adjoint functors ⇄ triple

def : A is algebraic over a base $\mathcal{C}$

iff $A \simeq \mathcal{C}^{T}$ T triple on $\mathcal{C}$

A algebraic on Set iff

① A exact

② $\exists P$ . . . reg gen
reg proj.

Ab | Grp

tuple $\overset{?}{\longmapsto}$ canonical

adjoint pair

$$\boxed{(T : \mathcal{C} \to \mathcal{C}, \eta, \mu)}$$

$$\mathcal{C}^T \underset{G^T}{\overset{F^T}{\underset{\perp}{\leftrightarrows}}} \mathcal{C}$$

$$\underbrace{\mathcal{C}^T}_{\text{cat of } T\text{-algeb.}} \qquad \eta^T \quad \varepsilon^T$$

$$G^T(\underbrace{X, h : TX \to X}) \overset{\text{def}}{=} X$$

$$f : (X, h) \to (Y, k)$$

$$\begin{array}{ccc} TX & \xrightarrow{h} & X \\ {\scriptstyle Tf}\downarrow & \circlearrowright & \downarrow{\scriptstyle f} \\ TY & \xrightarrow[k]{} & Y \end{array}$$

$X \in \mathcal{C}$

$$\boxed{F^T(X) \underset{\text{def}}{=} (TX, \mu_X)}$$

$\eta^T = \eta$ of the triple

ε^T $\forall\ (X, h) \in \mathcal{E}^T$

counit of $F^T \dashv G^T$

$\varepsilon^T_{(X,h)} : F^T G^T (X, h) \to (X, h)$

$\varepsilon^T_{(X,h)} : (TX, \mu_X) \to (X, h)$

$\overset{\text{def}}{=} h$

Conversely

$$\begin{tikzcd} A \arrow[r, bend left, leftarrow, "F"] \arrow[r, phantom, "\perp"] \arrow[r, "G"'] & \mathcal{C} \end{tikzcd} \quad \Longrightarrow \quad \text{have on } \mathcal{C} \quad (T, \eta^T, \mu^T)$$

adjoint

$$(F \dashv G, \eta, \varepsilon)$$

$$\boxed{T = GF}$$

$$\boxed{\eta^T = \eta}$$

$$\mu^T_X : GFGFX \to GFX \qquad \boxed{\mu^T_X = G\,\varepsilon_{FX}}$$

triple (T, η, μ) on $\mathcal{C}$ $\rightsquigarrow$ $F^T \dashv G^T$ $\cancel{\rightsquigarrow}$ $(\overline{T}, \overline{\eta}, \overline{\mu})$

$\mathcal{C}^T \underset{G^T}{\overset{F^T}{\rightleftarrows}} \mathcal{C} \circlearrowright \overline{T}$ $\qquad \mathcal{C} \circlearrowright \overline{T}$

$\overline{T}X = G^T F^T(X) = TX$ $\qquad$ same for morphisms

$\boxed{\overline{T} = T}$ $\qquad \overline{\eta} = \eta^T = \eta$

$F^T(X) = (TX, \mu_X)$

$\boxed{\overline{\mu}_X = G^T \varepsilon^T_{F^T X} = G^T \mu_X} = \mu_X$

$$\begin{tikzcd} \mathcal{E} \arrow[loop right, "T"] \end{tikzcd}$$

(T, η, μ)

$$\rightsquigarrow \quad \mathcal{E}^T \underset{U^T}{\overset{F^T}{\rightleftarrows}} \mathcal{E} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{E} \arrow[loop right, "\overline{T}"] \end{tikzcd}$$

$$(T, \eta, \mu) = (\overline{T}, \overline{\eta}, \overline{\mu})$$

$$\begin{tikzcd} \mathcal{E}^T \arrow[r, shift right, "U^T"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift right, "F^T"'] \arrow[loop right, "T"] \arrow[loop below, "\overline{T}"] \end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend right, "F"'] \arrow[loop right, "T"]
\end{tikzcd}
\rightsquigarrow (T, \eta, \mu) \rightsquigarrow
\begin{tikzcd}
\mathcal{E}^T \arrow[r, bend right, "G^T"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, bend right, "F^T"']
\end{tikzcd}
$$

$(F \dashv G, \eta, \varepsilon)$

ARE NOT THE SAME

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp"] \arrow[d, "K"'] & \mathcal{E} \arrow[l, bend right, "F"'] \arrow[loop right, "T"] \arrow[dl, bend right, "F^T"'] \\
\mathcal{E}^T \arrow[ur, bend right, "G^T"'] &
\end{tikzcd}
$$

K: comparison functor

$\mathcal{A}$ algebraic $\iff$ K is an iso

Ex

$$\begin{tikzcd}
\mathrm{Smon} \arrow[d, "U"', "\simeq"] & & \mathrm{Set} \arrow[ll, bend right, "F"'] \arrow[ll, phantom, "\perp"] \arrow[loop right, "T"] \arrow[dll, shift right, "F^T"'] \\
\mathrm{Set}^T \arrow[urr, shift right, "G^T"'] & &
\end{tikzcd}$$

G

$FX = \coprod X^n$

$$\boxed{TX = \coprod X^n}$$

a T-algebra

$$\boxed{X,\ h : \underbrace{\coprod X^n}_{TX} \longrightarrow X}$$

$$\begin{tikzcd}
X^n \arrow[d, "i_n"'] \arrow[dr, "\omega_n"] & \\
\coprod X^n \arrow[r, "h"'] & X
\end{tikzcd}$$

n-ary operation

axioms $\longrightarrow$

ω_1 is the identity

$\boxed{\text{ASSOC}}$ is ω_2

$\omega_3 - \omega_n$ is induced by ω_2

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] \arrow[d, "G"'] & \mathbf{Set} \arrow[l, shift right, "F"', "\text{DISCR.}"] \arrow[loop right, "T = \mathrm{Id}"] \arrow[dl, shift left, "G' = \mathrm{Id}"] \\ \mathbf{Set} = \mathbf{Set}^T \arrow[ur, shift left, "F^T"] & \end{tikzcd}$$

$T(X) = GF(x) = X$

$$\boxed{T = \mathrm{Id}}$$

T-alg $\quad (X,\ h : TX = X \longrightarrow X)$ such that

$$\begin{tikzcd} X \arrow[r, "\eta_X"', "\mathrm{Id}"] \arrow[dr, "1_X"'] & TX = X \arrow[d, "h"] \\ & X \end{tikzcd} \longrightarrow h = \mathrm{Id}$$

$$\boxed{\mathbf{Set}^T = \mathbf{Set}}$$

$$\boxed{K = G}$$

Theorem. $F \dashv G$, consider the corresponding category of T-algebras. Then $\exists$ a **COMPARISON FUNCTOR** $K : \mathcal{A} \to \mathcal{C}^T$ such that

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, shift right, "G"'] \arrow[d, "K"'] & \mathcal{C} \arrow[l, shift right, "F"'] \arrow[loop right, "T"] \arrow[dl, shift right, "F^T"'] \\
\mathcal{C}^T \arrow[ur, shift right, "G^T"']
\end{tikzcd}
$$

$$\boxed{KF = F^T}$$

$$\boxed{G^T K = G}$$

$$\boxed{A \in \mathcal{A} \quad ?? \quad K(A)}$$

$\forall A \in \mathcal{A}$ $\quad K(A) \overset{?}{=} (X, h : TX \to X)$

$K(A) \overset{?}{=} (X \quad h : GFX \to X)$

$\boxed{G^T K(A) = G(A)} \implies \boxed{X = G(A)}$

$h : G\underbrace{FGA} \to G\underline{A}$ $\qquad FGA \xrightarrow{\varepsilon_A} A$

$\boxed{h \underset{\text{Def}}{=} G\varepsilon_A}$ $\quad \boxed{K(A) \underset{\text{Def}}{=} (GA, \ G\varepsilon_A)}$

K is a functor

$$\mathcal{A} \xrightarrow{F} \mathcal{E}$$

K $\quad F^T$

$\mathcal{E}^T$

$$\boxed{G^T K = G} \quad \text{ok}$$

$$KF = F^T \quad ?$$

$$KF(x) = (GFx,\ G\varepsilon_{Fx})$$

$$F^T(x) = (Tx,\ \mu_x) = (GFx,\ G\varepsilon_{Fx})$$

When

A is algebraic over $\mathcal{E}$

$\Updownarrow$

K is an iso

Characterization theorem

K is iso $\Leftrightarrow$ G creates absolute coequalizers

$\boxed{F : \mathcal{A} \to \mathcal{B}}$ F functor

1. F preserves limits (or colimits)

2. F reflects limits (- —)

3. F creates limits (- -)

1. If D is a diag in $\mathcal{A}$, and $(L, p_i) = \lim_{\mathcal{A}} D$

then $(FL, Fp_i) = \lim_{\mathcal{B}} FD$

hom(X, -) preserve limits

$F \dashv G$ G preserves limits

F preserves colimits

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② $\mathcal{A} \xrightarrow{F} \mathcal{B}$ reflects LIMITS

$$
\begin{tikzcd}
 & D_i \arrow[d, "d_u"] \\
L \arrow[ur, "l_i"] \arrow[r, "l_j"'] & D_j
\end{tikzcd}
$$

$\mathbb{D}$ is a diagram in $\mathcal{A}$

$L \in \mathcal{A}$ $l_i : L \to D_i$

(L, l_i) is a cone such

that $(FL, Fl_i) \cong \lim_{\mathcal{B}} F\mathbb{D}$

$$
\begin{tikzcd}
 & D_i \arrow[dd, "u"] \\
L \arrow[ur, "l_i"] \arrow[dr, "l_j"'] & \\
 & D_j
\end{tikzcd}
\xrightarrow[\text{reflects limits}]{F}
\begin{tikzcd}
 & FD_i \arrow[dd, "Fu"] \\
FL \arrow[ur, "Fl_i"] \arrow[dr, "Fl_j"'] & \\
 & FD_j
\end{tikzcd}
$$

CONE

$\mathcal{A}$

by hypoth. (FL, Fl_i) is a **limit**

Then $\boxed{(L, l_i) = \mathrm{Lim}_{\mathcal{A}} D}$

$F : \mathcal{A} \to \mathcal{B}$

$\boxed{F \text{ creates limits}}$

$\underline{D}$ diag in $\mathcal{A}$ if FD has limits in $\mathcal{B}$,

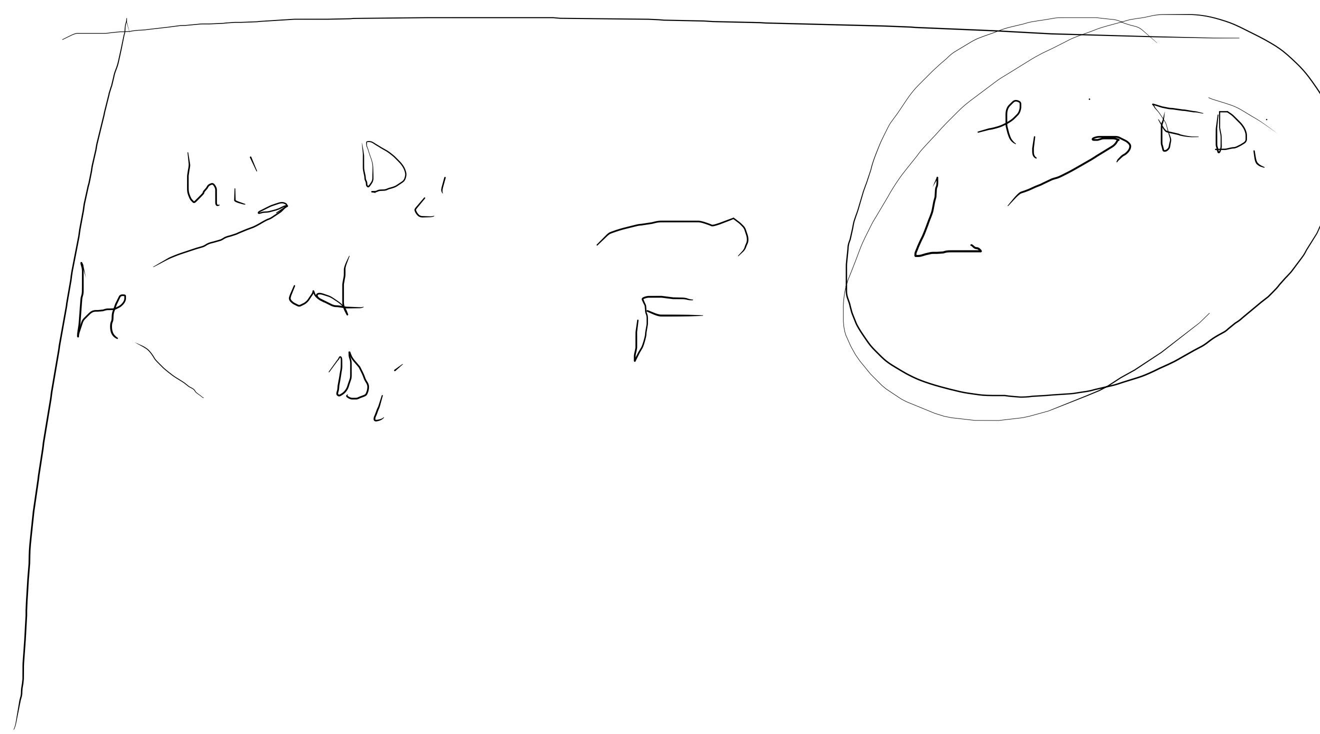
$(L, \ell_i) = \lim\limits_{\mathcal{B}} FD$ FORMALLY

If $(L, l_i) = \lim_{\mathcal{B}} FD$, then

$\Rightarrow \boxed{\exists! \, (H \in \mathcal{A}, \; h_i : H \to D_i)}$ s. that

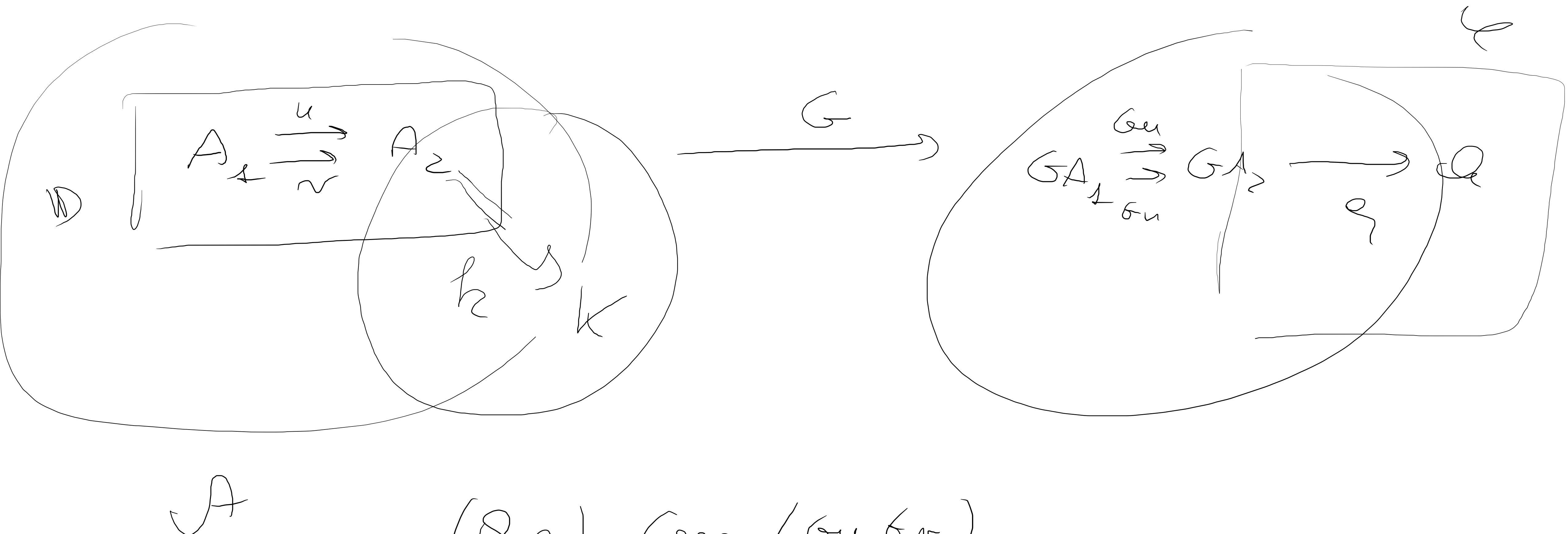
h_i is cone and $FH = L$, $Fh_i = l_i$

$\Rightarrow \boxed{(H, h_i) = \lim_{\mathcal{A}} D}$



$$\mathcal{A} \xrightarrow{G} \mathcal{C}$$

$\boxed{G \text{ creates coequalizers}}$



$$(Q, q) = \mathrm{Coeq}_{\mathcal{C}}(Gu, Gv)$$

$\exists!\,(k,\ k : A_2 \to K)$ in $\mathcal{A}$ s.that

$\Rightarrow \{\ Gk = Q\ ,\ \ Gk = q$

$\to (K,\ k) = \underset{\mathcal{A}}{\mathrm{Coeq}}(u, v)$

G must create $\boxed{\text{"good"}}$ coequalizer

Def 1. A coequalizer $p : Y \to Q$

$Q = \mathrm{Coeq}(u, v)$

$$X \overset{u}{\underset{v}{\rightrightarrows}} Y \xrightarrow[p]{} Q$$

(Q, p) is an $\boxed{\text{Absolute}}$ coeq. iff

it is preserved by any functor.

$\mathcal{A}$: $\cdot \overset{u}{\underset{v}{\rightrightarrows}} \cdot \xrightarrow[p]{} Q$ $\xrightarrow{H}$ $\cdot \overset{Hu}{\underset{Hv}{\rightrightarrows}} \cdot \xrightarrow{Hp} HQ$

s.t. $\boxed{(HQ, Hp) = \mathrm{Coeq}(Hu, Hv)}$

Coeq.

<u>Def 2</u> (p, q) is a <u>split</u> sequence iff

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[l, bend right=60, "k"'] \arrow[r, "p"'] & Q \arrow[l, bend right=60, "h"'] \end{tikzcd}$$

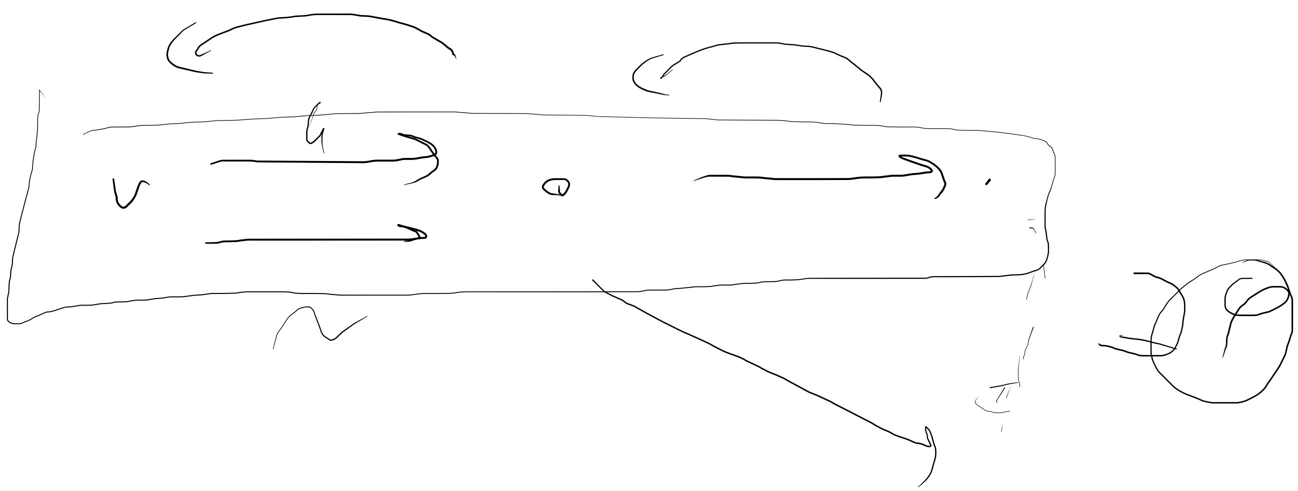
$\exists\, u, v, p$ s.th. (1) $pu = pv$

and $\exists$ $h : Q \to Y$, $k : Y \to X$ s.th.

(2) $ph = 1_Q$ (3) $uk = 1_Y$

(3) $hp = vk$

Lemma 1. Any split sequence is a coequalizer



Lemma 2. Any split sequence is absolute

Th.

$\boxed{A \xleftarrow{\ \ } \mathcal{E} \atop G}$

$$A \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{E}$$

- A is algebraic over $\mathcal{E}$

iff

- G creates split coeq

iff

- G creates absolute coeq

Corollary 1

$\boxed{\text{(S.T) algebras are algebraic on Set}}$

Corollary 2.

$\boxed{\text{Comp}T_2\text{Sp. are algebraic on Set}}$

$\boxed{\mathcal{C} = \text{Set}}$