algebraic categories on a base $\mathcal{C}$

$\boxed{\mathcal{A} \simeq \mathcal{C}^T}$ category of T-algebras for a triple T on $\mathcal{C}$

$\boxed{(\Omega, E) \text{ algebras}}$ always on Set

↳ operators → axioms

$\boxed{\text{Groups}}$ $\Omega = \{\omega^2, \omega^1, \omega^0\}$

$X \in \mathrm{Set}$ $X \times X \xrightarrow{\omega^2} X$ $X \xrightarrow{\omega^1} X$ $\{*\} \to X$

ex of ASSOCIATIVITY X

x, y, t

$$\boxed{(x \cdot y)t = x(y\,t)}$$

$$\begin{tikzcd}
X^3 \arrow[r, "\omega^2 \times Id"] \arrow[d, "Id \times \omega^2"'] \arrow[dr, bend left, "\delta^3"] \arrow[dr, bend right, "\sigma^3"'] & X^2 \arrow[d, "\omega^2_X"] \\
X^2 \arrow[r, "\omega^2_X"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
(x\,y\,t) \arrow[r] \arrow[d] & (x \cdot y)\,t \arrow[d] \\
x(y\,t) \arrow[r] & (x \cdot y) \cdot t = x(y\,t)
\end{tikzcd}$$

$\forall x\,y\,t$

$$\boxed{\delta^3 = \sigma^3}$$

equality of generalized operators.

identity axiom in G group

$\forall x \quad \boxed{x \cdot x^{-1} = 1_G}$

$\omega^2 \quad \omega^1 \quad \omega^0$

$$\omega^1_G : G \longrightarrow G$$
$$x \longmapsto x^{-1}$$

ax of inverse

$$\begin{tikzcd}
 & G \times G \arrow[dr, "\omega^2_G"] & \\
G \arrow[ur, "\xi"] \arrow[rr, "1_G"'] \arrow[rr, phantom, shift left=3, "||"] & & G
\end{tikzcd}$$

$$\begin{tikzcd}
 & G & G \\
 & & G \times G \arrow[ul, "p_1"'] \arrow[u, "p_2"'] \\
G \arrow[uu, "1_G"] \arrow[uurr, "\omega^1_G"] \arrow[urr, "\xi"'] & &
\end{tikzcd}$$

$\langle 1_G, \omega^1_G \rangle = \xi$

Theorem $\mathcal{A} \simeq \mathcal{C}^{T}$ $\mathcal{A}$ is algebraic IFF

$$\boxed{\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "G"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "F"'] \end{tikzcd}}$$

G creates split coeq iff

G creates absolute coeq

$\boxed{\text{Corollary 1}}$ (Ω, E) algebras are T algebras over Set

$\boxed{\text{Corollary 2}}$ Comp T_2 spaces is an algebraic categ over Set

$\boxed{\text{Recall 1}}$ $\forall$ $\boxed{A \overset{u}{\underset{v}{\rightrightarrows}} B}$ in (Ω, E) algebras

u, v morph of T algebras means that

$$\forall\, \omega^n \qquad \begin{tikzcd} A^n \arrow[r, "\omega^n_A"] \arrow[d, "u^n"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "u"] \\ B^n \arrow[r, "\omega^n_B"'] & B \end{tikzcd} \qquad u \text{ is a function}$$

u morph of algebras

$\Updownarrow$

$\forall\, \omega \in \Omega$ the

diagr is commutative

G creates <u>absolute</u> coeq

$$A \underset{v}{\overset{u}{\rightrightarrows}} B \overset{q}{\longrightarrow} C = (Q, ?)$$

(Ω, E) alg

$\overset{G}{\longmapsto}$

$$GA \underset{Gv}{\overset{Gu}{\rightrightarrows}} GB \overset{q}{\longrightarrow} Q$$

coeq

ABS.

Set

if $(Q, q) = \mathrm{Coeq}(Gu, Gv)$ in Set and it is ABSOLUTE then: $\exists!$ $C \in \Omega.E$ alg

$C \in (\Omega, E)$ alg : $\boxed{G(C) = Q}$

$\boxed{G(c) = q}$

$(\Omega.\bar{\Theta})\,\mathrm{alg} \xrightarrow{G} \mathrm{Set}$ G doesn't preserve coeq.

① Q is a $(\Omega.\bar{\Theta})$ algebra s.that ρ is an homomorph

fulfil

② $(Q, \rho) = \mathrm{Coeq}(A, B)$

in $(\Omega.\bar{\Theta})$ algebras

$A \overset{u}{\underset{v}{\rightrightarrows}} B$ $\quad\Big\Vert\quad$ ui(Σ, E) algebras

u, v homom $\iff$ $*$ are comm.

$$\begin{tikzcd}
A^n \arrow[r, shift left, "u^n"] \arrow[r, shift right, "v^n"'] \arrow[d, "\omega^n_A"'] \arrow[dr, phantom, "*"] & B^n \arrow[r, "q^n"] \arrow[d, "\omega^n_B"] \arrow[dr, phantom, "**"] & Q^n \arrow[d, "\omega^n_Q\,?"] \\
A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "q"'] & Q
\end{tikzcd}$$

is still a coeq

Set

ABSOLUTE coeq

refl.

$x \cdot y \longmapsto u(x), u(y)$

$x \cdot_A y \qquad u(x) \cdot_B u(y) = u(x \cdot_A y)$

q homom iff $**$ commutes $\forall\, \omega_Q$

$(Q, \rho) = \mathrm{Coeq}(u, v)$ apply the functor $\mathbf{Set} \xrightarrow{(\ )^n} \mathbf{Set}$

then $(Q^n, \rho^n) = \mathrm{Coeq}(u^n, v^n)$ Since ρ is ABSOLUTE

apply the universal property of $\underline{\underline{\rho^n}}$

to the new arrow $\rho \cdot \omega^n_B$

$$\begin{tikzcd}
\cdot \arrow[r, shift left, "u^n"] \arrow[r, shift right, "v^n"'] & B^n \arrow[r, "\rho^n"] \arrow[d, "\omega^n_B"'] \arrow[dr, bend left, "\rho\omega^n_B"] & Q^n \arrow[d, dashed, "??\exists!"] \\
& B \arrow[r, "\rho"'] & Q
\end{tikzcd}$$

we must prove that $q \cdot w^n_B$ is such that

$$(\rho \cdot w^n_B)\, u^n \overset{?}{=} (\rho \cdot w^n_B)\, v^n$$

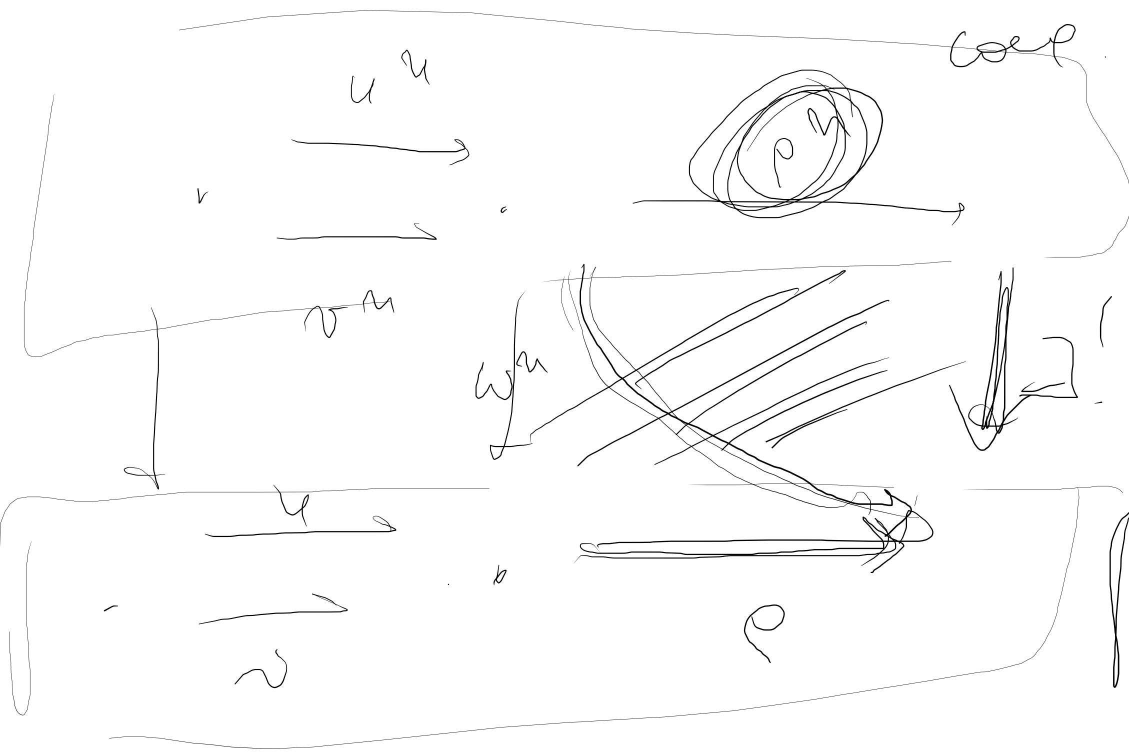
true

$$(q \cdot u) \cdot w^n_A = (q\, v)\, w^n_A = q\, w^n_B\, v^n$$

by univ property of the coeq q^n

$\exists !\ Q^n \longrightarrow Q \quad w^n_Q$

s.that $(q \cdot w^n_B) = w^n_Q \cdot q^n$



w_Q^M is unique

by univ property

ρ is anhomo

$$\boxed{w_Q^M \cdot \rho^M = \rho \cdot w_B^M}$$

$\forall w^M$

verify:

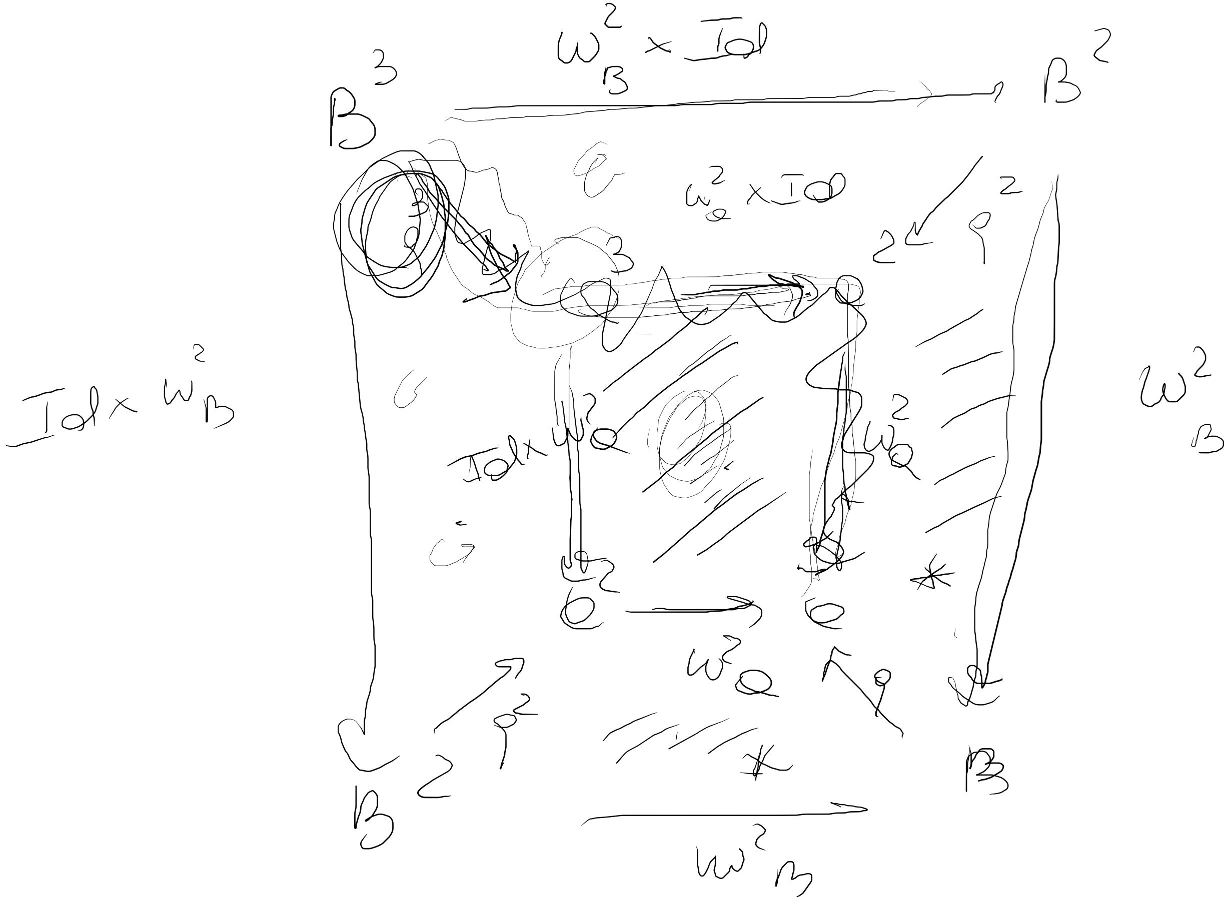
1 $(\Omega, \omega^n_\sigma)$ $\forall\, \omega \in \Omega$

is a $(\Omega \cdot \sigma)$ algebra

2 TRUE ϱ is an algebra however.

3 $(\Omega, \mathcal{A}) = \mathrm{Coop}(u, v)$ not $(\Omega \cdot \sigma)$ algebra

verify axioms   suppose to an associativity one



LP: $\boxed{\text{true in } B}$

?

$\boxed{\text{true in } Q}$

* commute by def of ρ

$$w^2_e \left(w^2_0 \times \overline{I}dl \right) \stackrel{?}{=} w^2_0 \left(\overline{I}dl \times w^2_e \right)$$

Compose by ρ^3

$$\left(w^2_e \cdot \left(w^2_e \times \overline{I}dl \right) \cdot \rho^3 \stackrel{?}{=} \left(\qquad \right) \cdot \rho^3$$

$$=$$

$$w^2_Q \cdot \rho^2 \cdot \left(w^2_B \times \overline{I}dl \right)$$

$$=$$

$$\boxed{q \cdot w^2_B} \left(w^2_B \times \overline{I}dl \right)$$

$$=$$

$$q \cdot w^2_e \cdot \left(\overline{I}dl \times w^2_B \right)$$

$$= w^2_e \cdot \rho^2 \cdot \left(\overline{I}dl \times w^2_B \right)$$

$$= w^2_e \left(\overline{I}dl \times w^2_e \right) \rho^3$$

q absolute coeq $\longrightarrow$ q^3 is still a coeq

$\longrightarrow$ q^3 is a reg epi $\longrightarrow$ can be cancelled

on the right.

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other way to prove

$\boxed{E}$

equality between generated operators

$$\begin{tikzcd}
A^3 \arrow[r, shift left, "u^3"] \arrow[r, shift right, "v^3"'] \arrow[d, shift right, "\delta_A"'] \arrow[d, shift left, "\sigma_A"] & B^3 \arrow[r, "q^3"] \arrow[d, shift right, "\delta_B"'] \arrow[d, shift left, "\sigma_B"] & Q^3 \arrow[d, shift right, "\delta_Q"'] \arrow[d, shift left, "\sigma_Q"] \\
A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "q"'] & Q
\end{tikzcd}$$

$\delta_A = \sigma_A$, $\delta_B = \sigma_B$

$\downarrow$ by unicity

$\delta_Q = \sigma_Q$

③ (Q, q) is $(D.E)$ algebra

$(Q,\rho) = \boxed{\text{coeq.}(u,v) \quad \text{in } (D.E) \text{ algebras}}$

$\rho = \mathrm{Coeq}(u,v)$ with

$\rho u = \rho v$ OK

we need prove the univ. prop. in $(D E)$ al

$$\begin{tikzcd}
A^n \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "w^n_A"'] & B^n \arrow[r, "\rho^n"] \arrow[d, "w^n_B"] \arrow[rr, bend left, "t^n"] & Q^n \arrow[r, "\varphi^n"] \arrow[d, "w^n_Q"] & T^n \arrow[d, "w^n_T"] \\
A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "\rho"] \arrow[rr, bend right, "t"'] & Q \arrow[r, "\varphi"] & T
\end{tikzcd}$$

$\boxed{\text{Coeq. univ}}$

$\forall T \in D.E$ alg $\quad \forall\, t : B \to T$ in $(D.E)$ alg

s.th. $\boxed{tu = tv} \longrightarrow \exists!\, \varphi : Q \to T$ s.th. $t = \varphi . \rho$ in $(D.E)$ alg

$\exists!\ \varphi$ by univ. property of the cone

φ in Set

φ s.th. $t = \varphi g$

We must have that φ is a factorization

also in $(\Omega\text{-}E)$ algebra.

$\Longleftrightarrow$ φ is a morph. of $(\Omega.E)$ algebra

$$
\begin{tikzcd}
\bullet \arrow[r, shift left, "u^n"] \arrow[r, shift right, "v^n"'] \arrow[d] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[r, "\rho^n"] \arrow[d] \arrow[rr, bend left, "t^n"] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[r, "\varphi^n"] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & \bullet \arrow[r, "\rho"'] \arrow[rr, bend right, "t"'] & \bullet \arrow[r, "\varphi"'] & \bullet
\end{tikzcd}
$$

~~then~~

then

$$\boxed{w^n_T \cdot t^n = t \cdot w^n_B}$$

t is a morph of Σ-Obj

$$\boxed{t = \varphi \cdot \rho} \longrightarrow t^n = \varphi^n \cdot \rho^n$$

to have

$w^n_T \cdot t^n = t \cdot w^n_B$

apply ρ^n

$w^n_T \cdot t^n \cdot \rho^n = t \cdot w^n_B \cdot \rho^n$

Corollary $\boxed{\text{Comp } T_2}$ are (β algebras on Set

$$\text{Comp } T_2 \underset{G}{\overset{F}{\rightleftarrows}} \text{Set} \qquad F \dashv G$$

G creates absolute coeq

$$\delta(X) \xrightarrow{\overline{(-)}} \delta(X)$$
$$S \longmapsto \overline{S}$$

$$\boxed{\delta(f)(S) = f(S)}$$

X, Y are Comp T_2 spaces then $f : X \to Y$ is continuous iff

$$\begin{array}{ccc} \delta(X) & \xrightarrow{\overline{(-)}} & \delta(X) \\ \delta(f)\downarrow & = & \downarrow \delta(f) \\ \delta(Y) & \xrightarrow[\overline{(-)}]{} & \delta(Y) \end{array}$$

$$\boxed{f(\overline{S}) = \overline{f(S)}}$$

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y$$

Z

Coeq T_2

$$GX \rightrightarrows GY \longrightarrow Q$$

obs. coeq

$$\begin{tikzcd}
\sigma(X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\overline{\mathrm{Cl}}_X"'] & \sigma(Y) \arrow[r] \arrow[d, "\overline{\mathrm{Cl}}_Y"] & \sigma(Q) \arrow[d, dashed, "?"] \\
\sigma(X) \arrow[r, shift left] \arrow[r, shift right] & \sigma(Y) \arrow[r] & \sigma(Q)
\end{tikzcd}$$

con

$\boxed{\text{coeq}}$

PARÉ