$A \,,\, B \qquad A \in M_{m,n}(\mathbb{K}) \,,\quad B \in M_{n,r}(\mathbb{K})$

$C = A \cdot B \qquad C = (c_{ij}) \in M_{m,r}(\mathbb{K})$

$$c_{ij} = A^{(i)} \cdot B_{(j)} = \sum_{h=1}^{n} a_{ih} b_{hj}$$

$$a = (a_1, \dots, a_n) \,,\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\underline{a \cdot b = \sum_{i=1}^{n} a_i b_i} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$A \in M_{m,n}(\mathbb{K}) \rightsquigarrow {}^tA \in M_{n,m}(\mathbb{K})$

$A = (a_{ij})$

$\uparrow$ matrice trasposta di A

${}^tA$ ottenuta da A scambiando le righe con le colonne

$A^t = (a'_{ij}) \qquad a'_{ij} \overset{\text{def}}{=} \underline{a_{ji}}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & & \cdots & & a_{mn} \end{pmatrix} \rightsquigarrow {}^tA = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ \vdots & & & & \\ a_{1n} & a_{2n} & \cdots & & a_{mn} \end{pmatrix}$$

$a \in M_{1,n}(\mathbb{K})$ $\qquad$ ${}^t a \in M_{n,1}(\mathbb{K})$

$a = (a_1 \ a_2 \ \dots \ a_n)$ $\qquad$ ${}^t a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$${}^t\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}$$

$A \in M_n(\mathbb{K}) \implies {}^t A \in M_n(\mathbb{K})$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

$a_{ij} = a_{ji}$ $\forall i,j$ $\iff$ A è detta simmetrica se ${}^t A = A$

$a_{ij} = -a_{ji}$ $\forall i,j$ $\iff$ se ${}^t A = -A$ antisimmetrica

$a_{ii} = -a_{ii}$ $\qquad$ $2a_{ii} = 0$ $\qquad$ se $2 \neq 0$ in $\mathbb{K} \implies a_{ii} = 0$

$(\mathbb{K} = \mathbb{R})$

$A \in M_{m,n}(\mathbb{K})$, $B \in M_{n,r}(\mathbb{K})$

$L_A(Y) = AY$

$L_B(X) = BX$

$$\begin{tikzcd} \mathbb{K}^r \arrow[r, "L_B"] \arrow[rr, bend right, "L_A \circ L_B"'] & \mathbb{K}^n \arrow[r, "L_A"] & \mathbb{K}^m \end{tikzcd}$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$$

$$(L_A \circ L_B)(X) = A \cdot (BX) = (AB)\,X$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$$

$$\underline{h \circ (g \circ f) = (h \circ g) \circ f}$$

$$(AB)C = A(BC) \quad (*) \qquad A \in M_{m,n}(\mathbb{K}),\ B \in M_{n,r}(\mathbb{K}),\ C \in M_{r,s}(\mathbb{K})$$

$$\mathbb{K}^s \xrightarrow{L_C} \mathbb{K}^r \xrightarrow{L_B} \mathbb{K}^n \xrightarrow{L_A} \mathbb{K}^m$$

$$(L_A \circ L_B) \circ L_C = L_A \circ (L_B \circ L_C)$$

$$\Longrightarrow (AB)C = A(BC)$$

$$\boxed{A(B+C) = AB + AC}$$

$$\boxed{(A+B)C = AC + BC}$$

(*)

$$V \underset{g}{\overset{f}{\longrightarrow}} W \xrightarrow{h} U \quad \text{lineari}$$

$$\boxed{h \circ (f+g) = (h \circ f) + (h \circ g)} \quad \text{(linearità di } h\text{)}$$

$$\big(h \circ (f+g)\big)(v) = h\big((f+g)(v)\big) = h\big(f(v) + g(v)\big) =$$

$$= (h \circ f)(v) + (h \circ g)(v) = \big((h \circ f) + (h \circ g)\big)(v)$$

$$\forall v \in V$$

$$V \xrightarrow{f} W \underset{g}{\overset{h}{\longrightarrow}} U$$

$$(h+g)\circ f = h\circ f + g\circ f$$

$$X \underset{g}{\overset{f}{\longrightarrow}} V$$

$$f+g : X \longrightarrow V$$

$$(f+g)(x) = f(x)+g(x)$$

$$((h+g)\circ f)(v) = (h+g)(f(v)) = h(f(v)) + g(f(v)) =$$
$$= ((h\circ f)+(g\circ f))(v)$$

$$\lambda \in \mathbb{K}, \quad \lambda(AB) = (\lambda A)B = A(\lambda B)$$

$$V \xrightarrow{f} W \xrightarrow{g} U \qquad \lambda\cdot(g\circ f) = (\lambda g)\circ f = g\circ(\lambda f)$$

LINEARI (almeno g)

$$A \in M_n(\mathbb{K}) \rightsquigarrow \mathbb{K}^n \xrightarrow{L_A} \mathbb{K}^n \qquad L_A \in \mathrm{End}(\mathbb{K}^n)$$

Se L_A è $\cong$ $\Longrightarrow \exists\, (L_A)^{-1} : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ lineare

$$\begin{array}{c} (L_A)^{-1} \\ \shortparallel \\ L_B \end{array} \qquad B \in M_n(\mathbb{K})$$

$$L_B = (L_A)^{-1}$$

$$\begin{array}{ccccc} L_A \circ L_B & = & L_B \circ L_A & = & id_{\mathbb{K}^n} \\ \shortparallel & & \shortparallel & & \\ L_{\underbrace{AB}} & = & L_{\underbrace{BA}} & = & id_{\mathbb{K}^n} = L_{\underline{I_n}} \end{array}$$

$I_n \in M_n(\mathbb{K})$

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\boxed{AB = BA = I_n}$$

B è detta matrice inversa di A

Si pone $\boxed{A^{-1} = B}$

<u>Prop</u> L_A è $\cong$ $\Longleftrightarrow$ A ammette un'inversa

(A è invertibile)

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ non è invertibile

$e_1 - e_2$ $\quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \ker\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

A invertibile $\iff L_A : \mathbb{K}^n \xrightarrow[\text{iso}]{} \mathbb{K}^n$

$\iff L_A$ manda la base canonica di $\mathbb{K}^n$ in una base $\iff$

$\{A_{(1)}, A_{(2)}, \dots, A_{(n)}\}$ base per $\mathbb{K}^n$

$\iff \{A_{(1)}, \dots, A_{(n)}\}$ linearmente indip

Matrici diagonali

$A \in M_n(\mathbb{K})$ è detta matrice diagonale se

$a_{ij} = 0 \quad \forall i \neq j$

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 \cdots & 0 \\ 0 & & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_n \end{pmatrix} = \operatorname{diag}(a_1, \dots, a_n)$$

$$(AB)_{ij} = \sum_{h=1}^{n} a_{ih} b_{hj} = \underline{a_{ii} b_{ij}} = \begin{cases} 0 & i \neq j \\ a_{ii} b_{ij} \end{cases}$$

A, B diagonali $\quad B = \operatorname{diag}(b_1, \dots, b_n)$

$$AB = \operatorname{diag}(a_1 b_1, a_2 b_2, \dots, a_n b_n)$$

$$A = \mathrm{diag}(a_1, \dots, a_n) \quad \text{invertibile} \iff a_i \neq 0 \quad \forall i \in \{1, \dots, n\}$$

$$\text{e} \quad \underline{A^{-1} = \mathrm{diag}(a_1^{-1}, \dots, a_n^{-1})}$$

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "h", "\cong"'] & W \arrow[d, "\cong", "k"'] \\
\mathbb{K}^n \arrow[r, "L_A"'] & \mathbb{K}^m
\end{tikzcd}$$

$v = \sum_j x_j v_j \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$v_j \mapsto e_j$ $\qquad$ $w_i \mapsto t_i$

$\mathcal{B} = (v_1, \dots, v_n)$ base di V, $\quad \mathcal{C} = (w_1, \dots, w_m)$ base di W

diagramma commutativo

$(e_1, \dots, e_n)$ $\quad (t_1, \dots, t_m)$ base canonica

$v = \sum_j x_j v_j$

$$f \rightsquigarrow A = M^{\mathcal{C}}_{\mathcal{B}}(f)$$

$$A = (a_{ij})$$

$$f(v) = f\left(\sum_{j=1}^n x_j v_j\right) = \sum_{j=1}^n x_j f(v_j) = \sum_{j=1}^n x_j \underline{\sum_{i=1}^m a_{ij} w_i}$$

$$= \sum_{i=1}^m \underline{\left(\sum_{j=1}^n a_{ij} x_j\right)} w_i$$

$f(v)$ ha componenti

$$AX = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

$$\| $$

$$L_A(X)$$

$$h(v) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad v = \sum_j x_j v_j$$

$$k(w) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \qquad w = \sum_i y_i w_i$$

$$\rightarrow \quad h(\ker f) = \ker L_A$$

$$k(\operatorname{im} f) = \operatorname{im} L_A \quad (*)$$

$$\ker f \ni v \longmapsto 0$$

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ h \downarrow \wr s & /\!/ & \wr s \downarrow k \\ \mathbb{K}^n & \xrightarrow{\ L_A\ } & \mathbb{K}^m \end{array}$$

$$X \longmapsto 0$$

$$k \circ f = L_A \circ h$$

$$\boxed{f = k^{-1} \circ L_A \circ h}$$

$$\mathcal{B} = (v_1, \dots, v_n) \qquad \mathcal{C} = (v_1', \dots, v_n')$$

$$\begin{tikzcd}
\sum_j x_j v_j = v \arrow[d, mapsto] \arrow[rrr, bend left, mapsto] & V \arrow[r, "id_V"] \arrow[d, "h"', "\wr"] & V \arrow[d, "\wr"', "k"] & v = \sum_i x_i' v_i' \arrow[d] \\
X \arrow[rrr, bend right, mapsto] & \mathbb{K}^n \arrow[r, "\cong", "L_A"'] & \mathbb{K}^n & AX = X'
\end{tikzcd}$$

$$\boxed{A = M^{\mathcal{C}}_{\mathcal{B}}(id_V) \quad \text{invertibile}}$$

matrice del cambiamento di base

$$\boxed{X' = AX}$$

$$L_A = k \circ h^{-1} \quad \text{iso}$$

$$\boxed{\begin{array}{c} X \xrightarrow[\text{biiett.}]{f} Y \xrightarrow[\text{biiett.}]{g} Z \\ g \circ f \ \text{biiettiva} \end{array}}$$

matrice del cambiamento di base
tra (t_1, t_2) e $\underline{(e_1, e_2)}$

$$A = \begin{array}{cc} t_1 & t_2 \\ \end{array}$$

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{R}) \quad \text{invertibile}$$

$\mathbb{R}^2 \quad (e_1, e_2)$ base canonica

(t_1, t_2) base

$$t_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\underset{(t_1, t_2)}{\mathbb{R}^2} \xrightarrow{id} \underset{(e_1, e_2)}{\mathbb{R}^2}$$