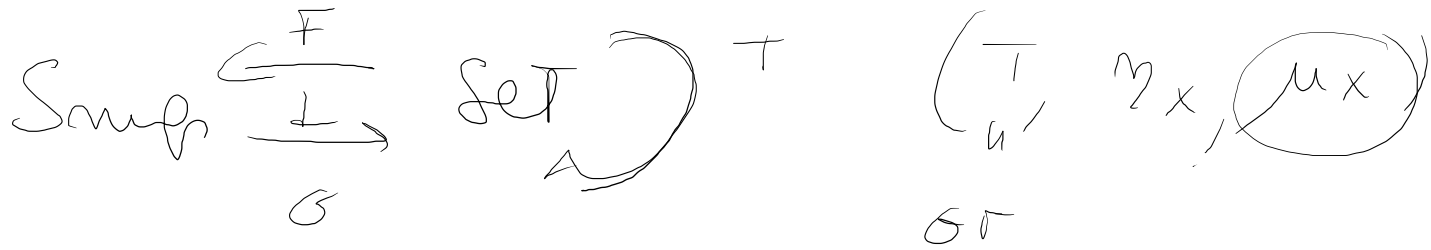
m words

$$\xrightarrow{\mu_x} (\alpha_1 \alpha_2 \alpha_3 \alpha_2 \alpha_2 \dots)$$

Oxions are fine



$$\begin{array}{l}
 \rho_X : X \rightarrow TX \\
 \mu_X : TX \rightarrow X
 \end{array}$$



Ex 2 $(X, \leq) = \mathcal{L} \leq \text{orden}$

$x \xrightarrow{T} Tx$

? type on \mathcal{L}

$\mathcal{L} \xrightarrow{T} \mathcal{L}$

$x \mapsto Tx$

T function that preserve order

1) $\gamma_x : x \rightarrow Tx$

$x \leq Tx$

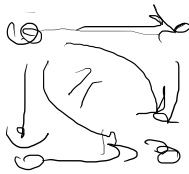
2) $\mu_x : Tx \rightarrow T^2x$

$T^2x \leq Tx$

$Tx \leq T^2x$

$\rightarrow Tx = T^2x$

~~$x \leq Tx$
are always true~~



T is a closure operator on X

Def T. algebra for a type T on \mathcal{C}

$$(T: \mathcal{C} \rightarrow \mathcal{C}, \mu)$$

is given by

$$(x \in \mathcal{C}, \underline{h: Tx \rightarrow x})$$

such that

axiom

$x \in \mathcal{C}$

$$\begin{array}{ccc} x & \xrightarrow{\mu_x} & Tx \\ & \searrow & \downarrow h \\ Tx & & x \end{array}$$

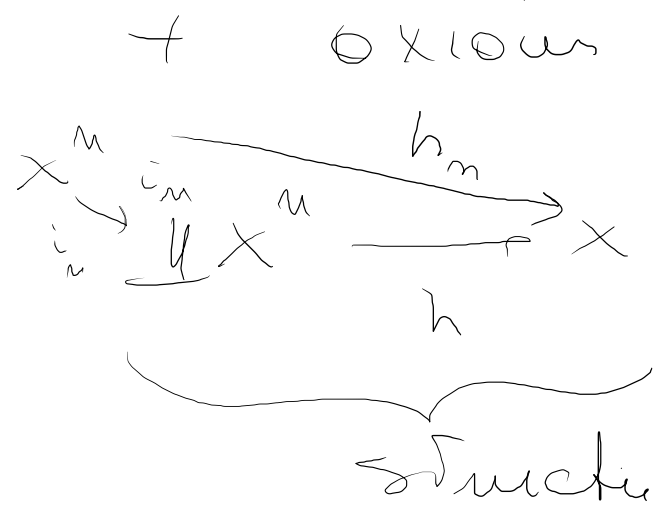
$$\begin{array}{ccc} Tx & \xrightarrow{\mu_x} & Tx \\ \downarrow h & & \downarrow h \\ Tx & \xrightarrow{h} & x \end{array}$$

Ex Samp $\begin{matrix} \xleftarrow{F} \\ \xrightarrow{I} \\ G \end{matrix}$ Set $\begin{matrix} T \\ \text{"} \\ GF \end{matrix}$ (T, η, μ)

T-algebra? $X \in \text{Set}$

$$h: TX \rightarrow X$$

$$h: \coprod X^n \rightarrow X$$



h_n is a
 canonical
 operation

The 2 axioms of T. algebra

They say that $h_1: X \rightarrow X$ is the identity

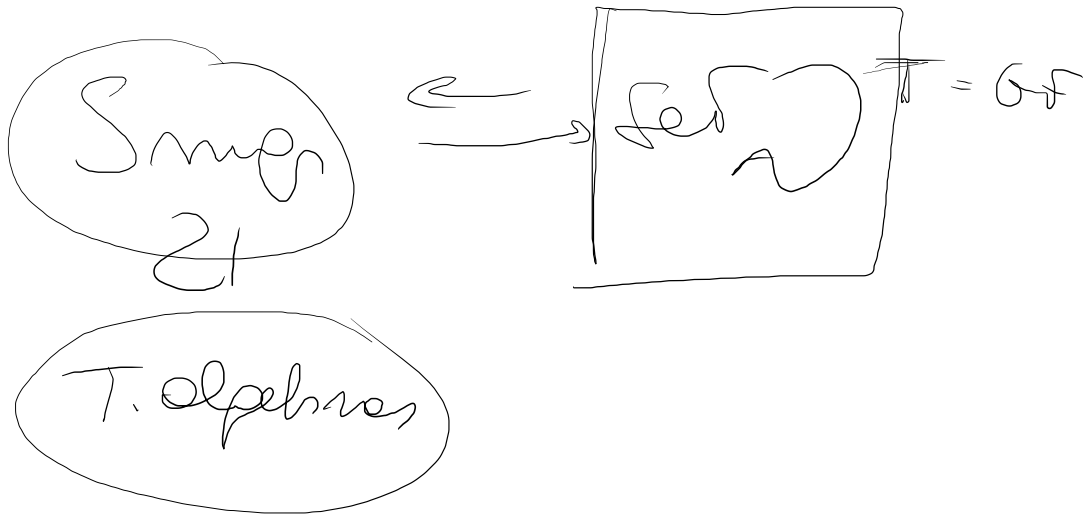
$$h_2: XXX \rightarrow X$$

by $ax. 2$ \rightarrow h_3 is obtained by h_2
its associativity

each h_n is obtained by h_2

$S_{\text{imp}} \rightleftharpoons \text{Set} \supset T = GF$

then T. algebras are exactly free groups



Ex $(X \subseteq \mathbb{R})$ colony

tuple $T: X \rightarrow X$ closure operator

? what is a T -algebra

$x \in X$

$h: Tx \rightarrow x$

$Tx \subseteq X$

$x \in Tx$

$x = Tx$

x is closed

T -algebras