Algebraic on Set

- EXACT
- $\exists P$ s.t. projective & generator

$\Longleftrightarrow$ $\boxed{A \simeq \mathrm{Set}^T}$

$\boxed{A \simeq \mathcal{C}^T}$ algebraic on $\mathcal{C}$

Beck theorem: $A \simeq \mathcal{C}^T$ iff

$\boxed{G : \mathcal{C}^T \to \mathcal{C}}$ creates absolute (split) coequalizers

Corollary 1   Any cat. of (Σ, E) algebras is a category of T-algebras over Set

Corollary 2 $\boxed{\text{Comp } T_2 \text{ spaces}}$ are algebras over Set, one of the form Set^T

R. Paré

__Prop__ $\mathrm{Comp}T_2 \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathrm{Set}$ $\qquad F \dashv G$

FX is the Stone-Čech compact. of (X discrete ~~topology~~)

__Beck theorem__ $\qquad G$ creates __ABSOLUTE coeq__

$$\underbrace{\boxed{X \underset{v}{\overset{u}{\rightrightarrows}} Y}}_{\mathrm{Comp}T_2} \overset{q}{\dashrightarrow} X_Q \qquad \overset{G}{\longmapsto} \qquad \underbrace{\boxed{GX \rightrightarrows GY \xrightarrow{\rho} Q}}_{\mathrm{Set}}$$

$\exists\, (Q, \rho)$ absolute coeq

$\exists!\ (Z, \omega)$ such that $G(Z) = Q$

$G(Z) = q$

$\longrightarrow Z = (Q, + \text{topology})$

$\longrightarrow Z$ is the function ρ — now is continuous

$\Rightarrow (Q, q) = \mathrm{Coeq}(u, v)$ in CoeqT_2

$\boxed{\text{Pari}}$ X top. space can be described in terms of a closure operator

$$\boxed{\mathcal{P}(X) \xrightarrow{\overline{(\ )}} \mathcal{P}(X)}$$

set closure operator

(X, τ) $\longleftrightarrow$ $S \subseteq X \longmapsto \overline{S}$

topology

- $S \subseteq \overline{S}$
- $\overline{\overline{S}} = \overline{S}$
- $\overline{S \cup T} = \overline{S} \cup \overline{T}$
- $\overline{\phi} = \phi$

Lemma X, Y comp T_2 spaces then

$u : X \to Y$ is continuous **IFF** u it is closed

$\forall S \in \mathcal{P}(X)$ $\boxed{u(\overline{S}) = \overline{u(S)}}$ $\longleftrightarrow$

$$
\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\mathcal{P}(u)"] \arrow[d, "\overline{(\,)}_X"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{P}(Y) \arrow[d, "\overline{(\,)}_Y"] \\
\mathcal{P}(X) \arrow[r, "\mathcal{P}(u)"'] & \mathcal{P}(Y)
\end{tikzcd}
$$

If $S \subset X$

$\mathcal{P}(u)(S) = u(S)$

direct image

$\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$

FUNCTOR

it is like homomorphism property

for ex $G \xrightarrow{u} H$ groups

$$\begin{tikzcd}
G \times G \arrow[r, "u \times u"] \arrow[d, "\cdot_G"'] \arrow[dr, phantom, "\circlearrowleft"] & H \times H \arrow[d, "\cdot_H"] \\
G \arrow[r, "u"'] & H
\end{tikzcd}$$

$\forall (x, y) \in G \times G$

$$\boxed{u(x) \cdot_H u(y) = u(x \cdot_G y)}$$

in CompT_2 $\quad$ u cont $\longleftrightarrow$ closed function u

coeq (ABS) in Set

proof $\quad X \underset{v}{\overset{u}{\rightrightarrows}} Y \dashrightarrow \boxed{Q}$ $\quad \boxed{e}$

$$S(X) \underset{S(v)}{\overset{S(u)}{\rightrightarrows}} S(Y) \xrightarrow{S(q)} S(Q)$$

$\overline{(\,)}_X \qquad \circlearrowleft \qquad \overline{(\,)}_Y \qquad \overline{(\,)}_Q \quad ?? \; \exists!$

$$S(X) \underset{S(v)}{\overset{S(u)}{\rightrightarrows}} S(Y) \xrightarrow{S(q)} S(Q)$$

$S(Q)$ coeq

(Q, e) is obs. coeq under
$\downarrow$ apply S
still a coeq.

in Set

the commut of the square

$$\boxed{\overline{(\,)}_Q \cdot \delta(q) = \delta(q) \cdot \overline{(\,)}_Y}$$ says

that $\boxed{q \text{ is continuous}}$

to verify $\left(Q, \overline{(\,)}_Q\right)$ is $\boxed{\text{a compact } T_2 \text{ space}}$

$\overline{(\,)}_Q$ is a closure operation

Now $(Q, \overline{\tau}_Q)$ are q they form a <u>coequalizer</u> in $\mathrm{Comp}\,\overline{T}_2$

we must verify the <u>universal property</u>

<u>idea</u>

CompT2

in $\boxed{\text{Set}}$

$$\begin{tikzcd}
X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[r, "q"] \arrow[dr, "\forall t"'] & Q \arrow[d, "\exists ! \varphi"] \\
 & & T
\end{tikzcd}$$

coeq in CompT2

t cont.

T comp. Haus.

$\exists ! \varphi : Q \to T$ such that

$\boxed{t = \varphi \cdot q}$

$\boxed{\varphi \text{ exists unique}}$?

$\boxed{\varphi \text{ is cont}}$

$$\begin{tikzcd}
\beta(X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\ )_X"] & \beta(Y) \arrow[r, Rightarrow] \arrow[d, "(\ )_Y"] \arrow[drr, "\beta(t)"] & \beta(Q) \arrow[r, "\beta(\varphi)"] \arrow[d] & \beta(T) \arrow[d, "(\ )_T"] \\
X \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r] \arrow[drr, "\beta(t)"'] & Q \arrow[r] & T \\
 & & & \beta(T)
\end{tikzcd}$$

we want to prove

$$\boxed{\overline{(\ )}_T \cdot \delta(\varphi) = \delta(\varphi) \cdot \overline{(\ )}_Q} \quad ?$$

compose with $\delta(q)$

$$\overline{(\ )}_T \cdot \delta(\varphi) \cdot \underline{\delta(q)} \overset{?}{=} \delta(\varphi) \cdot \overline{(\ )}_Q \cdot \delta(q)$$

true by the commut. of the square in the hypothesis.

$\delta(x) \longrightarrow \delta(x)$

$\overline{(\ )}_x$

algebraic on fct

exact ccl $\longleftrightarrow$ Set^T

P ---

if we want Set^T represent "fct with finite operations"

Set^T is "finitary algebraic" **iff**

which Set^T | T preserves filtered colimits

iff

P is "finitely presentable" i.e.

$\mathcal{A}(P,-)$ | preserves "filtered" colimits

$T = GF$

$$\mathcal{A} = Set^T \xleftarrow{F} Set = \mathcal{C}$$

$\mathcal{A}(P,-) = G$

filtered colimits iff the corresp. Diagram is filtered

D is $\boxed{\text{filtered}}$ iff :

$\forall\ D_i, D_j \quad \exists\ D_k$ and $u : D_i \to D_k$

$v : D_j \to D_k$

$$\begin{tikzcd} D_i \arrow[r] & D_k \\ D_j \arrow[ur] & \end{tikzcd}$$

$$D_i \rightrightarrows D_j \dashrightarrow D_k$$

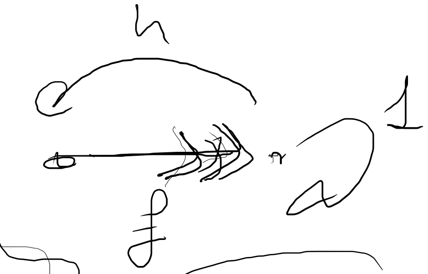
$\forall\ u, v : D_i \rightrightarrows D_j \quad \exists\ D_k,\ w : D_j \to D_k$

with $w . u = w v$

Theorem

$\mathcal{C}$ is well done $\longmapsto$ $\mathcal{C}^T$ is well done

T is a triple



$fh = 1$

Borceux

If $\mathcal{C}$ is complete, cocomplete, regular, any regular epi has a right inverse

then

- $\mathcal{C}^T$ is complete – cocomplete – regular
- $G : \mathcal{C}^T \to \mathcal{C}$ preserves & reflects regular epi

Lemma If $\mathcal{C}$ is complete (it has limits) then $\mathcal{C}^T$ is complete, (and G preserves limits)

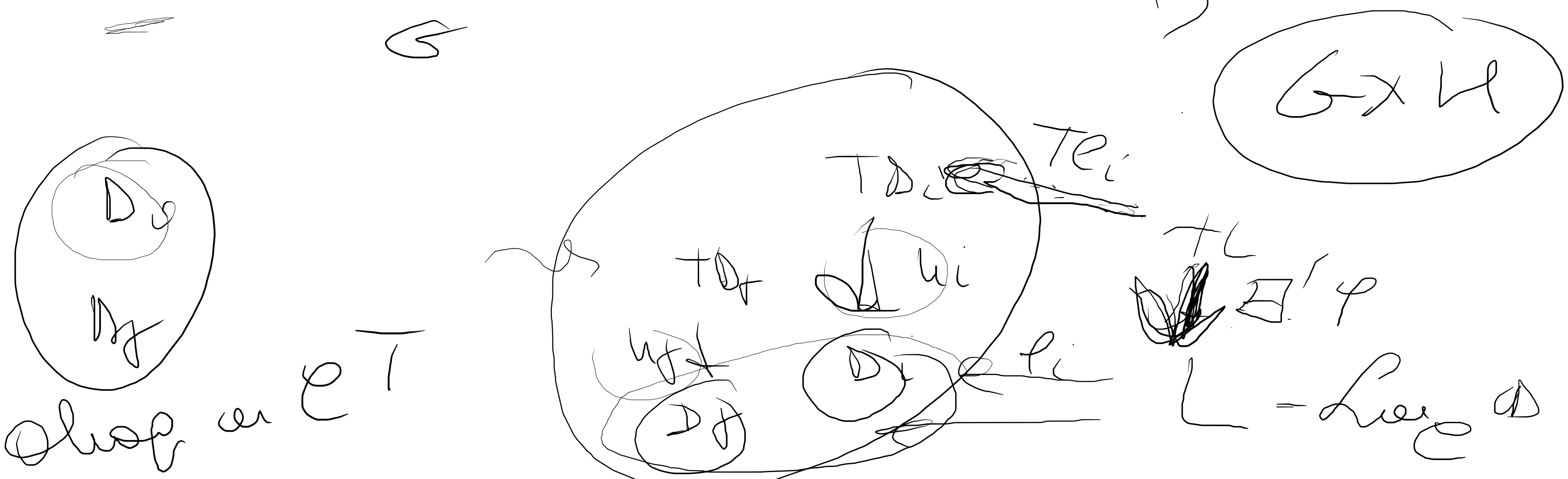
$$\begin{tikzcd} \mathcal{C}^T \arrow[r, shift right, "G"'] & \mathcal{C} \arrow[l, shift right] \end{tikzcd}$$

$\mathrm{Grp} \longrightarrow \mathrm{Set}$

$G, H \in \mathrm{Grp}$

$G \times H$ in Set

it is a group



diag on $\mathcal{C}^T$