copowers

coprodotti o somme

$$\underbrace{X \times Y \times Z}_{\text{prod. cat}}$$

duale coprodotti

$$\underbrace{X \sqcup Y \sqcup Z}_{\text{coprod}}$$

$X \times X \times X$

X^3 X^n

potenze

$X \sqcup X \sqcup X$

$\coprod X^3$ $\coprod X^n$

$n \in \mathbb{N}^+$

copotenze

Algebraic categories on an arbitrary base $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \arrow[loop right, "T"] \end{tikzcd}$$

Def $\boxed{\text{triple on } \mathcal{C}}$ (T, η, μ) $X \in \mathcal{C}$

$$\boxed{T : \mathcal{C} \to \mathcal{C}}$$

$$\boxed{\eta_X : X \to TX \quad \text{nat. transf}}$$

$$\boxed{\mu_X : T^2 X \to TX}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Id}"{name=U}] \arrow[r, bend right, "T"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$$

$$\begin{tikzcd} & T \\ \mathcal{C} \arrow[r, bend left, "T^2"{name=U}] \arrow[r, bend right, "T"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, Rightarrow, "\mu"] \end{tikzcd}$$

+ axioms

Axiomes.

$$\begin{tikzcd}
TX \arrow[r, "T\eta_X"] \arrow[dr, bend right, "1_{TX}"'] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & T^2X \arrow[d, "\mu_X"] & TX \arrow[l, "\eta_{TX}"'] \arrow[dl, "1_{TX}"] \arrow[l, phantom, shift right=6ex, "\circlearrowleft"] \\
 & TX &
\end{tikzcd}$$

$$\boxed{\mu_X \cdot T\eta_X = 1_{TX}}$$

$$\begin{tikzcd}
T^3X \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2X \arrow[d, "\mu_X"] \\
T^2X \arrow[r, "\mu_X"'] & TX
\end{tikzcd}$$

$$\boxed{\begin{array}{l} \mu_X \cdot T\mu_X = \\ = \mu_X \cdot \mu_{TX} \end{array}}$$

Def $\boxed{T\text{. algebra}}$ $\quad \mathcal{C} \xrightarrow{T} \mathcal{C} \quad (T, \eta, \mu)$

is a pair $\boxed{\left(X \in \mathcal{C},\ h : TX \to X\right)}$ $\quad$ h is the structure

+ 2 axioms

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "h"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2X \arrow[r, "\mu_X"] \arrow[d, "Th"'] \arrow[dr, phantom, "\circlearrowleft"] & TX \arrow[d, "h"] \\
TX \arrow[r, "h"'] & X
\end{tikzcd}$$

T-algebras form a category : $\mathcal{C}^T$ category of T-algebras

Def T-algebra MORPHISM $\boxed{f : (X, h) \to (Y, k)}$

$h : TX \to X$ $\qquad$ $k : TY \to Y$

is a morphism in $\mathcal{C}$ $\boxed{f : X \to Y}$ such that $\forall x, y$ $f(x \cdot y) = f(x) \cdot f(y)$

$$\begin{tikzcd}
TX \arrow[r, "h"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\
TY \arrow[r, "k"'] & Y
\end{tikzcd}$$

$G \xrightarrow{f} H$ group hom.

$$\begin{tikzcd}
G \times G \arrow[r, "\cdot"] \arrow[d, "f \times f"'] \arrow[dr, bend left] & G \arrow[d, "f"] \\
H \times H \arrow[r, "\cdot"'] & H
\end{tikzcd}$$

$\mathcal{C}^T$ — obj : T. algebra

— morph morph of T. algebra.

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category of T. algebra and morphism

$f : (X, h) \longrightarrow (Y, k)$ $\quad$ $g : (Y, k) \longrightarrow (T, t)$

$\boxed{f : X \longrightarrow Y}$ $\quad$ $\boxed{g : \underline{Y \longrightarrow T}}$

gf is T. algebras morphism.

Def: a category $\mathcal{A}$ is algebraic over $\mathcal{C}^T$ iff $\mathcal{A} \simeq \mathcal{C}^T$ T triple on $\mathcal{C}$

if $\mathcal{C} = Set$

$\mathcal{A}$ algebraic over Set

iff

$\mathcal{A}$ exact

$\exists\ P$ reg proj

reg gen

ADJOINT PAIRS $\longleftrightarrow$ T algebraic cats on $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "F"'] \end{tikzcd}$$

Theorem. If I have a triple (T, η, μ) on $\mathcal{C}$ then we get an adjoint pair

Then. If I have an adjoint $F \dashv G$, then we get a triple on $\mathcal{C}$

$$\begin{tikzcd} \boxed{\text{ADJ PAIR}} \arrow[r, leftrightarrow] & \boxed{\text{TRIPLE}} \arrow[l, Leftarrow, bend left] \end{tikzcd}$$

Thm. If (T, η, μ) is a triple on $\mathcal{C}$, then we get an adjoint pair

$$\begin{tikzcd} \mathcal{E}^T \arrow[r, bend right, "G^T"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F^T"'] \end{tikzcd}$$

Proof

$$\begin{tikzcd} \boxed{\mathcal{E}^T} \arrow[r, bend right, dashed, "G^T"'] \arrow[r, phantom, "\perp"] & \boxed{\mathcal{C}} \arrow[l, bend right, "F^T"'] \end{tikzcd}$$

category of T algebras

We must define G^T, F^T, and prove $F^T \dashv G^T$

$$\mathcal{C}^T \longrightarrow \mathcal{C}$$

$$= G^T$$

$$\boxed{G^T(\underbrace{X,h}_{T\text{.algebra}}) = X}$$

G^T is a

functi

$$\begin{tikzcd} (X,h) \arrow[d, "f"'] & \rightsquigarrow & X \arrow[d, "f"] \\ (Y,h_2) & & Y \end{tikzcd}$$

morphism of T.algebra

$$\boxed{G^T \text{ forgetful functor}}$$

$\mathcal{C} \xrightarrow{F^T} \mathcal{C}^T$ F^T is the free T-algebra functor.

$\forall X \in \mathcal{C}$ $\boxed{F^T(X) = (TX, \mu_X)}$

$\mu_X : T(TX) \to TX$

Ex $\mathrm{Set} \to \mathrm{Grp}$

$X \to \underline{F(X)}$

Verify :

① (TX, μ_X) is a T-algebra

② F^T is a functor

③ $F^T + G^T$ are adjoints

① (TX, μ_X) is a T-algebra on $\mathcal{C}$

$(A, h : TA \to A)$

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "h"] \arrow[dl, phantom, "\circlearrowright"] \\
 & A
\end{tikzcd}$$

$$\begin{tikzcd}
TX \arrow[r, "\eta_{TX}"] \arrow[dr, "1_{TX}"'] & T^2X \arrow[d, "\mu_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & TX
\end{tikzcd}$$

axiom of triple

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true

① $\boxed{(TX, \mu_X)}$ is a T-alg.

ex 2

$$\begin{tikzcd}
(A, & h) \\
TX \arrow[u, equal] & \mu_X \arrow[u, equal]
\end{tikzcd}$$

$$\begin{tikzcd}
T^2A \arrow[r, "\mu_A"] \arrow[d, "Th"'] & TA \arrow[d, "h"] \\
TA \arrow[r, "h"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
T^3X \arrow[r, "\mu_{TX}"] \arrow[d, "T\mu_X"'] \arrow[dr, phantom, "\circlearrowright"] & T^2X \arrow[d, "\mu_X"] \\
T^2X \arrow[r, "\mu_X"'] & TX
\end{tikzcd}$$

true by def. of TRIPLE

② F^T is a functor

$$\mathcal{C} \longrightarrow \mathcal{C}^T$$

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[r, phantom, "\xrightarrow{F^T}"] & (TX, \mu_X) \arrow[d, "Tf"] \\
Y & (TY, \mu_Y)
\end{tikzcd}$$

in $\mathcal{C}$ $\qquad\qquad$ $\mathcal{C}^T$

$$\boxed{F^T(f) \overset{\text{def}}{=} Tf}$$

Tf is a morphism of T-algebras

??

$$\begin{tikzcd}
T^2X \arrow[r, "\mu_X"] \arrow[d, "T^2f"'] \arrow[dr, phantom, "\circ"] & TX \arrow[d, "Tf"] \\
T^2Y \arrow[r, "\mu_Y"] & TY
\end{tikzcd}$$

$\boxed{\mu \text{ is a natural transf.}}$

③ $\boxed{F^T \dashv G^T}$

$$\begin{tikzcd} \mathcal{E}^T \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend right, "F^T"'] \arrow[l, bend left, leftarrow, "G^T"] \arrow[loop right, "T"] \end{tikzcd}$$

use the charact. theorem of adjoints

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F"'] \end{tikzcd}$$

$F \dashv G$ iff

- ∃ ⓷ $\boxed{\text{nat. transf.}}$ $\boxed{\eta_X : X \to GFX}$ $\forall X \in \mathcal{C}$
- ∃ ⓔ $\boxed{\text{nat. transf.}}$ $\forall A \in \mathcal{A}$ $\boxed{\varepsilon_A : FGA \to A}$

\+ 2 triangular IDENTITIES

$$\begin{tikzcd} FX \arrow[r, "F\eta_X"] \arrow[dr, "1_{FX}"'] & FGFX \arrow[d, "\varepsilon_{FX}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & FX \end{tikzcd}$$

$$\begin{tikzcd} GA \arrow[r, Rightarrow, "\eta_{GA}"] \arrow[dr, "1_A"'] & GFGA \arrow[d, "G\varepsilon_A"] \arrow[dl, phantom, "\circlearrowleft"] \\ & GA \end{tikzcd}$$

$$\boxed{F^T \dashv G^T}$$

$$\eta^T_X : X \longrightarrow G^T F^T(X) = G^T(\underline{TX, \mu_X})$$

$$= TX$$

$$\boxed{\eta^T_X : X \longrightarrow TX}$$

DEF.

$$\boxed{\eta^T_X = \eta_X}$$

η_X is the η of the triple (T, η, μ) ✓ well kno

$$\boxed{A = (C, h : TC \to C)}$$

$$\boxed{\varepsilon^T_A : F^T G^T A \to A}$$

$$\boxed{\varepsilon^T_{(C,h)} : F^T G^T (C, h) \longrightarrow (C, h)}$$

$$\boxed{\varepsilon^T_{(C,h)} : \underline{(TC, \mu_C)} \longrightarrow \underline{(C,h)}}$$ a T. algebra morphism

Def $$\boxed{\varepsilon^T_{(C,h)} = h}$$

This is a morphism of T. algebra.

$\#$ means that the diagram commute

$$\begin{tikzcd}
T^2C \arrow[r, "\mu_C"] \arrow[d, "Th"'] \arrow[dr, phantom, "\#"] & TC \arrow[d, "h"] \\
TC \arrow[r, "h"'] & C
\end{tikzcd}$$

— $\boxed{\text{axiom of T. algebra}}$ (C,h)

ε^T is a natural transf

Verify that the identity equations of an adjoint pair are true.

$$\begin{tikzcd}
FX \arrow[r, "F\eta X"] \arrow[dr, "1_{FX}"'] \arrow[r, phantom, shift right=6ex, "\Rightarrow"] & FGFX \arrow[d, "\varepsilon_{FX}"] \\
& FX
\end{tikzcd}$$

$$\begin{tikzcd}
F^T(X) \arrow[r, "F^T\eta^T_X"] \arrow[dr, "1_{F^TX}"'] & F^TG^T\underline{F^T(X)} \arrow[d, "\varepsilon^T_{F^TX}"] \\
& F^T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
(TX, \mu X) \arrow[r, "T\eta_X"] \arrow[dr, "1_{(TX\mu X}"'] \arrow[r, phantom, shift right=6ex, "\Rightarrow"] & (T^2X, \mu_{TX}) \arrow[d, "\mu_X"] \\
& (TX, \mu X)
\end{tikzcd}$$

true is an axiom of TRIPLE

identity for adjoint form.

$$\begin{tikzcd}
GA \arrow[r, "\eta_{GA}"] \arrow[dr, "1_{GA}"'] & GFGA \arrow[d, "G\varepsilon_A"] \\
 & GA
\end{tikzcd}$$

$$\boxed{A = (X, h)}$$

$\in \mathcal{C}$

$$\begin{tikzcd}
G^T(X,h) = X \arrow[r, "\eta^T_X = \eta_A"] \arrow[dr, "1_X"'] & G^TF^TG^T(X,h) \\
 & TX \arrow[d, "h"] \\
 & X
\end{tikzcd}$$

true by def of (X, h) T-algebra

we proved

TRIPLE (T, η, μ) $\longrightarrow$ ADJOINT PAIR

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, bend right, "G^T"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "F^T"'] \end{tikzcd}$$

T-alg

$$\boxed{F^T(X) = (TX, \mu_X)}$$

ADJOINT PAIR $\longrightarrow$ TRIPLE

$$\begin{tikzcd} A \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "F"'] \arrow[loop right, "T = GF"] \end{tikzcd}$$

hypothesis $F \dashv G$

$\overset{?}{\rightsquigarrow}$

I want to construct a triple on $\mathcal{C}$

Proof $T : \mathcal{C} \to \mathcal{C}$ T is defined by $\boxed{T = GF}$

$$\boxed{\eta^T \underset{\text{DEF}}{=} \eta} \qquad \boxed{\eta^T_X : X \to TX = GFX}$$

$$A \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{C}$$

$$\boxed{F \dashv G, \quad \eta, \quad \varepsilon}$$

$$\boxed{\begin{array}{c} \underline{(T, \eta^T, \mu^T)} \\ T = GF \\ \eta^T = \eta \end{array}}$$

$$\mu^T_X = G\varepsilon_{FX}$$

$$\varepsilon_A : FGA \to A$$

$$\mu^T ?$$

$$\boxed{\mu^T_X : T^2 X \longrightarrow TX}$$

$$A = FX$$

$$\boxed{\varepsilon_{FX} : FGFX \to FX}$$

?? $\mu^T_X : GFGFX \longrightarrow GFX$

$$= \quad \boxed{\mu^T_X \overset{def}{=} G\varepsilon_{FX}}$$

verify $\boxed{\mu^T \text{ is natural}}$

$\mu^T_X \stackrel{\text{def}}{=} G\varepsilon_{FX}$ $\qquad$ ε is natural

verification: $\boxed{(T, \eta^T, \mu^T) \text{ is a TRIPLE}}$

3 axioms of a triple are true . . .

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "K"'] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend right, "F"'] \arrow[loop right, "T=GF"] \arrow[dl, bend left, "F^T"] \\
\mathcal{E}^T \arrow[ur, bend right, "G^T"'] &
\end{tikzcd}$$

COMPARISON FUNCTOR

$$\begin{tikzcd}
\mathrm{Sup} \arrow[r, bend right, "G"'] & \mathrm{Set} \arrow[l, bend right, "F"'] \arrow[loop right, "T"] \arrow[dl, bend left] \\
\mathrm{Set}^T \arrow[ur, bend right] &
\end{tikzcd}$$

Sup

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Set^T

adjoint $F \dashv G$

$\Downarrow$ hnpe

$C^T, \eta, \mu,$

$\Downarrow$

adjoint

$F^T \dashv G^T$