

copowers

$$X \times Y \times Z$$



prod. cat

$$X \times X \times X$$

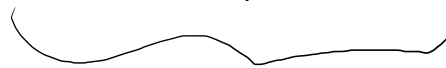
$$X^3 \quad X^m$$

for sense

coproducts of  $S$  or  $U$

dual coproduct

$$X \amalg Y \amalg Z$$



coprod

$$X \amalg X \amalg X$$

$$\amalg X^3$$

coproduct

$$\amalg X^m$$

$m \in \mathbb{N}^k$

# Algebraic categories on an arbitrary base $\mathcal{C}$

def tuple on  $\mathcal{C}$

$$T: \mathcal{C} \rightarrow \mathcal{C}$$

$$\mu_x: T^2 X \rightarrow TX$$

$$(T, \eta, \mu)_{x \in \mathcal{C}}$$

$$\eta_x: X \rightarrow TX$$

mul. heap

$$\mathcal{C} \circ T$$

$$\mathcal{C} \xrightarrow{\text{Id}} \mathcal{C}$$

$$\downarrow \eta$$

$$\mathcal{C}$$

$$\mathcal{C} \xrightarrow{T} \mathcal{C}$$

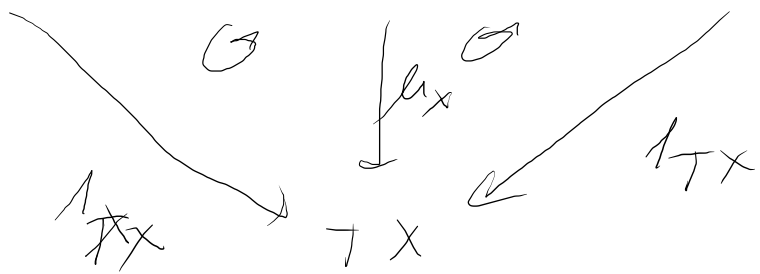
$$\downarrow \eta$$

$$T$$

+ axioms

Axioms.

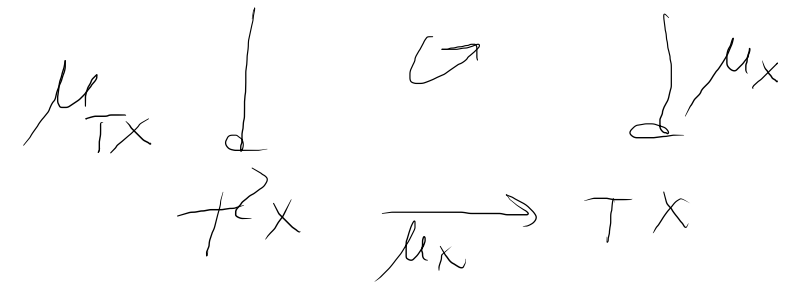
$$TX \xrightarrow{T\mu_X} T^2X \xleftarrow{\mu_{TX}} TX$$



$$\mu_X \cdot T\mu_X = \mu_{TX}$$

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$$T^3X \xrightarrow{T\mu_X} T^2X$$



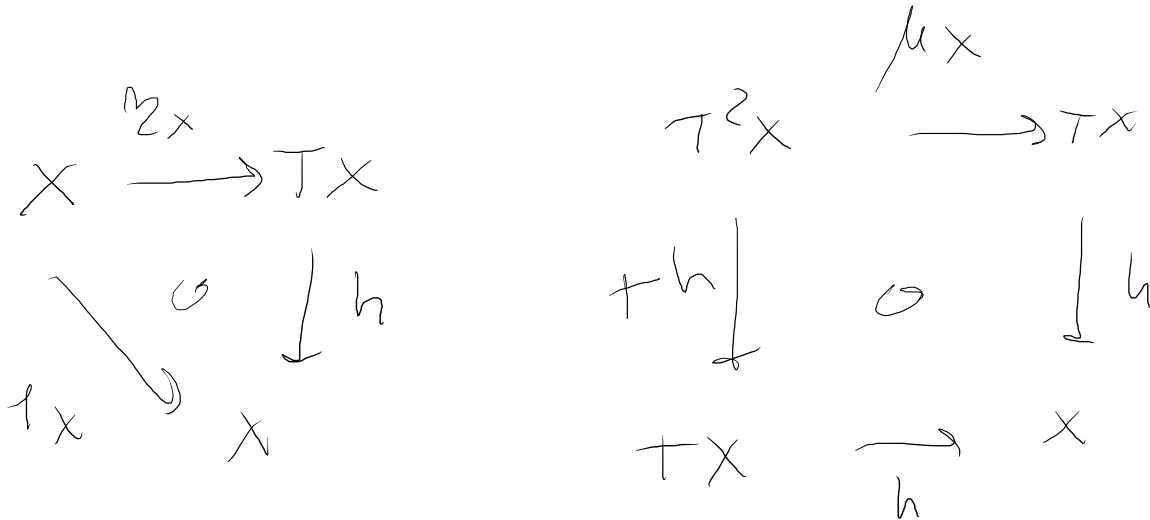
$$\mu_X \cdot T\mu_X = \mu_X \cdot \mu_{TX}$$

Def T. algebra  $\mathcal{C} \xrightarrow{T} \mathcal{C} \quad (T, \eta, \mu)$

is a pair  $(X \in \mathcal{C}, \underline{h: TX \rightarrow X})$

$h$  is the structure

+ 2 axioms



T-algebras form a category:  $\mathcal{C}^T$  category of T-algebras

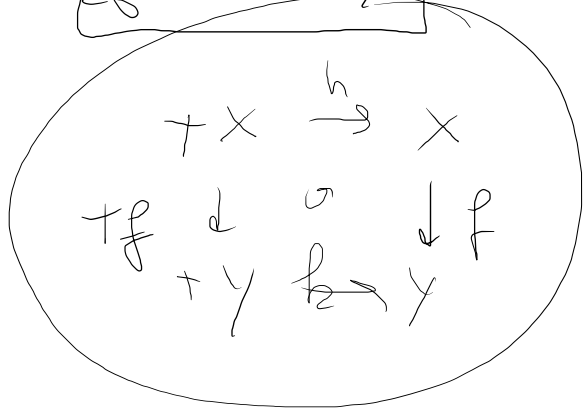
A T-algebra morphism  $f: (X, h) \rightarrow (Y, k)$

$$h: TX \rightarrow X$$

$$k: TY \rightarrow Y$$

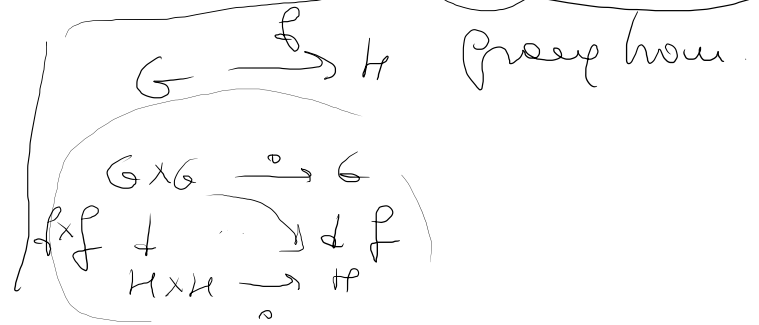
is a morphism in

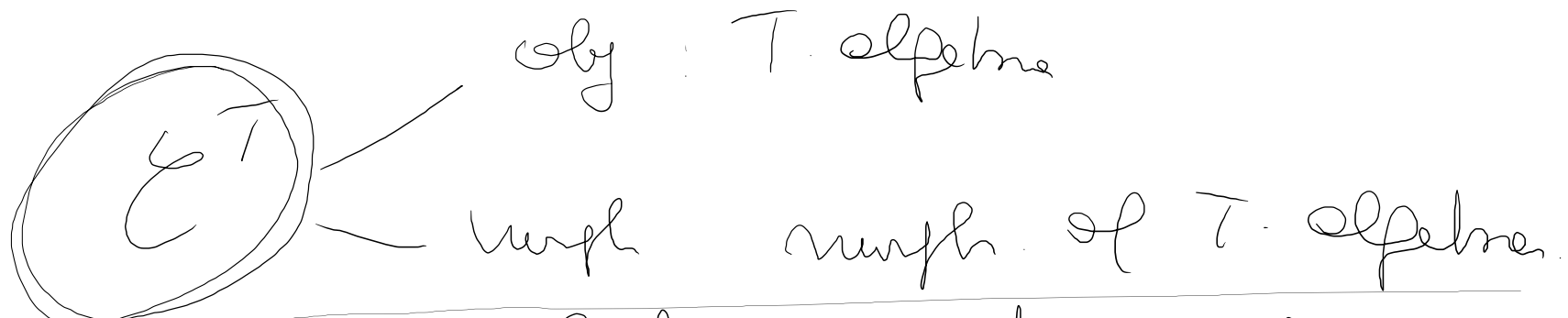
$$\mathcal{C} \quad f: X \rightarrow Y$$



such that

$$f(x \cdot y) = f(x) \cdot f(y)$$





category of T-algebras and morphisms

$$f: (x, h) \rightarrow (y, k) \quad g: (y, k) \rightarrow (T, t)$$

$$f: x \rightarrow y$$

$$g: y \rightarrow T$$

$gf$  is T-algebras morphism.

def a category  $\mathcal{A}$  is algebraic over  $\mathcal{E}$   
iff  $\mathcal{A} \simeq \mathcal{E}^T$   $T$  triple on  $\mathcal{E}$

f  $\mathcal{E} = \text{Set}$

$\mathcal{A}$  algebraic over  $\text{Set}$

iff

$\mathcal{A}$  exact

$\exists P$

rep pres

rep pres

ADJOINT PAIRS

$\longleftrightarrow$  T algebraic cols  
on  $\mathcal{C}$

$$A \xrightarrow[\sigma]{F} \mathcal{C}$$

Thm 1.

If I have a triple  $(T, \eta, \mu)$  on  $\mathcal{C}$  then we get an adjoint pair

then

If I have an adjoint pair  $F \dashv G$  then we get a triple on  $\mathcal{C}$

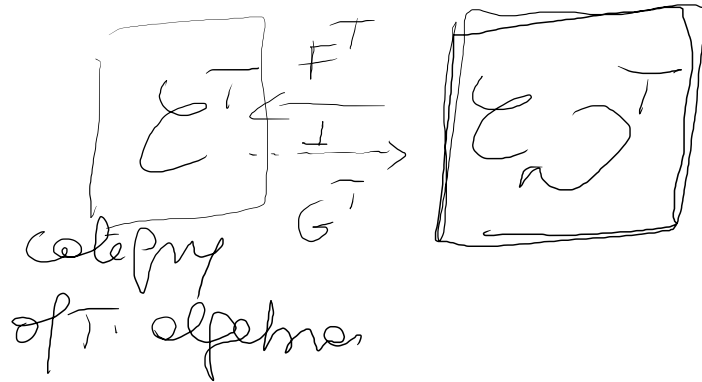




Then if  $(T, \eta, \mu)$  is a triple on  $\mathcal{C}$ , then we get an adjoint pair.

$$\mathcal{C} \begin{array}{c} \xleftarrow{F^T} \\ \perp \\ \xrightarrow{G^T} \end{array} \mathcal{C}$$

Proof



we must define  $G^T, F^T$  and map  $F^T \rightarrow G^T$

$$\mathcal{L}^T \xrightarrow{G^T} \mathcal{L}$$

$$G^T(x, h) = x$$

T-algebra

$G^T$  is a  
functor

$(x, h)$

$x$

$\rightsquigarrow \downarrow f$

$\mathcal{L}$

$y$

$(y, h_2)$

members  
of T-algebra

$G^T$  forgetful  
functor

$\mathcal{C} \xrightarrow{F^T} \mathcal{C}^T$   $F^T$  is the free

~~T. algebra~~ T. algebra functn.

$\forall X \in \mathcal{C}$

$$F^T(X) = (TX, \mu_X)$$

Ex Set  $\rightarrow$  Grp

$$\mu_X: T(TX) \rightarrow TX$$

$X \rightarrow \underline{F(X)}$

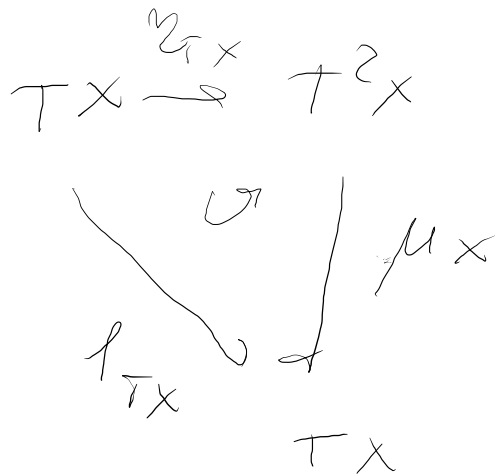
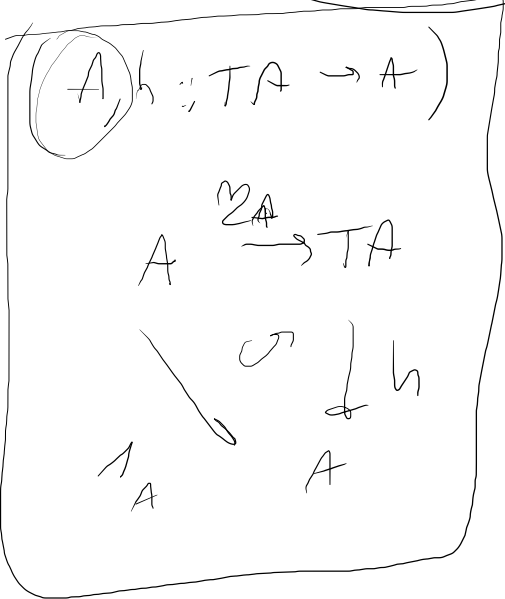
Verify  $\circ \rightarrow$  ①  $(TX, \mu_X)$  is a T. algebra

②  $F^T$  is a functor.

③  $F^T \dashv G^T$  on objects

$\lambda$   $(TX, \mu_X)$

$\omega \in T\text{algebra on } \mathcal{L}$



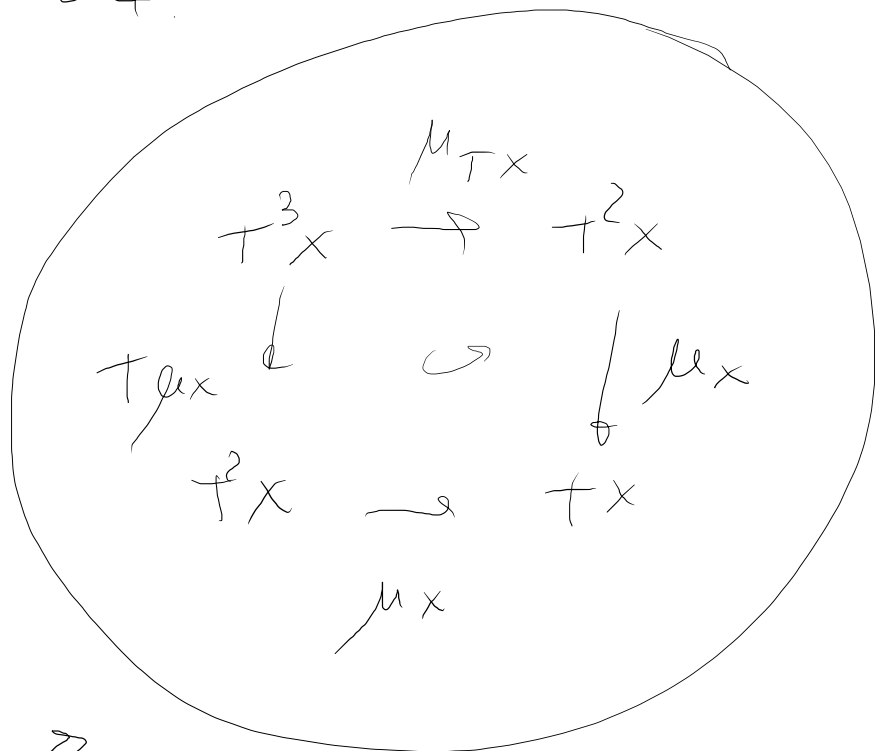
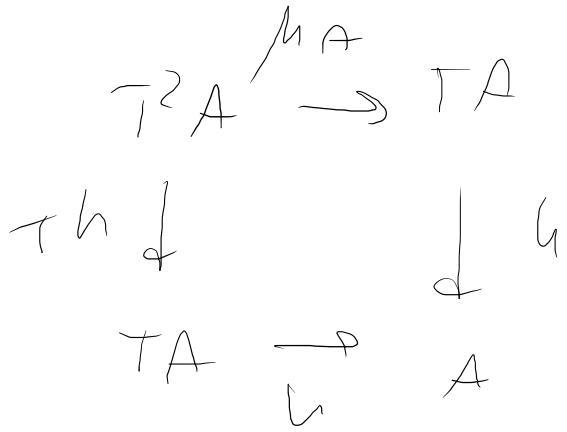
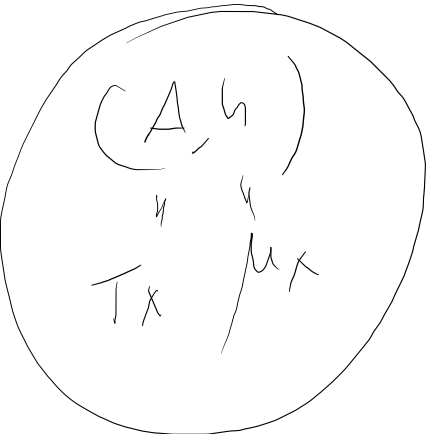
existence of  
triple

triple

①

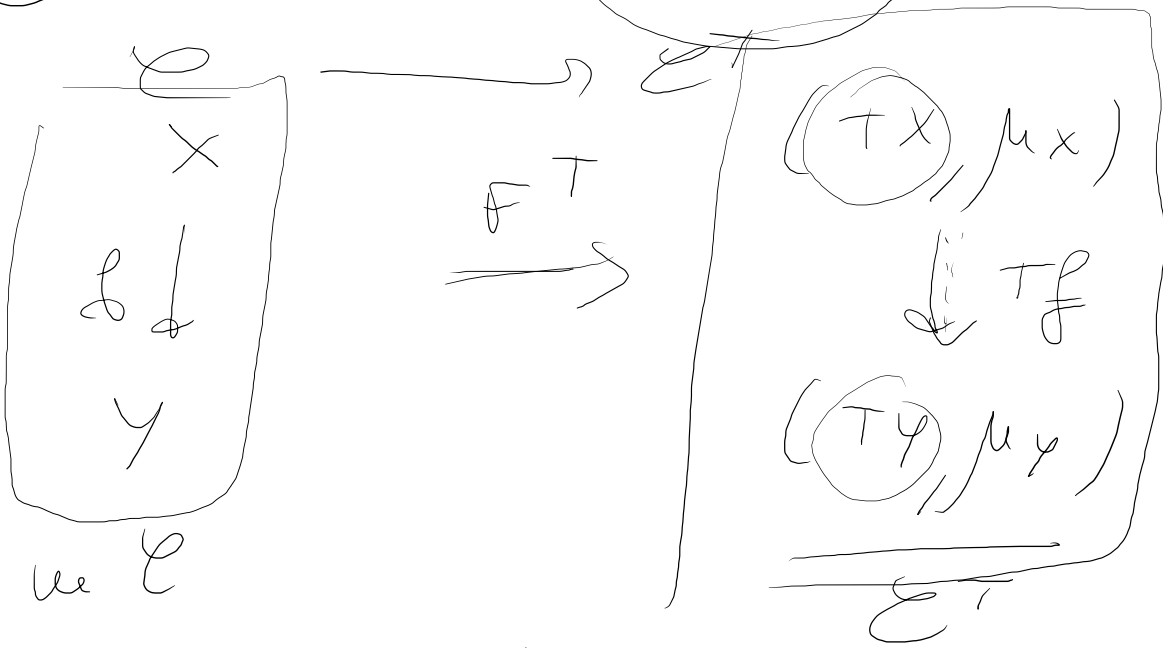
$(Tx, \mu_x)$  is a T. alp.

ex 2



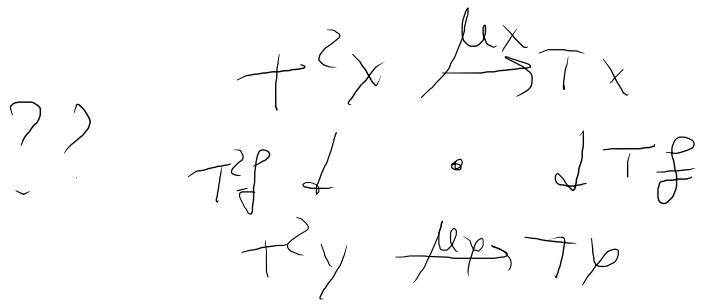
here by def. of TRIPLO

② FT is a functor



$$FT(f) = Tf$$

$Tf$  is a morphism of  $T$ -algebras



$\mu$  is a natural transformation

③  $F^T \rightarrow G^T$

$E^T \begin{matrix} \xleftarrow{F^T} \\ \perp \\ \xrightarrow{G^T} \end{matrix} E^T$

use the character theorem of adjoints

$A \begin{matrix} \xleftarrow{F} \\ \perp \\ \xrightarrow{G} \end{matrix} E$

$F + G \quad \text{iff}$

$\exists \eta$  mod. hom.

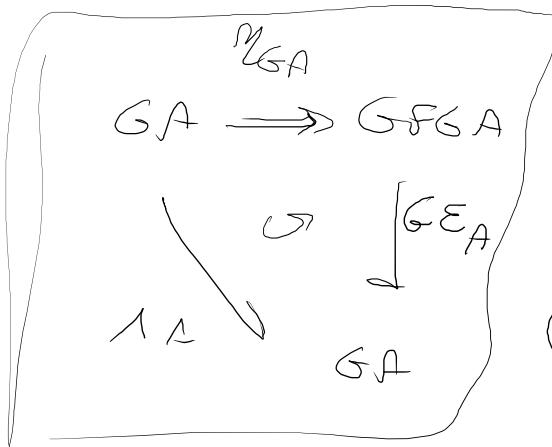
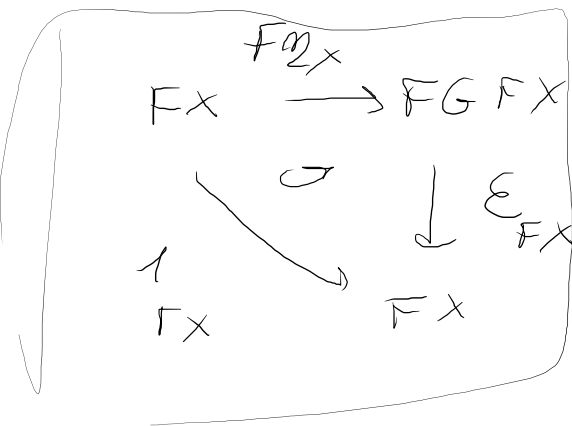
$\forall x: x \rightarrow \sigma Fx \quad \forall x \in \rho$

$\exists \epsilon$  mod. hom.

$\forall a \in A$

$E_A \circ FGA \rightarrow A$

+ 2 more identities







$$F^T \rightarrow G^T$$

$$\sum_x^T : x \rightarrow G^T F^T(x) = G^T(\underbrace{Tx}_{\mu_x}) = Tx$$

$$\sum_x^T : x \rightarrow \cancel{Tx}$$

DEF

$$\eta_x^T = \eta_x$$

$\eta_x$  is the  $\eta$  of the tuple  $(T, \eta, \mu)$  ↙ lead to

$$\varepsilon_A^T : F^T G^T A \rightarrow A$$

$$A = (c, h, \dots \rightarrow c)$$

$$\varepsilon_{(c, h)}^T : F^T G^T (c, h) \rightarrow (c, h)$$

$$\varepsilon^T_{(c,h)} : (TC, \mu_c) \rightarrow (c, h)$$

a T-algebra morphism

Def

$$\varepsilon^T_{(c,h)} = h$$

This is a morphism of T-algebras.

/// means that the degree counts

$$\begin{array}{ccc} TC & \xrightarrow{\mu_c} & TC \\ \downarrow h & \equiv & \downarrow h \\ TC & \xrightarrow{h} & C \end{array}$$

axiom of T-algebra  $(c, h)$

$\varepsilon^T$  is a natural transformation

Verify that the identity equations of an adjoint pair are true.

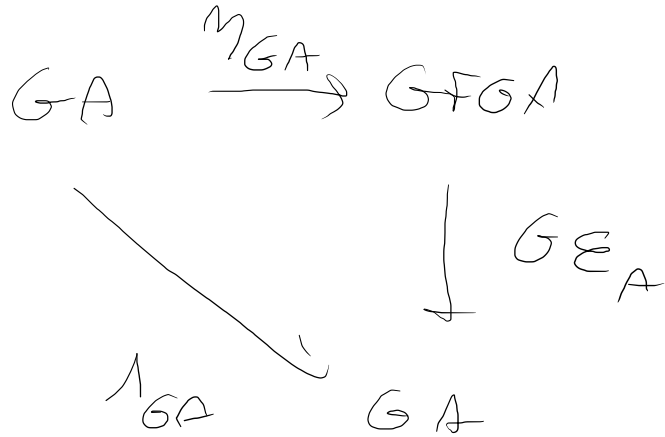
$$\begin{array}{ccc}
 FX & \xrightarrow{F\eta_X} & FGFX \\
 \searrow \downarrow \varepsilon_{FX} & \circlearrowright & \\
 & & FX \\
 \swarrow \downarrow 1_{FX} & & 
 \end{array}$$

$$\begin{array}{ccc}
 F^T(X) & \xrightarrow{F^T\eta_X^T} & F^T G^T F^T(X) \\
 \searrow \downarrow 1_{F^T X} & & \\
 & & F^T(X)
 \end{array}$$

$$\begin{array}{ccc}
 (TX, \mu_X) & \xrightarrow{T\eta_X} & (T^2 X, \mu_{TX}) \\
 \searrow \downarrow 1_{(TX, \mu_X)} & \circlearrowright & \\
 & & (TX, \mu_X)
 \end{array}$$

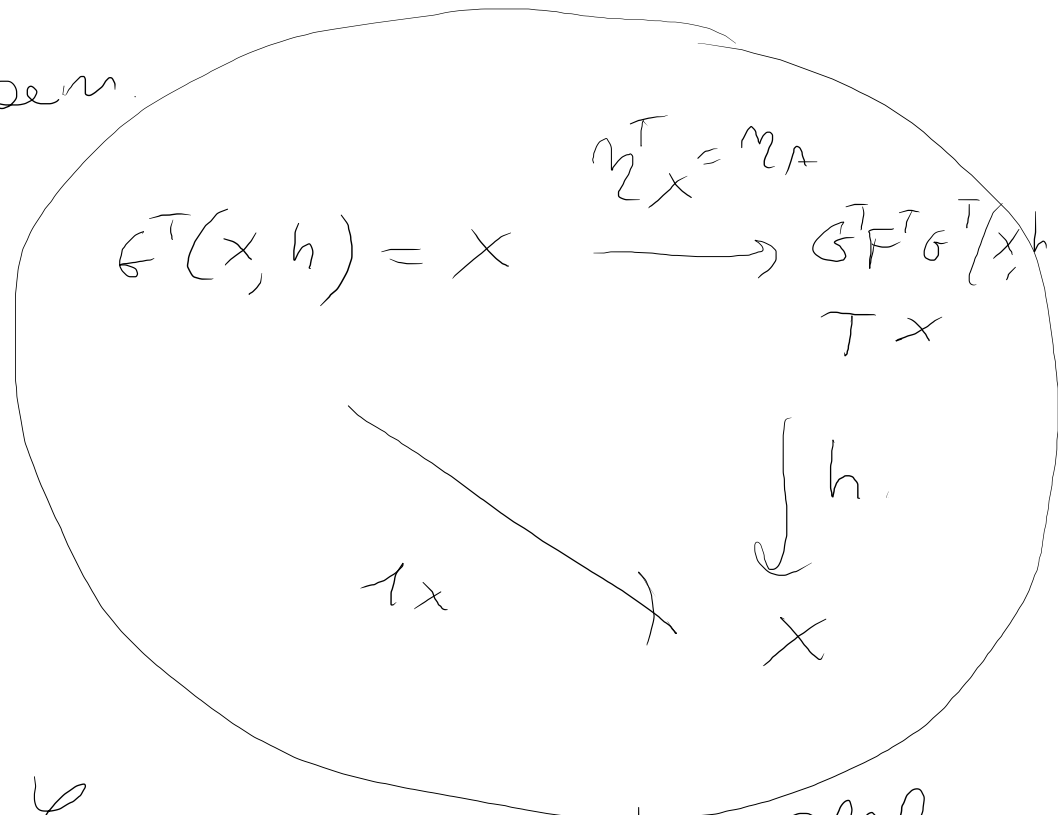
here is an axiom of TRIPLES

identity for adjoint form.



$$A = (x, h)$$

$\in \mathcal{L}$



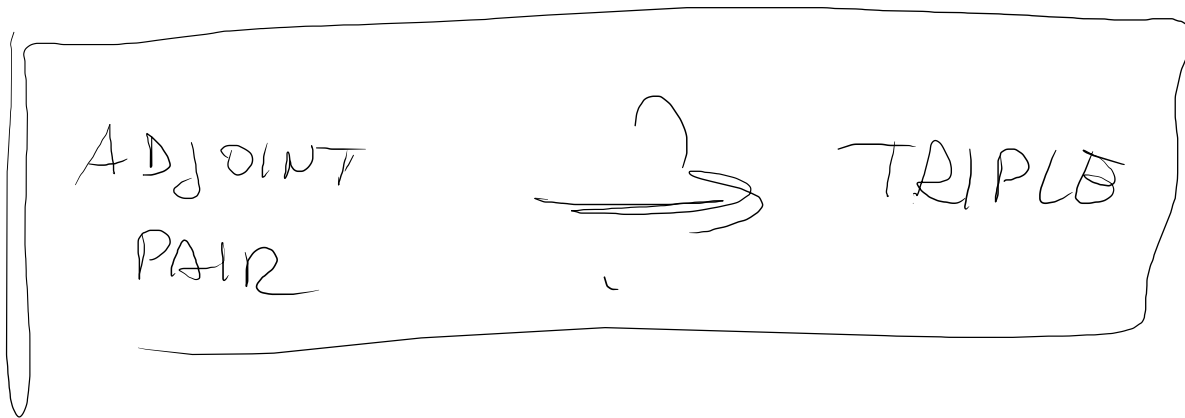
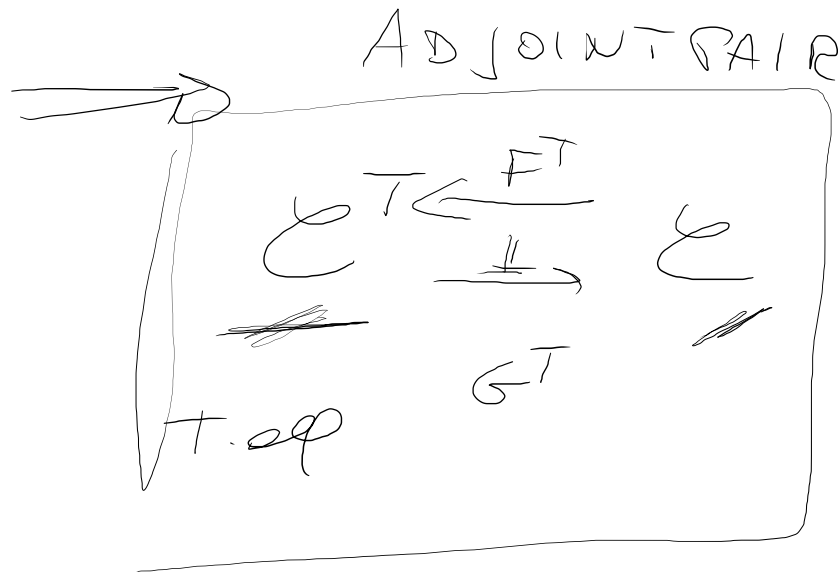
true by def of  $(x, h)$   
T. alpha

we have

TRIPLE

$$(T, \eta, \mu)$$

$$F^T(x) = (Tx, \mu_x)$$





I want to construct a triple on  $\mathcal{C}$

hypothesis  $F \circ G$

Proof  $T: \mathcal{C} \rightarrow \mathcal{C}$   $T$  is defined by  $T = GF$

$$\eta_{DEF}^T = \eta$$

$$\eta_X^T: X \rightarrow TX = GF X$$

$$A \xrightarrow{\delta} \mathcal{E}$$

$\delta$

$$F \dashv G, \eta, \epsilon$$

$$(T, \eta^T, \mu^T)$$

$$T = GF$$

$$\eta^T = \eta$$

$$\mu_x^T = G \epsilon_{FX}$$

$$\epsilon_A : FGx \rightarrow A$$

$$A = FX$$

$$\epsilon_{FX} : FGFX \rightarrow FX$$

$\mu^T?$

$$\mu_x^T : T^2x \rightarrow Tx$$

$$\mu_x^T : GF GFx \rightarrow GFx$$

$$\mu_x^T \stackrel{\text{def}}{=} G \epsilon_{FX}$$

verify

$\mu^T$  is natural

$\mu_X^T \stackrel{e_i}{=} G \stackrel{E}{=} FX$

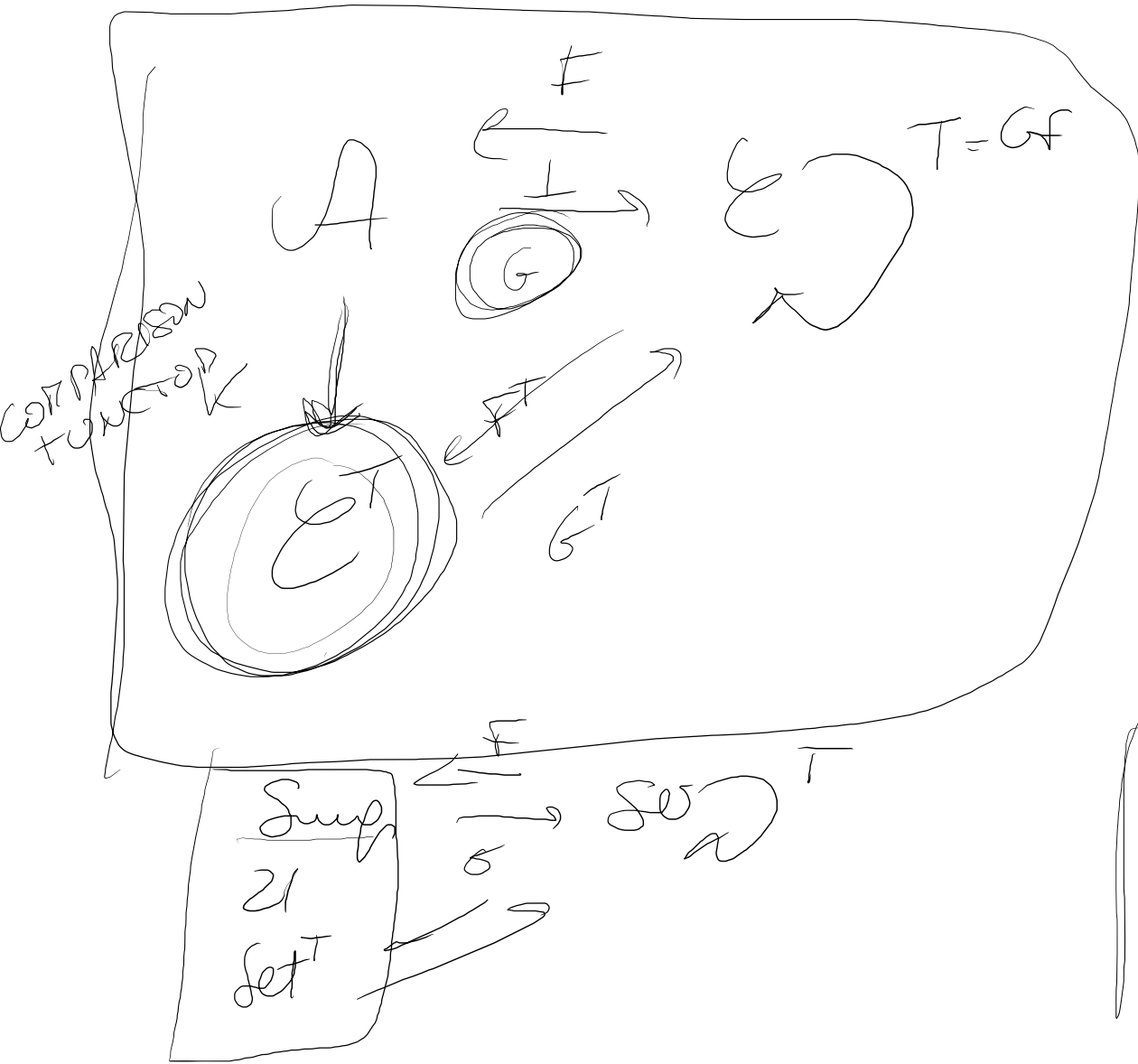
$\varepsilon$  is natural

verification:

$(T, \eta^T, \mu^T)$  is a TRIPLE

3 axes of a triple are two ...





adjoint  $F \rightarrow G$

$\downarrow$  adj

$(T, \eta, \mu)$

$\downarrow$

adjoint

$F^T \rightarrow G^T$