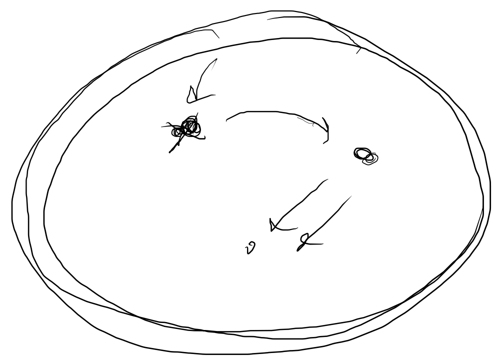
Algebraic category

working with elements

No elements

products

products



universal properties

Set Group Ring Latt Bool

Universal algebra

in terms of elements

(Ω, E) - algebras

Ω: operators

E: axioms

X set

operation

axioms are equations

$\boxed{\Omega \ni \omega}$

ω_2

ω_1

ω_0

Group

$\forall X$

$$\boxed{X \times X \xrightarrow{\omega_2^X} X}$$

$$X \xrightarrow{\omega_1^X} X$$

$$X^0 \xrightarrow{\omega_0^X} X$$

$$X^0 = \{*\}$$

fix a constant

associativity $\quad \boxed{\forall x, y, z \in X \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)}$

$$
\begin{tikzcd}
X \times X \times X \arrow[r, "w_2^X \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times w_2^X"'] \arrow[dr, phantom, "="] & X \times X \arrow[d, "w_2^X"] \\
X \times X \arrow[r, "w_2^X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
(x, y, z) \arrow[r] \arrow[d] & ((x \cdot y), z) \arrow[d] \\
x (y \cdot z) \arrow[r] & (x y) \cdot z = x \cdot (y \cdot z)
\end{tikzcd}
$$

$$\underbrace{\boxed{w_2 \cdot (w_2 \times \mathrm{Id})}}_{w_3} = \underbrace{\boxed{w_2 \cdot (\mathrm{Id} \times w_2)}}_{w_3}$$

$\boxed{(\Omega, E) \text{ algebra}}$

Ω : operators

$\underline{E}$ are identities between composed operators

$\forall$ (Ω, E) algebras

$$\begin{tikzcd}
(\Omega, E)\text{-alg} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, bend right, "F"']
\end{tikzcd}$$

There exists the free (Ω, E) algebra

$F(1)$

Smgr

$X \in Set$

$F(X)$ free semigroup

$\Downarrow$

$\coprod_{n \in \mathbb{N}^+} X^n$

$x_1 x_2 x_2$

Grp

$F(X)$ free group

words $x_1 x_2 x_2 x_1$

$\bar{x}$ inverse

$x_1 \bar{x}_2 x_3 \bar{x}_4 \bar{x}_5$

$\bar{x} x$ = empty word

Categorical approach

What do they have in common

Set $\boxed{\text{Regular category}}$ $\longleftrightarrow$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, two heads, "\text{reg epi}"'] \arrow[dr, phantom, "\text{pb}"] & \cdot \arrow[d, two heads, "\text{reg epi}"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

$\Longrightarrow$

$$\boxed{\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\ & I \arrow[ur, hook, "m"'] & \end{tikzcd}}$$

def $\boxed{\text{EXACT CAT}}$ — $\boxed{\text{regular}}$

$\boxed{\text{any equiv. relation is effective}}$

def $R \rightrightarrows X$ $\boxed{\text{equiv. relat}}$

$$\boxed{R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X} \xrightarrow{q} \mathrm{Coeq}(r_1, r_2)$$

equiv rel.

$\boxed{R = \text{kernel pair of the coeq.}}$

kernel pair of q

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_2"'] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & \cdot \end{tikzcd}$$

Counterex

Top

non regular

non exact

equiv. rel. on
not effective

$$R \rightrightarrows X \to X/R$$

$\check{d}$

R

Alg_{TF}

P_2 $\quad \mathbb{Z}\times\mathbb{Z} \quad$ 2 eq. relat. with the same quotient.

Grp $\boxed{\text{exact category}}$ ✓

$\mathbb{Z} \in$ Grp $\quad \boxed{\mathbb{Z} = F(1)}$

$\mathbb{Z}$ is a $\boxed{\text{rep. projective ob} \\ \text{rep. generator.}}$

Def $\mathcal{C}$ with finite limits and colimits

$P \in \mathcal{C}$ is called a $\boxed{\text{regular PROJECTIVE}}$

iff

$\forall f : X \twoheadrightarrow Y$ reg. ep.

$\forall h : P \to Y$

$\exists \bar{h} : P \to X$ s. that $\boxed{f \cdot \bar{h} = h}$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\bar{h}"'] \arrow[d, "h"] \\ X \arrow[r, two heads, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & Y \end{tikzcd}$$

Ex: Set   Any object is reg. projective

Grp

$$\begin{tikzcd} & P = \mathbb{Z} \ni 1 \arrow[dl, dashed, "\bar{h}"'] \arrow[d, "h"] \\ x \in X \arrow[r, two heads, "f"', "\text{reg epi}"] & Y \ni \xi = h(1) \end{tikzcd}$$

reg epi in Grp is a surjective homom.

$\mathrm{Grp}(\mathbb{Z}, Y) \simeq |Y|$

$\mathbb{Z} \xrightarrow{h} Y \qquad h(1) = \xi \in Y$

$n \in \mathbb{Z} \qquad h(n) = n\,h(1) =$

$\exists x :$

$f(x) = \xi = h(1)$

$\boxed{\bar{h}(1) = x}$

$\forall$ (Σ, E) algebras the free (Σ, E) algebra on 1 is a regular projective objects

$$\begin{tikzcd}
(\Sigma, E)\text{ algebra} \arrow[rr, bend left, leftarrow, "F"] \arrow[rr, phantom, "\perp"] \arrow[rr, bend right, leftarrow, "U"'] & & \mathrm{Set}
\end{tikzcd}$$

$$A \in (\Sigma, E)\text{ algebra} \qquad X \in \mathrm{Set}, \quad X = 1$$

$\boxed{F \text{ left adjoint}}$

$$\boxed{\mathrm{Set}(X, UA) \simeq (\Sigma E)\mathrm{alg}(FX, A)}$$

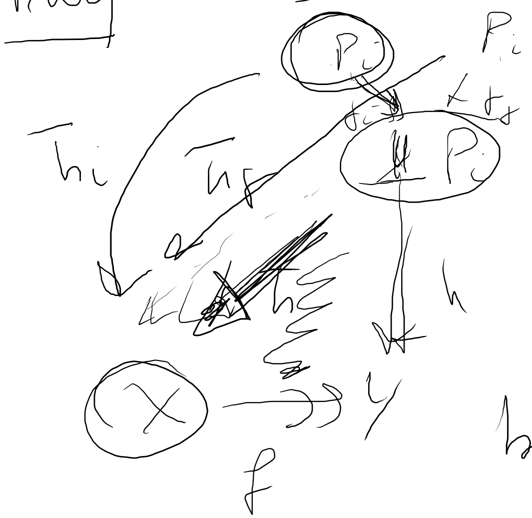
$$\underbrace{\mathrm{Set}(1, UA)}_{\cong\ |A|} \simeq (\Sigma E)\mathrm{alg}(\underline{F(1)}, A)$$

Proposition $P_i \in \mathcal{C}$ proj. object

then $\coprod P_i$ is again a proj. proj.

Set $\{x\}$ proj. proj. $\qquad \coprod \{x\} = X$

Proof IP. P_i proj. proj.



$$\boxed{h_i = h \circ f_i}$$

P_i proj. $\to \bar{h}_i$

$$\boxed{\bar{h}_i : P_i \to X}$$

by univ. prop of $\coprod \to \exists \bar{h}$

Th $\coprod P_i$ proj. proj.

$\forall f : X \twoheadrightarrow Y$ epi

$\forall h : \coprod P_i \to Y$

$\exists \bar{h} : \coprod P_i \to X$ with

$f \bar{h} = h$

$h \circ f_i = \bar{h}_i$

the diagram

$$\boxed{\bar{h} \cdot f = h}$$

is true

by univ. property of $\amalg P_i$

$$\bar{h} f = h \iff \forall i : \boxed{\mu_i \bar{h} f = \mu_i h}$$

$$\bar{h}_i f = \mu_i h$$

$\mathbb{Z} \in \mathrm{Grp}$

free prod

$\mathbb{Z} + \mathbb{Z}$

free prod.

$\mathbb{Z} \in \mathrm{Ab}$

$\mathbb{Z} + \mathbb{Z} = \mathbb{Z} \times \mathbb{Z}$

Proposition $P \in \mathcal{C}$ reg. proj. $\Longleftrightarrow$ (IFF) $\mathcal{C}(P,-) : \mathcal{C} \to Set$ pres. reg. epis

Proof $\mathcal{C} \twoheadrightarrow Set$

$\mathcal{C}(P,-)$

representable functors (they preserve limits)

$X \overset{f}{\twoheadrightarrow} Y$ is a reg. ep. $\longrightarrow$ $\mathcal{C}(P,X) \twoheadrightarrow \mathcal{C}(P,Y)$, ξ, $\mathcal{C}(P,f)$

?? $\mathcal{C}(P,f)$ must be surjective

$$\begin{tikzcd} & P \arrow[dl, dashed, "\xi"'] \arrow[d, "h"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$

$\underline{\mathcal{C}(P,f)(\xi) = f\xi}$

$\forall h : P \to Y$
$\exists\, \xi : \mathcal{C}(P,f)(\xi) = h$

Conversely ... the same

$\boxed{P \text{ rep uniq}}$ $\longleftrightarrow$ $\boxed{\mathcal{C}(P,-) \text{ preserves reg ep}}$

$P = \mathbb{Z}$

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift right, "F"'] \end{tikzcd}$$

$$\boxed{\mathrm{Grp}(\mathbb{Z}, -) \simeq U}$$

def $T \in \mathcal{C}$ is a $\boxed{\text{regular GENERATOR}}$

iff
- T has arbitrary copowers $\coprod T$
- $\boxed{\forall X \in \mathcal{C} \ \exists \text{ a reg. epi } k \quad k : \coprod T \twoheadrightarrow X}$

"any X is a quotient of a coproduct of T"

Cor $\mathbb{Z} = F(1)$ is a reg. generator

Ab $\boxed{\coprod \mathbb{Z} \twoheadrightarrow G}$

Proposit. T is the generator.

take all possible $f : T \to X$

$$\begin{tikzcd} & T \arrow[dl, "i_f"'] \arrow[dr, "f"] & \\ \coprod_f T \arrow[rr] & & X \end{tikzcd}$$

$\exists h$ by univ. property

h is the epi

Proposition. $P \in \mathcal{C}$ $\boxed{P \text{ rep gen.}}$ ✓

iff

$\boxed{\mathcal{C}(P,-) : \mathcal{C} \longrightarrow \mathbf{Set}}$ ✓

reflects isomorphisms

$\mathcal{A} \xrightarrow{F} \mathcal{B}$ F reflects isos $\Longleftrightarrow$

$$\begin{array}{ccc} X & & FX \\ f \downarrow & \rightsquigarrow & \downarrow Ff \\ Y & & FY \end{array}$$

if Ff is an iso in $\mathcal{B}$

then f is an iso in $\mathcal{A}$

Grp

$P = \mathbb{Z}$

$\mathbf{Grp} \longrightarrow \mathbf{Set}$

$\mathbf{Grp}(\mathbb{Z}, -) \cong U$

U reflects isos

$$\begin{array}{c} G \\ f \downarrow \\ H \end{array} \quad \rightsquigarrow \quad \begin{array}{c} UG \\ \downarrow Uf \\ UH \end{array}$$

$$\begin{array}{c} G \\ f \downarrow \uparrow f^{-1} \\ H \end{array}$$

f homom

if f has an inverse function f^{-1}

f^{-1} is again an homomorph

underlying function

If Uf is a bijection then f is an iso in Grp.

Definition $\mathcal{C}$ is called algebraic category over Set iff

1) $\mathcal{C}$ is EXACT

2) $\exists P \in \mathcal{C}$ with copowers $\amalg P$

s.t. • P is reg. projective

• P is reg. generator

$\mathcal{C}$ algebraic,

$(\mathcal{S} \cdot \overline{\mathcal{B}})$ algebraic

Ex Grp – Ring – Bool are $\boxed{\text{algebraic cat}}$

Ex $\boxed{\text{CoAlg } T_2}$ they are algebraic

$P = \{*\}$

If we want an algebraic cats to

describe exactly (2-S) algebras

then we must add one more

condition to P

P is called finite presentable object

iff $\mathcal{C}(P,-)$ preserves filtered colimits

$\mathcal{E}(P,-)$ preserves colimits

FILTERED

$\mathbb{D}$ diagram $\mathbb{D}$ is filtered iff

- $\forall \begin{cases} D_1 \\ D_2 \end{cases} \exists D_3$

$$\begin{array}{ccc} D_1 & & \\ & \searrow & \\ & & D_3 \\ & \nearrow & \\ D_2 & & \end{array}$$

- $\forall\ D_1 \underset{v}{\overset{u}{\rightrightarrows}} D_2 \quad \exists w\ D_2 \xrightarrow{w} D_3$

$$D_1 \rightrightarrows D_2 \longrightarrow D_3$$

$\mathcal{C}$ algebraic $\Longleftrightarrow$ • EXACT ✓

~~on Set~~ DEF • P well def ✓

Exercise:

$\mathcal{C}$ $\quad P \in \mathcal{C}$ $\quad$ P has coproducts

then

$$\mathcal{C} \underset{\amalg}{\overset{F = \amalg P}{\rightleftarrows}} \mathrm{Set}$$

$\mathcal{C}(P, -)$

$$FX = \coprod_{x \in X} P_x$$

$\boxed{F(1) = P}$

$\boxed{F \dashv \mathcal{C}(P, -)}$