<u>Def</u> An <u>ABSOLUTE</u> coequalizer in $\mathcal{C}$ is a coequalizer $(Q, q) = \mathrm{Coeq}(u, v)$ s.th. $\forall F : \mathcal{C} \to \mathcal{D}$ then (FQ, Fq) is again a coequalizer

$$\underbrace{\bullet \underset{v}{\overset{u}{\rightrightarrows}} \bullet \xrightarrow{q} Q}_{\mathcal{C}} \quad \xrightarrow{F} \quad \underbrace{\bullet \underset{Fv}{\overset{Fu}{\rightrightarrows}} \bullet \xrightarrow{Fq} FQ}_{\mathcal{D}}$$

Def A SPLIT sequence is

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[l, bend right=60, "k"'] \arrow[r, "q"'] & Q \arrow[l, bend right=60, "h"'] \end{tikzcd}$$

s.t.

- $qu = qv$
- $uk = 1_Y$
- $qh = 1_Q$
- $vk = hq$

Lemma 1

Any SPLIT epi is a COEQUALIZER

Proof

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[l, bend right=50, "k"'] \arrow[r, "q"'] & Q \arrow[l, bend right=50, "h"'] \end{tikzcd}$$

is split

we must prove that $(Q, q) = \mathrm{Coeq}(u, v)$

$qu = qv$ ✓

we must verify the universal property

$\forall (L, \ell : Y \to L)$ with $\ell u = \ell v \longrightarrow$

$\Rightarrow$ $\boxed{\exists!\ \varphi : Q \to L}$ such that

Pf? $\boxed{l = \varphi\rho}$

$$
\begin{tikzcd}
X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[l, bend right, "k"'] \arrow[r, "\rho"] \arrow[dr, "l"'] & Q \arrow[l, bend right, "h"'] \arrow[d, dashed, "\varphi ?"] \arrow[d, bend left, "\psi"] \\
 & & L
\end{tikzcd}
$$

$\boxed{\varphi = lh}$ DEF

to prove $\boxed{l = \varphi\rho}$

$\Rightarrow l\underbrace{h\rho} = l\underbrace{vk} =$

$= l\underbrace{uk} = l 1_Y = l$

φ is unique — if ψ is such

$\boxed{\psi\rho = l} \overset{?}{\Rightarrow} \boxed{\varphi = \psi}$

$\psi\rho = \varphi\rho \longrightarrow \psi\underbrace{\rho h} = \varphi\underbrace{\rho h} \longrightarrow$

$\to \psi \cdot 1_Q = \varphi \cdot 1_Q \longrightarrow \psi = \varphi$

<u>Lemma 2</u>

split coeq $\xrightarrow{?}$ <u>absolute coeq</u>

Lemma 1

split sequence

$$\mathcal{C} \xrightarrow{H} \mathcal{D}$$

<u>Proof</u>

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[l, bend right] \arrow[r, "\rho"'] & Q \arrow[l, bend right] \end{tikzcd} \xrightarrow{H} \begin{tikzcd} HX \arrow[r, shift left, "Hu"] \arrow[r, shift right, "Hv"'] & HY \arrow[l, bend right, "Hk"'] \arrow[r, "H\rho"'] & HQ \arrow[l, bend right, "Hh"'] \end{tikzcd}$$

in $\mathcal{C}$ is split, 4 equations apply

in $\mathcal{D}$ then 4 equations are still valid

$$\rho u = \rho v$$

$$\downarrow$$

$$H(\rho u) = H\rho . Hu = H\rho . Hv$$

any functor preserves compositions and identities

so the 4 equations of a split sequence

are preserved (they are coded with Δ)

$\Rightarrow$ the sequence with Δ is split

$\Rightarrow$ so it is a coequalizer

$\to$ (q, e) is an absolute coeq.

$\boxed{\text{Lemma 3}}$ $\mathcal{C} \curvearrowright T$ $\quad (T, \eta, \mu)$ is a triple on $\mathcal{C}$

$\mathcal{C}^T$ categ. of T-algebras

$\forall$ T-algebra $(X, h : TX \to X)$ $\exists$ a canonical associated split sequence,

that is called the "presentation" of the T-algebra

$\forall (X,h) \in \mathcal{C}^T$ we can define a split sequence:

$$\begin{tikzcd} T^2X \arrow[r, shift left, "Th"] \arrow[r, shift right, "\mu_X"'] & TX \arrow[l, bend right, "\eta_{TX}"'] \arrow[r, "h"'] & X \arrow[l, bend right, "\eta_X"'] \end{tikzcd}$$

it is SPLIT coeq

hyp: (X,h) is a T-algebra

- $\boxed{h \cdot Th = h \cdot \mu_X}$ ax 2 of T-alg.
- $\boxed{h \cdot \eta_X = 1_X}$ ax 1 of T-alg
- $\boxed{\mu_X \cdot \eta_{TX} = 1_{TX}}$ ax 1 of triple
- $\boxed{Th \cdot \eta_{TX} = \eta_X h}$ naturality of η_X

Caracterization theorem

Let be
$$
\begin{tikzcd}
A \arrow[r, shift right, "G"'] \arrow[d, "K"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "F"'] \arrow[dl, shift right, "F^T"'] \\
\mathcal{C}^T \arrow[ur, shift right, "G^T"']
\end{tikzcd}
\qquad T
$$

T the monad we get and $\mathcal{C}^T$ the category of T-algebras

the following facts are equivalent:

1) $\boxed{\text{A is algebraic on } \mathcal{C}}$ (that means K is an iso)

2) $\boxed{\text{G creates absolute coequalizers}}$

3) $\boxed{\text{G creates split coeq.}}$

$\boxed{G \text{ creates coeq}}$

$\boxed{\mathcal{A} \xrightarrow{G} \mathcal{C}}$

$D \longmapsto D_i \longrightarrow \text{colim}$

D_j

$\forall \; X \overset{u}{\underset{v}{\rightrightarrows}} Y$ in $\mathcal{A}$ s.th.

$\boxed{GX \overset{Gu}{\underset{Gv}{\rightrightarrows}} GY \xrightarrow{q} Q}$

has a coeq in $\mathcal{C}$ then

① $\exists! \; T \in \mathcal{A} \quad t : Y \to Q$ s.th.

$G(T) = Q \qquad G(t) = q$

② $(T, t) = \mathrm{Coeq}(u, v)$

G creates ABSOLUTE (SPLIT) coeq in $\mathcal{C}$

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y \in T \quad \xrightarrow{\;G\;} \quad GX \rightrightarrows GY \xrightarrow{e} Q$$

is absolute

$\forall\ (u, v)$ s.t. (Gu, Gv) has an ABSOLUTE coeq

then $\exists\ (T, t)$ ... unique ... $\boxed{\mathrm{Quot}(T, t) = \mathrm{Coeq}(u, v)}$

Proof ① → ②

$\boxed{K \text{ is iso}}$

G creates abs. coeq.

K is an iso → G and G^T are isomorphic

$$\boxed{\begin{tikzcd} A \arrow[r, shift right, "G"'] \arrow[d, "K"'] & C \arrow[l, shift right, "F"'] \arrow[dl, shift left, "F^T"] \\ C^T \arrow[ur, shift left, "G^T"] & \end{tikzcd}}$$

to prove ② G creates abs. coeq.

we can prove $\boxed{G^T \text{ creates abs. coeq.}}$

$$\forall\ (X,h) \overset{u}{\underset{v}{\rightrightarrows}} (Y,b) \xrightarrow{r}$$

$\mathcal{C}^T$

$\xrightarrow{G^T}$ $X \overset{u}{\underset{v}{\rightrightarrows}} Y \xrightarrow{q} Q$ in $\mathcal{C}$

$R = (Q, ?)$

$\forall\ (u,v)$ s.th. $(G^T(u), G^T(v))$ has an abs. coeq

$(Q, \rho) = \mathrm{Coeq}(u,v)$ in $\mathcal{C}$

then ① $\exists!\ R \in \mathcal{C}^T$ $(Y,b) \xrightarrow{r} R$ $\quad$ $G^T(R) = Q$

② $(R, r) = \mathrm{Coeq}(u,v)$ in $\mathcal{C}^T$

we must prove: $\exists$ a structure of T-algebra on Q s.t. q is a T-alg. morphism.

$(X,h) \underset{v}{\overset{u}{\rightrightarrows}} (Y,k)$ in $\mathcal{C}^T$

$\supset$ rewrite in φ: it is a morph of T-alg

$$\begin{tikzcd} TX \arrow[r, shift left, "Tu"] \arrow[r, shift right, "Tv"'] \arrow[d, "h"] & TY \arrow[r, "Tq"] \arrow[d, "k"] & TQ \arrow[d, dashed, "\xi ?"] \\ X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[r, "q"] & Q \end{tikzcd}$$

$u^{\mathcal{C}}$

is a coeq

coeq in $\mathcal{C}$

$k \cdot Tv = v \cdot h$

$k \, Tu = u \cdot h$

q is an absolute coeq $\Rightarrow Tq = \mathrm{Coeq}(Tu, Tv)$

we must define ξ in such a way

that $(Q, \xi : TQ \to Q)$ is a T-algebra

- $q : (Y, u) \to (Q, \xi)$ is a morph. of T-algebra

- (Q, ξ) is the coeq of (u, v) in $\mathcal{C}^T$

$$\begin{tikzcd}
TX \arrow[r, shift left, "Tu"] \arrow[r, shift right, "Tv"'] \arrow[d, "h"'] & TY \arrow[r, "Tp"] \arrow[d, "k"] \arrow[dr, bend right, "\xi k"] & TQ \arrow[d, "\xi"] \\
X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[r, "p"] & Q
\end{tikzcd}$$

λ

univ. prop. of Tp as coequaliser

$\boxed{p k Tu = p k Tv} \rightarrow \exists!\ \xi$ such that $\boxed{\xi Tp = p . k}$

We must prove that (Q, $\xi : TQ \to Q$) is a T-algebra: axioms are valid

(Y, k) is a T-alg

$\downarrow$?

$\boxed{\xi \cdot \eta_Q = 1_Q}$

ax1

$\xi \eta_Q \overset{?}{=} 1_Q$

$$\begin{tikzcd}
Y \arrow[rrr, "\eta_Y"] \arrow[dr, "q"] \arrow[ddrr, bend right, "1_Y"'] & & & TY \arrow[dl, "Tq"'] \arrow[ddl, "k"] \\
& Q \arrow[r, "\eta_Q"] \arrow[dr, "1_Q"'] & TQ \arrow[d, "\xi"] & \\
& & Q & \\
& & Y \arrow[u, "q"] &
\end{tikzcd}$$

$\boxed{\xi \eta_Q \cdot q} =$

$= \xi T_q \eta_Y$

$= q k \eta_Y$

$= q \cdot 1_Y = \boxed{q}$ $\longrightarrow$ q is a coeq $\longrightarrow$ $\boxed{\xi \eta_Q = 1_Q}$

$V(\mathcal{O},\xi)$ is a T. algebra V_ρ is a module of T.alg

to prove $\boxed{(\mathcal{O},\xi) = \mathrm{Coeq}(u,v) \text{ in } \mathcal{C}^T}$

$$
\begin{tikzcd}
TX \arrow[r, shift left, "Tu"] \arrow[r, shift right, "Tv"'] \arrow[d, "h"'] & TY \arrow[r, "T\rho"] \arrow[d, "h"] \arrow[ddr, "T\beta"] & TQ \arrow[d, "\xi"] \arrow[dd, bend left, "T\varphi"] \\
X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[r, "\rho"] \arrow[ddr, "\beta"'] & Q \arrow[dd, bend right, "\varphi"'] \\
& & TA \arrow[d, "\alpha"] \\
& & A
\end{tikzcd}
$$

$(A, \alpha) \in \mathcal{C}^T$

$\beta : (Y, h) \to (A, \alpha)$

s. that

$\beta u = \beta v$

in T algebra

$\exists!\, \varphi :$

$(Q, \xi) \to (A, \alpha)$

with

$\boxed{\beta = \varphi \rho}$

φ $\exists$ since $\rho = \mathrm{Coeq}(u\, v)$

in $\mathcal{C}$

φ is a morph of T algebra

φ is a morph of T algebra $\longleftrightarrow$

$$\boxed{\varphi \cdot \xi = \alpha\, T\varphi}$$

$$\begin{tikzcd}
TQ \arrow[r, "T\varphi"] \arrow[d, "\xi"'] & TA \arrow[d, "\alpha"] \\
Q \arrow[r, "\varphi"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
TY \arrow[r, "Te"] \arrow[rr, bend left, "T\beta"] \arrow[d, "h"'] & TQ \arrow[r, "T\varphi"] \arrow[d, "\xi"] \arrow[dr, phantom, "????"] & TA \arrow[d, "\alpha"] \\
Y \arrow[r, "e"'] \arrow[rr, bend right, "\beta"'] & Q \arrow[r, "\varphi"'] & A
\end{tikzcd}$$

β is a morph of T. alg. $\longrightarrow$

Te is a coeq

$$\boxed{\begin{gathered} \varphi\, \xi\, \cancel{Te} \\ \shortparallel \\ \alpha\, T\varphi\, \cancel{Te} \end{gathered}}$$

Tews

$2 \Rightarrow 3$ obvious

G creates ob coeq $\longrightarrow$ G creates split coeq

any split coeq is absolute.

$3 \Rightarrow 1$

G creates split coeq $\xrightarrow{?}$ A is algebraic.

K is an iso.

$T = GF$ we will construct a functor H inverse of K

\begin{tikzcd}
\mathcal{A} \arrow[r, shift right, "G"'] \arrow[d, shift right, "K"'] & \mathcal{C} \arrow[l, shift right] \arrow[dl] \\
\mathcal{C}^T \arrow[u, shift right, "H"'] &
\end{tikzcd}

$\forall (X, h) \in \mathcal{C}^T$ we must define $H(X, h)$

$\mathcal{L}$ 3

$$\boxed{T^2X \rightrightarrows TX \to X}$$ is split

$T = GF$

$$\begin{tikzcd} T^2X \arrow[r, shift left, "Th"] \arrow[r, shift right, "\mu_X"'] & TX \arrow[l, bend right, "\eta_{TX}"'] \arrow[r, "h"'] & X \arrow[l, bend right, "\eta_X"'] \end{tikzcd}$$

$(X,\ h : TX \to X)$

$$\begin{tikzcd} GFGFX \arrow[r, shift left, "GFh"] \arrow[r, shift right, "G\varepsilon_{FX}"'] & GFX \arrow[l, bend right, "\eta_{GFX}"'] \arrow[r, "h"'] & X \arrow[l, bend right, "\eta_X"'] \end{tikzcd}$$

~~sp~~ split coeq

in $\mathcal{C}$

G

G create split coeq

$$\begin{tikzcd} FGFX \arrow[r, shift left, "Fh"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, dashed, "q"'] & A \end{tikzcd}$$

$\mathcal{A}$

$\exists A \in \mathcal{A} \quad q$

$GA = X$

$A = \mathrm{Coeq}(-,-)$

def $\boxed{H(x,h) \equiv A}$

$\boxed{H : \mathcal{E}^T \to A}$

H is a function

H and K are inverse function

$\Longrightarrow A \cong \mathcal{E}^T$ A is algebraic

$$\underline{\mathbf{Grp}} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Set}$$

$\boxed{G \text{ cre coeq ASS.}}$

$$u \rightrightarrows \cdot \quad \xmapsto{G} \quad \cdot \rightrightarrows \cdot$$