$f : V \longrightarrow W$ $\qquad$ V, W $\mathbb{K}$-spazi vett. $\qquad$ $\underline{f \in \mathrm{Hom}(V, W)}$

$n = \dim V$, $\quad m = \dim W$

$\underline{\mathcal{B} = (v_1, \dots, v_n)}$ base di V, $\quad \mathcal{C} = (w_1, \dots, w_m)$ base di W

$v \in V \rightsquigarrow \boxed{v = x_1 v_1 + \dots + x_n v_n}$, $x_i \in \mathbb{K}$

$$\updownarrow$$

$$\boxed{(x_1, \dots, x_n) \in \mathbb{K}^n}$$

$\Phi_{\mathcal{B}} : V \longrightarrow \mathbb{K}^n$ $\quad$ lineare biiettiva $\Rightarrow$ $\Phi_{\mathcal{B}}$ l'isomorfismo

$v = x_1 v_1 + \dots + x_n v_n \longmapsto \Phi_{\mathcal{B}}(v) = (x_1, \dots, x_n)$

$(x_1, \dots, x_n)$ componenti o coordinate di v rispetto alla base $\mathcal{B}$.

$f : V \longrightarrow W$ lineare

$\mathcal{B} = (v_1, \dots, v_n)$

$\mathcal{C} = (w_1, \dots, w_m)$

$$W \ni f(v_j) = a_{1j} w_1 + \dots + a_{mj} w_m = \sum_{i=1}^{m} a_{ij} w_i \quad , a_{ij} \in \mathbb{K}$$

$$\rightsquigarrow A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in M_{m,n}(\mathbb{K})$$

$f(v_1) \quad f(v_2) \quad f(v_n)$

Spazio delle matrici di tipo $m \times n$ e entrate in $\mathbb{K}$

$$\boxed{A =: M^{\mathcal{C}}_{\mathcal{B}}(f)}$$

$M^{\mathcal{C}}_{\mathcal{B}} : \mathrm{Hom}(V, W) \longrightarrow M_{m,n}(\mathbb{K})$ biiettiva

E_{ij} = matrice $m \times n$ con $\begin{cases} 1 & \text{al posto } (i,j) \\ 0 & \text{altrove} \end{cases}$

$$\underline{E_{ij}} = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & & & \\ 0 & \cdots & 1 & \cdots 0 \\ & & & \\ 0 & \cdots & & 0 \end{pmatrix}$$

(i,j)

$\{E_{ij}\}_{(i,j)}$ base di $M_{m,n}(k)$

$(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$ $\Rightarrow$ $\boxed{\dim M_{m,n}(k) = mn}$

$$\boxed{\dim M_n(\mathbb{K}) = n^2}$$

$$\mathbb{K}^{m \times n} \overset{\text{def}}{=} M_{m,n}(\mathbb{K})$$

altra notazione

$$M_n(\mathbb{K}) \overset{\text{def}}{=} M_{n,n}(\mathbb{K})$$

spazio delle matrici quadrate

$$A = (a_{ij}) \in M_{m,n}(\mathbb{K}) \cong \mathbb{K}^{mn}$$

$$\updownarrow$$

$$(a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{m1}, \dots, a_{mn}) \in \mathbb{K}^{mn}$$

$$M^{\mathcal{C}}_{\mathcal{B}} : \mathrm{Hom}(V, W) \longrightarrow M_{m,n}(\mathbb{K})$$

$$\boxed{M^{\mathcal{C}}_{\mathcal{B}}(f+g)}$$

$$A = M^{\mathcal{C}}_{\mathcal{B}}(f) = (a_{ij})$$

$$B = M^{\mathcal{C}}_{\mathcal{B}}(g) = (b_{ij})$$

$f, g : V \longrightarrow W$ lineari.

$$C = M^{\mathcal{C}}_{\mathcal{B}}(f+g) = (c_{ij})$$

$$(f+g)(v_j) = \underline{f(v_j)} + \underline{g(v_j)} = \underline{\sum_{i=1}^{m} a_{ij} w_i} + \sum_{i=1}^{m} b_{ij} w_i =$$

$$= \sum_{i=1}^{m} \underbrace{(a_{ij} + b_{ij})}_{c_{ij}} w_i = \sum_{i=1}^{m} \underline{c_{ij}} w_i \Longrightarrow \boxed{c_{ij} = a_{ij} + b_{ij}}$$

$$\boxed{M^{\mathcal{C}}_{\mathcal{B}}(f+g) = M^{\mathcal{C}}_{\mathcal{B}}(f) + M^{\mathcal{C}}_{\mathcal{B}}(g)}$$

$$\Longrightarrow C = A + B$$

$\lambda \in \mathbb{K}$, $f : V \longrightarrow W$ lineare

$$A = M^{\mathcal{C}}_{\mathcal{B}}(f) = (a_{ij})$$

$$M^{\mathcal{C}}_{\mathcal{B}}(\lambda f) = \lambda\, M^{\mathcal{C}}_{\mathcal{B}}(f) = (\lambda a_{ij})$$

$$(\lambda f)(v_j) = \lambda f(v_j) = \lambda \sum_{i=1}^{m} a_{ij} w_i = \sum_{i=1}^{m} \boxed{\lambda a_{ij}}\, w_i$$

Quindi $M^{\mathcal{C}}_{\mathcal{B}} : \mathrm{Hom}(V, W) \longrightarrow \underline{M_{m,n}(\mathbb{K})}$

è <u>isomorfismo</u>

<u>Cor.</u> $\dim \mathrm{Hom}(V, W) = \dim V \cdot \dim W$.

$\dim \mathrm{End}(V) = (\dim V)^2$.

Prodotto di matrici

$B \cdot A$ definito se $\boxed{r = m}$

$$A \in M_{m,n}(\mathbb{K}) \quad , \quad B \in M_{n,r}(\mathbb{K})$$

$$\#\{\text{colonne di } A\} = \#\{\text{righe di } B\}$$

$$C = A \cdot B = (c_{ij}) \in M_{m,r}(\mathbb{K})$$

Prodotto righe per colonne

$$\boxed{c_{ij} \overset{def}{=} \sum_{h=1}^{n} a_{ih} b_{hj} = a_{i1} b_{1j} + \dots + a_{in} b_{nj}}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} \vdots & b_{1j} & \vdots \\ \cdots & \vdots & \cdots \\ \vdots & b_{mj} & \vdots \end{pmatrix}$$

A

Es $\quad A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 0 \\ 4 & 1 \end{pmatrix} \in M_2(\mathbb{Q})$

$$A \cdot B = \begin{pmatrix} -9 & -1 \\ -7 & 2 \end{pmatrix} \in M_2(\mathbb{Q})$$

$$A, B \in M_n(\mathbb{K}) \implies A \cdot B \in M_n(\mathbb{K})$$

$A \cdot B = B \cdot A$? $\qquad$ In generale NO

(SI se $n = 1$)

$$B \cdot A = \begin{pmatrix} -5 & 5 \\ 7 & -2 \end{pmatrix} \neq A \cdot B$$

$$B = \begin{pmatrix} -5 & 0 \\ 4 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

$$A \cdot 0 = 0$$

$$0 \cdot A = 0$$

Matrice identica $n \times n$

o identità

$$\mathbb{1}_n = I_n = E_n = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = (\delta_{ij})$$

δ di Kronecker

$$\delta_{ij} = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

$$A \in M_{m,n}(\mathbb{K}), \quad C = A \cdot I_n = (c_{ij})$$

$$c_{ij} = \sum_{h=1}^{n} a_{ih} \cdot \delta_{hj} = a_{ij} \cdot \delta_{jj} = a_{ij}$$

$$\delta_{jj} = 1$$

$$C \in M_{m,n}(\mathbb{K}) \qquad C = A.$$

$$\boxed{A \cdot I_n = A}$$

$$I_m \cdot A = A \qquad A \in M_{m,n}(\mathbb{K})$$

$$C = I_m \cdot A = (c_{ij}) \in M_{m,n}(\mathbb{K})$$

$$c_{ij} = \sum_{h=1}^{m} \delta_{ih} \cdot a_{hj} = a_{ij}$$

$$C = A$$

$$\boxed{I_m \cdot A = A}$$

$$A \in M_n(\mathbb{K}) \qquad \underline{I_n} \cdot A = A \cdot \underline{I_n} = A$$

$I \in M_n(\mathbb{K})$ elemento neutro (o identità) per $\cdot$

$$a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = M_{n,1}(\mathbb{K})$$

$$\mathbin{\rotatebox[origin=c]{90}{$\in$}}$$

$$M_{1,n}(\mathbb{K})$$

$$\boxed{a \cdot b = a_1 b_1 + \cdots + a_n b_n} \in M_1(\mathbb{K}) \cong \mathbb{K}$$

$$b \cdot a = \begin{pmatrix} a_1 b_1 & \cdots & a_n b_1 \\ a_1 b_2 & \cdots & a_n b_2 \\ - & - & - \\ a_1 b_n & \cdots & a_n b_n \end{pmatrix} \in M_n(\mathbb{K})$$

$A \in M_{m,n}(\mathbb{K}) \quad , \; B \in M_{n,r}(\mathbb{K})$

$$(A \cdot B)_{ij} = A^{(i)} \cdot B_{(j)}$$

$\uparrow$ i-enme rigen

$\uparrow$ j-enme colonne

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & V \arrow[r, "g"] & W \end{tikzcd}$$

U, V, W $\mathbb{K}$-spazi vett.

f, g lineari $\Rightarrow$ $g \circ f$ lineare

infatti

$$(g \circ f)(\lambda_1 u_1 + \lambda_2 u_2) = g(f(\lambda_1 u_1 + \lambda_2 u_2)) =$$

$$= g(\lambda_1 f(u_1) + \lambda_2 f(u_2)) = \lambda_1 g(f(u_1)) + \lambda_2 g(f(u_2))$$

$$= \lambda_1 (g \circ f)(u_1) + \lambda_2 (g \circ f)(u_2)$$

$$\left. \forall \lambda_1, \lambda_2 \in \mathbb{K}, \ \forall u_1, u_2 \in U \right\} \Rightarrow g \circ f \text{ lineare}$$

$id_V : V \longrightarrow V$ lineare $\forall V$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & V \arrow[r, "g"] & W \end{tikzcd}$$

$\mathcal{A} = \{u_1, \dots, u_r\}$ base di U

$\mathcal{B} = (v_1, \dots, v_n)$ base di V

$\mathcal{C} = (w_1, \dots, w_m)$ base di W

$M^{\mathcal{C}}_{\mathcal{A}}(g \circ f) = C = (c_{ij})$, $A = M^{\mathcal{C}}_{\mathcal{B}}(g) = (a_{ij})$, $B = M^{\mathcal{B}}_{\mathcal{A}}(f) = (b_{ij})$

$$(g \circ f)(u_j) = g(f(u_j)) = g\left(\sum_{h=1}^{n} b_{hj} v_h\right) =$$

$$= \sum_{h=1}^{n} b_{hj}\, g(v_h) = \sum_{h=1}^{n} b_{hj} \sum_{i=1}^{m} a_{ih} w_i = \sum_{i=1}^{m}\left(\sum_{h=1}^{n} a_{ih} b_{hj}\right) w_i =$$

$$= \sum_{i=1}^{m} c_{ij}\, w_i$$

$$\boxed{c_{ij} = \sum_{h=1}^{n} a_{ih}\, b_{hj}}$$

$$\boxed{C = A \cdot B}$$

$$\boxed{M^{\mathcal{C}}_{\mathcal{A}}(g \circ f) = M^{\mathcal{C}}_{\mathcal{B}}(g) \cdot M^{\mathcal{B}}_{\mathcal{A}}(f)}$$

$f: \mathbb{K}^n \longrightarrow \mathbb{K}^m$ lineare

$\mathcal{E} = (e_1, \dots, e_n) \qquad \mathcal{T} = (t_1, \dots, t_m)$

base canonica

$$f(e_j) = \underline{\sum_{i=1}^{m} a_{ij}\, t_i} \qquad \rightsquigarrow \quad A = (a_{ij})$$

$$v = x_1 e_1 + \dots + x_n e_n = (x_1, \dots, x_n) \in \mathbb{K}^n$$

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$f(v) = f\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j\, f(e_j) = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij}\, t_i =$$

$$f(v) = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad = \underline{\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}\, x_j \right) t_i}$$

$$f(v) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) t_i =$$

$$= \begin{pmatrix} a_{11} x_1 + \cdots + a_{1n} x_n \\ a_{21} x_1 + \cdots + a_{2n} x_n \\ \cdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n \end{pmatrix} = A \cdot X$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boxed{f(X) = A \cdot X}$$

$$A \rightsquigarrow L_A : \mathbb{K}^n \longrightarrow \mathbb{K}^m$$

$$\underset{M_{m,n}(\mathbb{K})}{\hat{A}}$$

$$L_A(X) = AX$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$