Theorem The following are equivalent:

① $\boxed{\mathcal{A} \simeq \mathrm{Set}^T}$

② $\mathcal{A}$ is an $\boxed{\text{exact}}$ category with $P \in \mathcal{A}$ with copowers $\coprod P$

|| P is resp. projective ||

P is resp. generator

Proof 1) $\Rightarrow$ 2)

Theorem If $\mathcal{E}$ is a complete, cocomplete, regular, any reg epimorphism has a right inverse, then

1) $\mathcal{E}^T$ is complete, cocomplete, regular

2) $\mathcal{E}^T \underset{G^T}{\overset{F^T}{\underset{\perp}{\rightleftarrows}}} \mathcal{E}$ $\|$ G^T preserves reg epis

G^T reflects reg epis

3) if $\mathcal{E}$ is $\boxed{\text{exact}}$ then $\boxed{\mathcal{E}^T \text{ is exact}}$

in our case $\underline{\mathcal{C} = \mathrm{Set}}$

Set exact plus

the theorem applies to $\mathcal{C} = \mathrm{Set}$

any reg. epi has a right inverse

$$\begin{tikzcd} X \arrow[r, two heads, "f"'] & Y \arrow[l, bend right, "h"'] \end{tikzcd}$$

$\boxed{f \cdot h = 1_Y}$

in Set this is

the axiom of choice

$\forall y$

$\exists x \in X$: $f(x) = y$

$\boxed{\mathcal{A} \simeq Set^T}$ is $\boxed{\text{EXACT}}$

$$Set^T \simeq \mathcal{A} \underset{G}{\overset{F}{\underset{\perp}{\leftrightarrows}}} Set \quad T = GF$$

def $P \in \mathcal{A}$ as $\boxed{P = F(1)}$

we must verify:

- P is a reg proj
- P is a reg. generator

we must define $P \in \mathcal{A}$

$$Grp \underset{}{\overset{F}{\leftrightarrows}} Set$$

$$\cup$$

$$\mathbb{Z} \quad \boxed{\mathbb{Z} = F(1)}$$

$1 \in Set$

P reg proj. in $\mathcal{A} \simeq Set^T$

$$\begin{tikzcd} & P \arrow[dl, dashed, "k"'] \arrow[d, "h"] \\ X \arrow[r, two heads] & Y \end{tikzcd}$$

$\forall f$ regep

~~Set~~

$\forall f$ reg ep. – $\forall h : P \to Y$ $\exists k : P \to X$ such that

$f k = h$

$\boxed{P = F(1)}$

by adjoint properties.

$$\boxed{\mathcal{A}(F(1), Y) \underset{\varphi_{1,Y}}{\simeq} Set(1, GY)}$$

$Y \in \mathcal{A}$
$1 \in Set$

$$\begin{array}{c}
\text{In } A: \quad P = F(1),\quad k: P \to X,\quad h: P \to Y,\quad X \overset{f}{\twoheadrightarrow} Y
\end{array}$$

$\xrightarrow{G}$

$$\begin{array}{c}
1,\quad \bar{k}: 1 \to G(X),\quad \bar{h}: 1 \to G(Y),\quad G(X) \xrightarrow{G(f)} G(Y)
\end{array}$$

$\bar{h} \overset{def}{=} \varphi_{Y,1}(h)$

Set

$$\begin{array}{ccc}
A(F(1), Y) & \xrightarrow{\varphi_{Y,1}} & Set(1, GY) \\
\uparrow {\scriptstyle A(F(1), f)} & & \uparrow {\scriptstyle Set(1, Gf)} \\
A(F(1), X) & \xrightarrow{\varphi_{X,1}} & Set(1, GX)
\end{array}$$

$k \mapsto \bar{k}$

$G f$ still an epi

$\downarrow$

1 is reproj. in Set

$\downarrow$

$\exists\, \bar{k} : 1 \to G(X)$

$G(f)\,\bar{k} = \bar{h}$

$$\boxed{k \overset{DEF}{=} (\varphi_{X,1})^{-1}(\bar{k})}$$

to prove

$f k = h$??

$$\begin{tikzcd} \boxed{fk = h} & \boxed{Gf \circ \bar{k}} = \bar{h} \arrow[l, "\varphi_{1,Y}"'] \\ k \arrow[u, "\wr", "A(F(1), f)"'] & \bar{k} \arrow[l] \arrow[u, "\wr"', "Set(1, Gf)"] \end{tikzcd}$$

1 is reg. under

$k : F(1) \to X$

hence $F(1)$ is reg. projective

$F(1) = P$ is a $\boxed{\text{reg. generator}}$

$\forall X \in Set^T = \mathcal{A}$ $X = (S, h : TS \to S)$

$\exists\, \rho : \coprod P \twoheadrightarrow X$ reg epi

<u>Lemma.</u> $\forall$ T-algebra (S, h) $\exists$ a $\boxed{\text{split sequence}}$ in Set (characteristic presentation of (S, h))

$$\begin{tikzcd} T^2S \arrow[r, shift left, "\mu_S"] \arrow[r, shift right, "Th"'] & TS \arrow[l, bend right, "\eta_{TS}"'] \arrow[r, "h"'] & S \arrow[l, bend right, "\eta_S"'] \end{tikzcd}$$

- $h \cdot \eta_S = 1_S$
- $h \cdot Th = h \cdot \mu_S$
- $\mu_S \cdot \eta_{TS} = 1_{TS}$
- $Th \cdot \eta_{TS} = \eta_S \cdot h$

$\boxed{\text{in Set}}$

$TS = GFX \qquad T^2S = GFGFX$

$\cancel{G}FGFX \rightrightarrows \cancel{G}FX$ - - - -

G creates split coeq

$$\begin{tikzcd} FGFS \arrow[r, shift left, "Fh"] \arrow[r, shift right, "\varepsilon_{FS}"'] \arrow[r, phantom, "\perp"] & FS \arrow[r, dashed, "\exists"] \arrow[r, phantom, "\bar{h}"'] & A \end{tikzcd}$$

$A = \text{Coeq}(Fh, \varepsilon_{FS})$

$\underline{A = \text{Coeq}^T} \xrightarrow{G} \text{Coeq}$

$$\boxed{(A, \bar{h}) = (S, h)}$$

Lemma: $\forall\, (S, h)$ T-algebra (in FV)

The following is a coequalizer

$$(T^2S, \mu_{TS}) \underset{\mu_S}{\overset{Th}{\rightrightarrows}} (TS, \mu_S) \xrightarrow{h} (S, h) \quad \Big| \text{ in } T\text{-algebra}$$

Proof.

$\forall\, (S, h) = X$ $\quad$ $\boxed{h \text{ is a reg epi}}$

$\exists\, g : \coprod P \to X$

$\coprod F(1)$

$P = F(1)$

T preserves coproducts.

$$S = \coprod_{S} 1 \qquad \text{in Set}$$

F pres. coproducts

$$F(S) = F\Big(\coprod_{S} 1\Big) = \coprod_{S} F(1) = \coprod_{S} P$$

$$\begin{gathered}(TS, \mu_X) \\ \shortparallel \\ F(S),\end{gathered}$$

$\boxed{1 \to 2}$

ALGEBRAS

$\mathcal{A} \simeq \mathrm{Set}^T$ $\longrightarrow$ $\mathcal{A}$ EXACT ✓

$\to$ P "good" obj!!

$\mathcal{A}$ is an $\boxed{\text{ABELIAN}}$ cat

$\Downarrow$

$\mathcal{A}$ EXACT

$\mathcal{A}(X,Y) \in \mathrm{Ab}$

$\mathcal{A}(X,Y) \times \mathcal{A}(Y,Z)$

$\mathcal{A}(X,Z)$

is a hom. "~~X~~ Y one well def."

$\boxed{\text{Grothendieck tops}}$ $\mathcal{A}$

cat of functors over Set

$\Downarrow$

$\mathcal{A}$ EXACT

$\boxed{2 \Rightarrow 1}$ HP: $\mathcal{A}$ is exact + P - - -

Th. $\mathcal{A} \simeq \mathrm{Set}^T$

Proof $P \in \mathcal{A}$ P has $\amalg P$, then

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "\mathcal{A}(P,-)=G"'] & \mathrm{Set} \arrow[l, shift right, "\amalg P = F"'] \arrow[loop right, "T=GF"] \end{tikzcd}$$

$\amalg P \dashv \mathcal{A}(P,-)$ comes from the univ. property of $\amalg P$

Apply the Beck th.:

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[r, dashed] & \overline{Q} \end{tikzcd} \quad \overset{G}{\longmapsto} \quad \begin{tikzcd} GX \arrow[r, shift left, "Gu"] \arrow[r, shift right, "Gv"'] & GY \arrow[l, bend right, "t"'] \arrow[r, "q"'] & Q \arrow[l, bend right, "s"'] \end{tikzcd} \quad \mathrm{SSF}$$

in $\mathcal{A}$

iff in SSF (Gu, Gv) has a split coequalizer (Q, q)

then $\exists\, \overline{Q} \in \mathcal{A}$ $\quad \overline{q} : Y \to \overline{Q}$
such that $G(\overline{Q}) = Q$ $\quad G(\overline{q}) = q$ and $\overline{Q} = \underset{\mathcal{A}}{\mathrm{Coeq}}(u, v)$

hy hypoth $\mathcal{A}$ is EXACT

P gen/proj $\quad G = \mathcal{A}(P, -)$

- P rep proj $\iff$ $G = \mathcal{A}(P,-)$ preser. rep epi
- P rep gen $\iff$ G reflect isos
- G preserves limits
- G preserves monos

$F \dashv G$

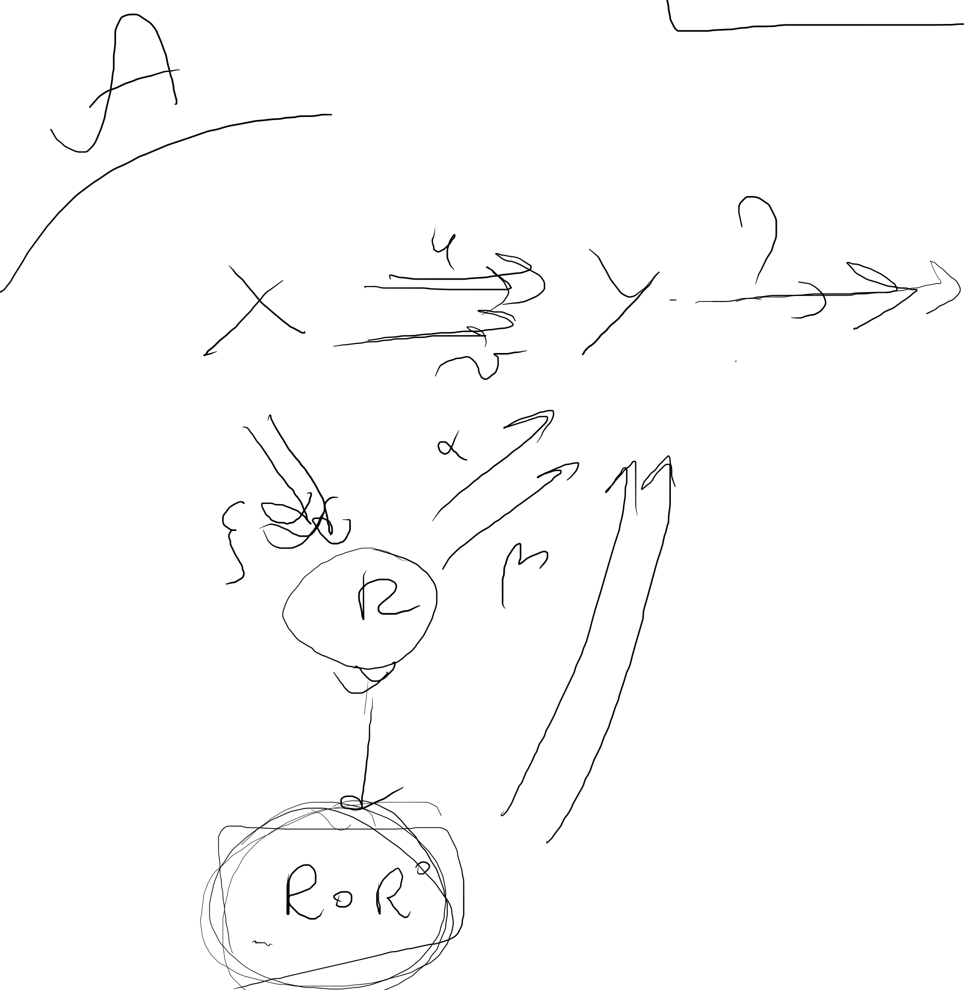
(any mono can be described by a pb)

u, v, pb, m

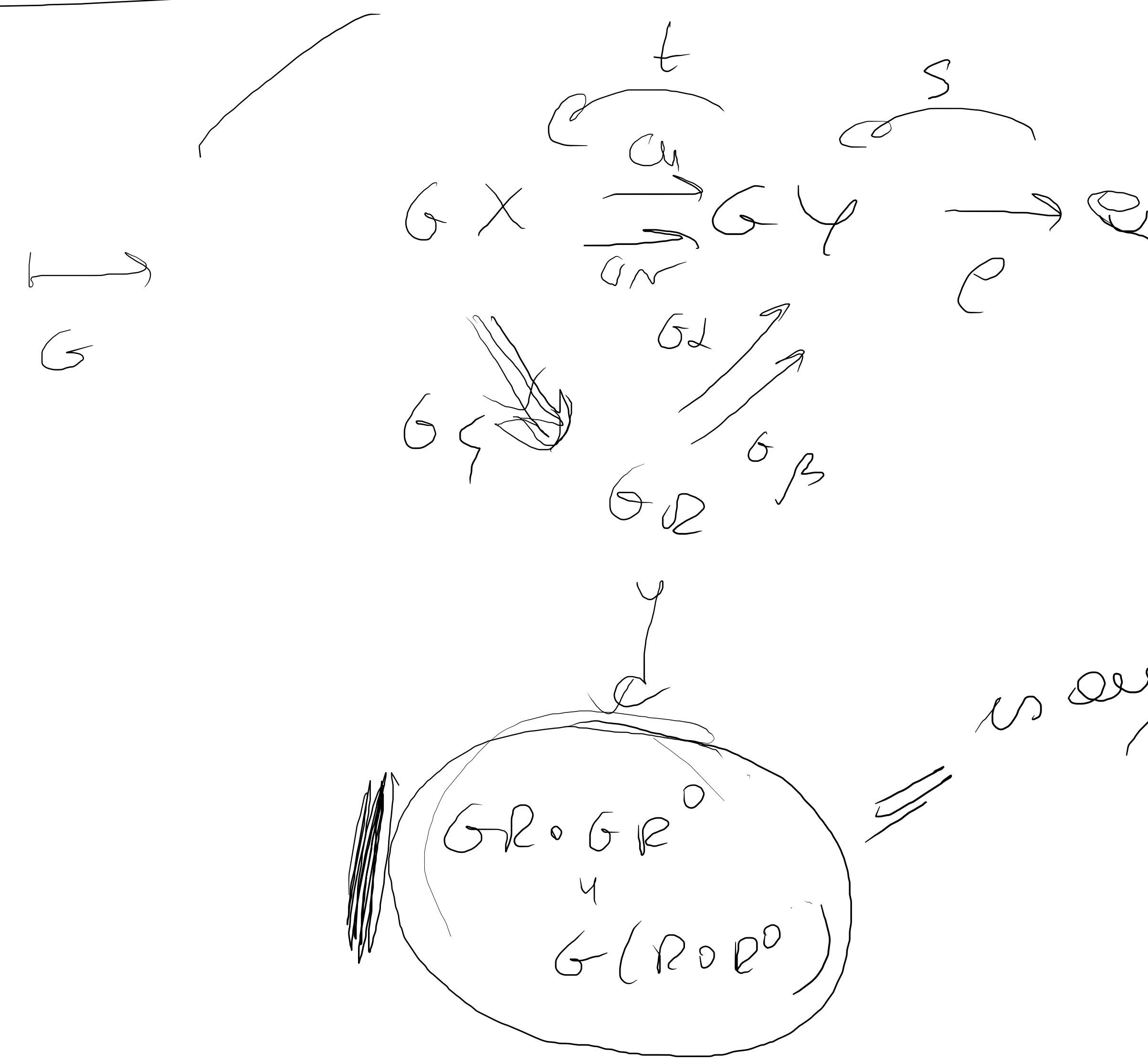
m mono $\iff$ $u = v$

property:

G preserve coequ.
G reflects isos $\longrightarrow$ G reflects limits



is still an equi rel.



is an equi relat.

factorization

$$\begin{tikzcd}
X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, "\rho"'] & Y \arrow[r, dashed, "\rho"'] & Q \\
R \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\beta"'] & &
\end{tikzcd}$$

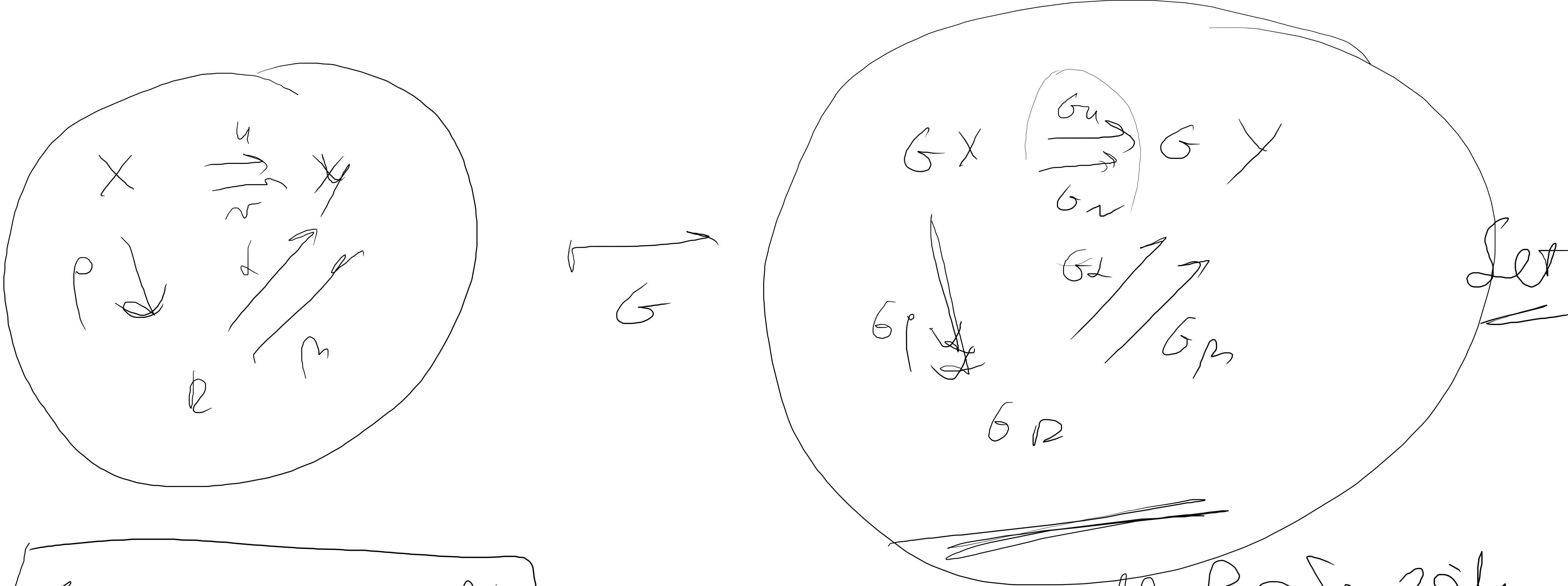
$$\boxed{\begin{array}{c} (Q,\rho) = \mathrm{Coeq}(u,v) \\ \Updownarrow \\ (Q,\rho) = \mathrm{Coeq}(\alpha,\beta) \end{array}}$$

$$\begin{tikzcd}
 & & Y & Y \\
X \arrow[rr, "\langle u, v\rangle"] \arrow[urr, bend left, "u"] \arrow[urrr, bend left=40, "v"] \arrow[dr, two heads, "\rho"'] & & Y \times Y \arrow[u, "p_1"'] \arrow[ur, "p_2"'] & \\
 & R \arrow[ur, tail, "m"'] & &
\end{tikzcd}$$

relation

$(\rho$ is a (reg) epi$)$

A



this is the factorization in Set

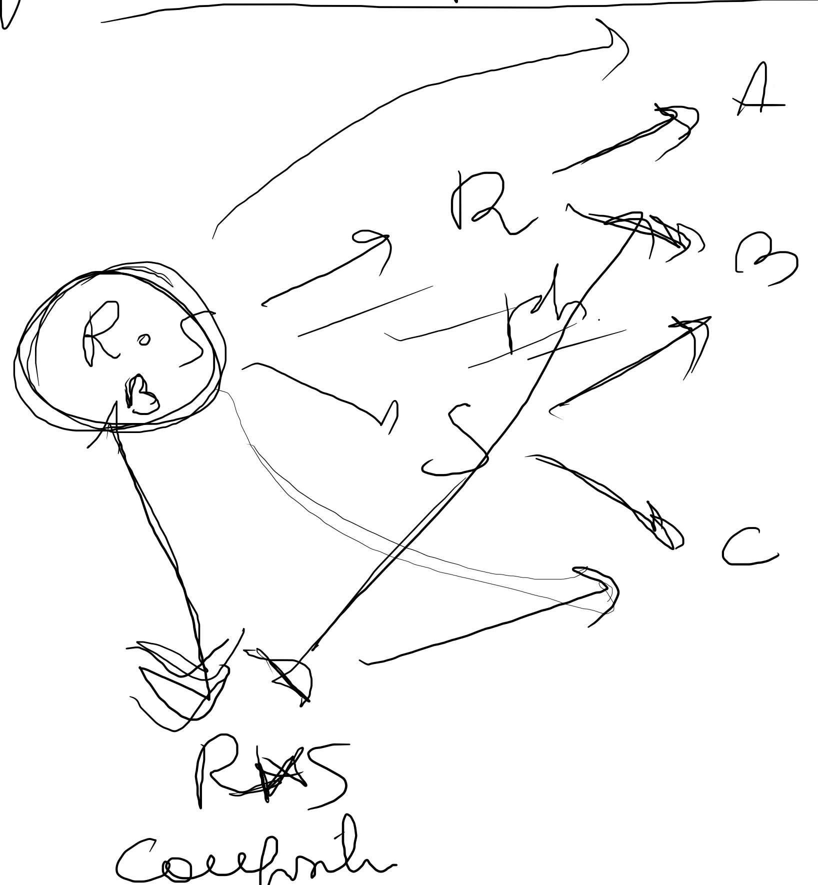
G pres. eq.

G pres. products

G pres. monos.

$R \qquad R \cdot R^{\circ}$ composition of relations

$\forall$ regular category $\mathcal{A}$ relations can be composed



we set $x\,(R\cdot S)\,y \iff$

$\exists z : \quad xRz$

$\qquad\qquad zSy$

R^{o} opposite relation

$R \xrightarrow{u} Y$ $\quad$ $R^{o} \xrightarrow{v} X$

$R \xrightarrow[v]{} X$ $\quad$ $R^{o} \xrightarrow[u]{} X$

$\underbrace{R \circ R^{o}}$

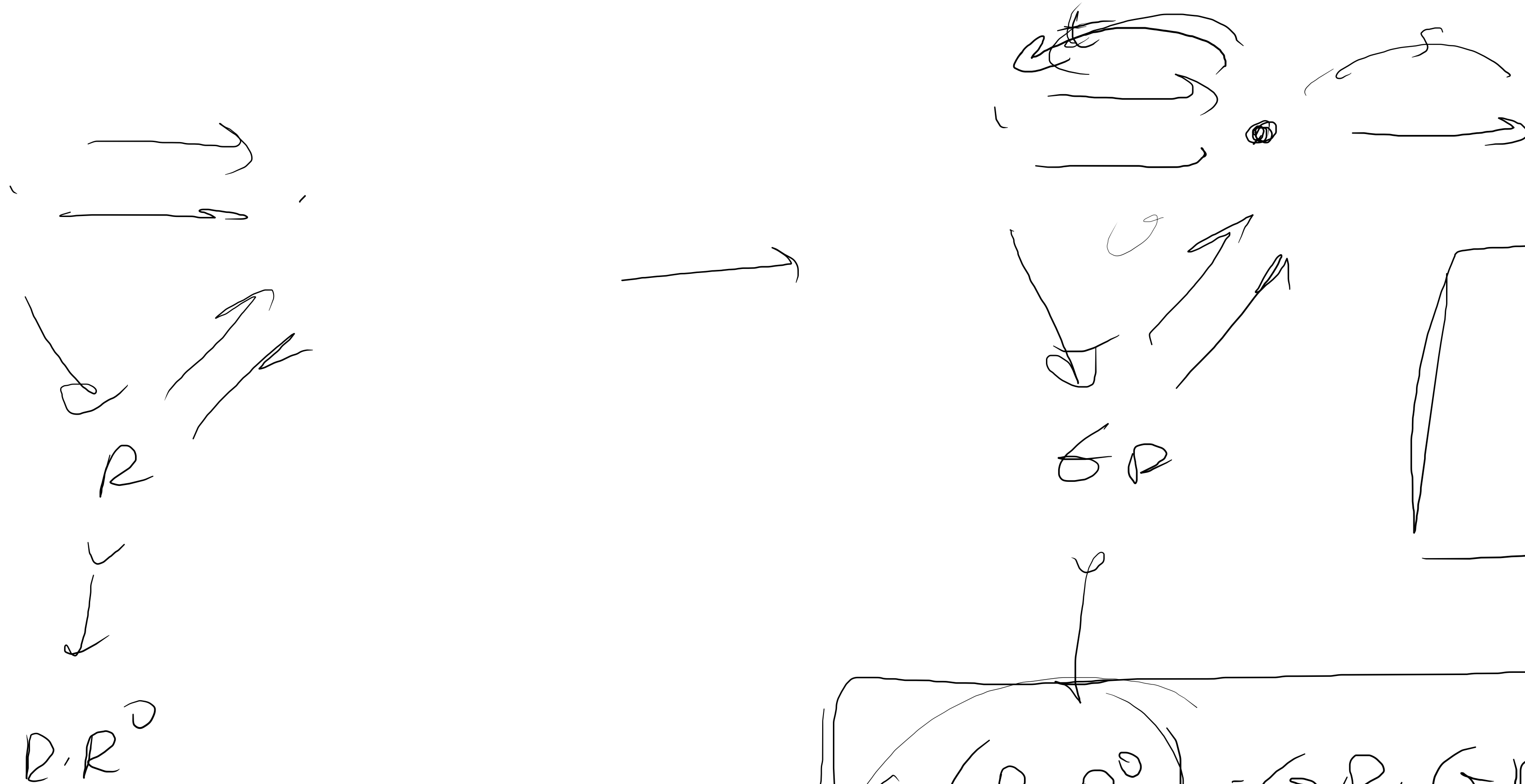
Set $\quad$ $\boxed{a\, R \cdot R^{o}\, b} \iff$

$\exists c \quad aRc \quad cR^{o}b \iff$

$\exists c \quad \boxed{\begin{matrix} aRc \\ bRc \end{matrix}}$

G preserves factors and composition of relations

G has limits



SPLIT

$\forall x$

$x\,(GR \cdot GR^{\circ})\,x$

$G(R \cdot R^{\circ}) = GR \cdot GR^{\circ}$

- SPLIT dep

$\longrightarrow$ $GR \cdot GR^{\circ}$ is an equiv. relation

G preserves - - —

$\Longrightarrow$ a relation V in A is an equiv. iff GV is an equiv. in Set

$$\boxed{A \xrightarrow{G} Set}$$

$T \Longrightarrow \; \rightsquigarrow \; GT \Longrightarrow 0$

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y \xrightarrow{\bar{p}} \bar{Q}$$

$$\rho : X \to R, \quad \alpha, \beta : R \rightrightarrows Y$$

$R \cdot R^{o}$

also is a EQUIV

$\sim\!\!\to G$

$$GX \rightrightarrows GY \xrightarrow{p} Q$$

GR

$GR \circ GR^{o}$

EQUIV

$G\bar{Q} = \bar{O}$

$G\bar{p} = p$

$\boxed{\text{A is EXACT}} \longrightarrow \exists (\bar{O}, \bar{p}) = \mathrm{Coeq}(\bar{\alpha}, \bar{\beta}) = \mathrm{Coeq}(\alpha, \beta) = \mathrm{Coeq}(u, v)$

A exact means every equiv relation is effective

G preserves $\boxed{G\bar{Q} = Q}$

$$\underbrace{R \cdot R^{0} \underset{\bar{\beta}}{\overset{\bar{\alpha}}{\rightrightarrows}} Y}_{\text{kernel pair of } \bar{\rho}} \overset{\bar{\rho}}{\twoheadrightarrow} \bar{Q}$$

$$\begin{tikzcd} \cdot \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\beta}"'] \arrow[dr, phantom, "\mathrm{pb}"] & \cdot \arrow[d, "\bar{\rho}"] \\ \cdot \arrow[r, "\rho"'] & \cdot \end{tikzcd}$$

The square is a p.b.