

A, B

$$A \in M_{m,n}(K), \quad B \in M_{n,r}(K)$$

$$C = A \cdot B$$

$$C = (c_{ij}) \in M_{m,r}(K)$$

$$c_{ij} = A^{(i)} \cdot B_{(j)} = \sum_{h=1}^n a_{ih} b_{hj}$$

$$a = (a_1, \dots, a_n), \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$a \cdot b = \sum_{i=1}^n a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$A \in M_{m,n}(\mathbb{K}) \rightsquigarrow {}^t A \in M_{n,m}(\mathbb{K})$$

$$A = (a_{ij})$$

↑
matrice trasposta
di A

${}^t A$ ottenuta da A scambiando le righe con le colonne

$$A^t = (a'_{ij}) \quad a'_{ij} \stackrel{\text{def}}{=} \underline{a_{ji}}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & & & & a_{mn} \end{pmatrix}$$

$$\rightsquigarrow {}^t A = \begin{pmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ \textcircled{a_{12}} & a_{22} & a_{32} & \dots & a_{m2} \\ \vdots & & & & \\ a_{1n} & a_{2n} & \dots & & a_{mn} \end{pmatrix}$$

$$a \in M_{1,n}(K)$$

$$a = (a_1 \ a_2 \ \dots \ a_n)$$

$${}^t a \in M_{n,1}(K)$$

$${}^t a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$${}^t \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = (b_1 \ b_2 \ \dots \ b_n)$$

$$A \in M_n(K) \Rightarrow {}^t A \in M_n(K)$$

$$a_{ij} = a_{ji} \iff A \text{ è matrice simmetrica se } {}^t A = A$$

$$\forall i, j$$

$$a_{ij} = -a_{ji} \iff \text{se } {}^t A = -A \text{ antisimmetrica}$$

$$\forall i, j$$

$$a_{ii} = -a_{ii}$$

$$2a_{ii} = 0$$

$$\text{Se } 2 \neq 0 \text{ in } K \Rightarrow a_{ii} = 0$$

$$(K = \mathbb{R})$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

$$A \in M_{m,n}(\mathbb{K}) \quad , \quad B \in M_{n,r}(\mathbb{K})$$

$$\mathbb{K}^r \xrightarrow{L_B} \mathbb{K}^n \xrightarrow{L_A} \mathbb{K}^m$$



$$L_A \circ L_B$$

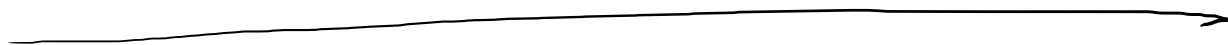
$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}$$

$$L_A(Y) = AY$$

$$L_B(X) = BX$$

$$(L_A \circ L_B)(X) = A \cdot (BX) = \underbrace{(AB)} X$$



$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$$

$$\underline{h \circ (g \circ f) = (h \circ g) \circ f}$$

$$\underline{\underline{(A B) C = A (B C)}} \quad (*)$$

$$A \in M_{m,n}(\mathbb{K}), \quad B \in M_{n,r}(\mathbb{K})$$

$$C \in M_{r,s}(\mathbb{K})$$

$$\mathbb{K}^s \xrightarrow{L_C} \mathbb{K}^r \xrightarrow{L_B} \mathbb{K}^n \xrightarrow{L_A} \mathbb{K}^m$$

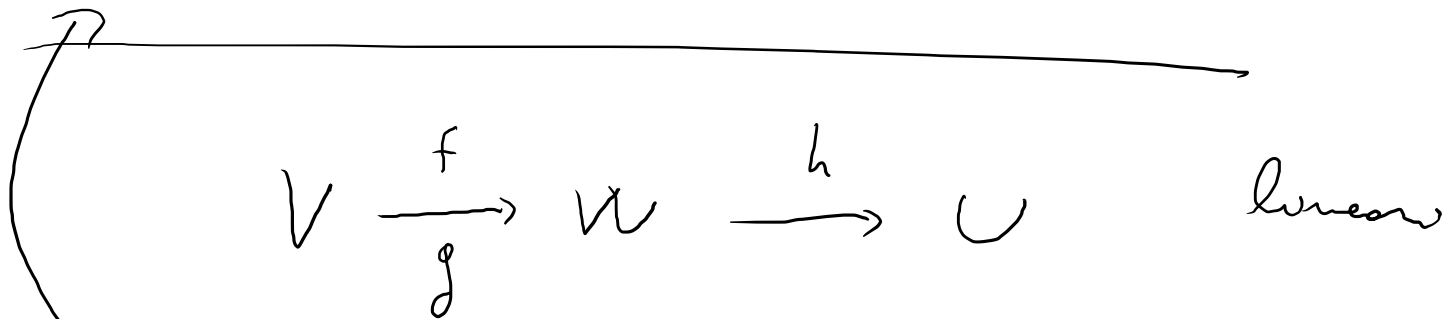
$$\underline{\underline{(L_A \circ L_B) \circ L_C = L_A \circ (L_B \circ L_C)}}$$

$$\Rightarrow \underline{\underline{(A B) C = A (B C)}}$$

$$A(B+C) = AB+AC$$

(*)

$$(A+B)C = AC+BC$$



$$h \circ (f+g) = (h \circ f) + (h \circ g) \quad (\text{linearité de } h)$$

$$\begin{aligned} \underline{(h \circ (f+g))}(v) &= h((f+g)(v)) = h(f(v) + g(v)) = \\ &= (h \circ f)(v) + (h \circ g)(v) = \underline{(h \circ f) + (h \circ g)}(v) \end{aligned}$$

$\forall v \in V$

$$V \xrightarrow{f} W \xrightarrow{g} U$$

$$(h+g) \circ f = h \circ f + g \circ f$$

$$\begin{aligned} ((h+g) \circ f)(v) &= (h+g)(f(v)) = h(f(v)) + g(f(v)) = \\ &= ((h \circ f) + (g \circ f))(v) \end{aligned}$$

$$X \xrightarrow{f} Y$$

$$f+g : X \rightarrow Y$$

$$(f+g)(x) = f(x) + g(x)$$

$$\lambda \in \mathbb{K}, \quad \lambda(A \circ B) = (\lambda A) \circ B = A \circ (\lambda B)$$

$$V \xrightarrow{f} W \xrightarrow{g} U$$

LINEAR (elemento f)

$$\lambda \cdot (g \circ f) = (\lambda g) \circ f = g \circ (\lambda f)$$

$$A \in M_n(\mathbb{K}) \rightsquigarrow \mathbb{K}^n \xrightarrow{L_A} \mathbb{K}^n \quad L_A \in \text{End}(\mathbb{K}^n)$$

$$\text{Se } L_A \text{ e } \cong \Rightarrow \exists (L_A)^{-1} : \mathbb{K}^n \longrightarrow \mathbb{K}^n \text{ linear}$$

||

$$L_B \quad B \in M_n(\mathbb{K})$$

$$L_B = (L_A)^{-1}$$

$$L_A \circ L_B = L_B \circ L_A = \text{id}_{\mathbb{K}^n}$$

||

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$$\underline{L_{AB}} = \underline{L_{BA}} = \text{id}_{\mathbb{K}^n} = \underline{L_{I_n}}$$

$$I_n \in M_n(\mathbb{K})$$

$$\underline{I_n} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$AB = BA = \underline{I}_n$$

B è detta matrice inversa di A

Si pone $A^{-1} = B$

Prop L_A è $\cong \iff A$ ammette un'inversa
(A è invertibile)

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ non è invertibile

$e_1, -e_2 \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \text{ker} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

A invertible $\Leftrightarrow L_A : \mathbb{K}^n \xrightarrow{\text{iso}} \mathbb{K}^n$

$\Leftrightarrow L_A$ maps the base canonico di \mathbb{K}^n in
una base \Leftrightarrow

$\{A_{(1)}, A_{(2)}, \dots, A_{(n)}\}$ base per \mathbb{K}^n

$\Leftrightarrow \{A_{(1)}, \dots, A_{(n)}\}$ linearmente indep

Matrice diagonali

$A \in M_n(K)$ è detta
matrice diagonale se

$$(AB)_{ij} = \sum_{h=1}^n a_{ih} b_{hj} = \underline{a_{ii} b_{ij}} = \begin{cases} 0 & i \neq j \\ a_{ii} b_{ii} & i = j \end{cases} \quad \left. \begin{array}{l} a_{ij} = 0 \quad \forall i \neq j \\ a_{ii} b_{ii} \end{array} \right\}$$
$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & & & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & & a_n \end{pmatrix} = \text{diag}(a_1, \dots, a_n)$$

A, B diagonali

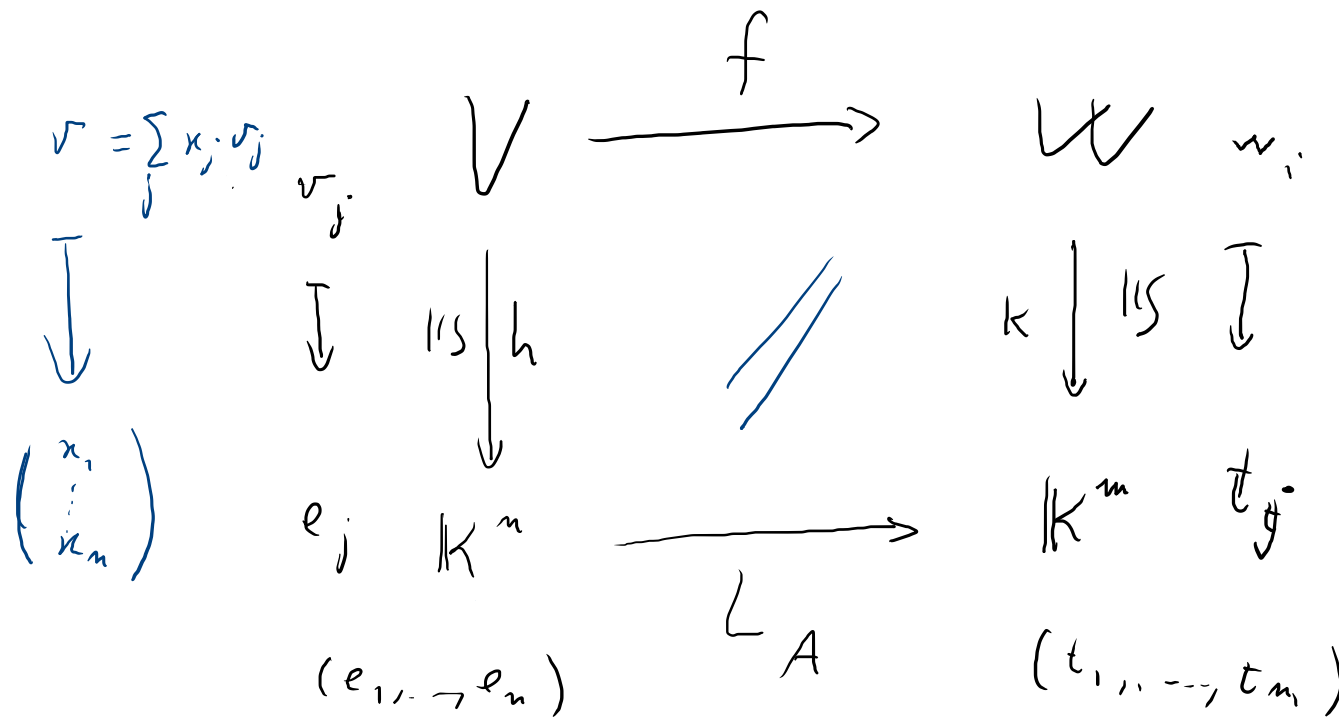
$$B = \text{diag}(b_1, \dots, b_n)$$

$$AB = \text{diag}(a_1 b_1, a_2 b_2, \dots, a_n b_n)$$

$A = \text{diag}(a_1, \dots, a_n)$ invertible $\Leftrightarrow a_i \neq 0 \forall i \in \{1, \dots, n\}$

e $A^{-1} = \text{diag}(a_1^{-1}, \dots, a_n^{-1})$

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$



$B = (v_1, \dots, v_n)$, $C = (w_1, \dots, w_m)$
 base of V base of W

commutative
 diagram

$v = \sum_j x_j v_j$

base of W

$f \rightsquigarrow A = M_{CB}^C(f)$

$A = (a_{ij})$

$f(v) = f\left(\sum_{j=1}^n x_j v_j\right) = \sum_{j=1}^n x_j f(v_j) = \sum_{j=1}^n x_j \sum_{i=1}^m a_{ij} w_i$

$= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j\right) w_i$

$f(v)$ has components

$$AX = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

$$h(v) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

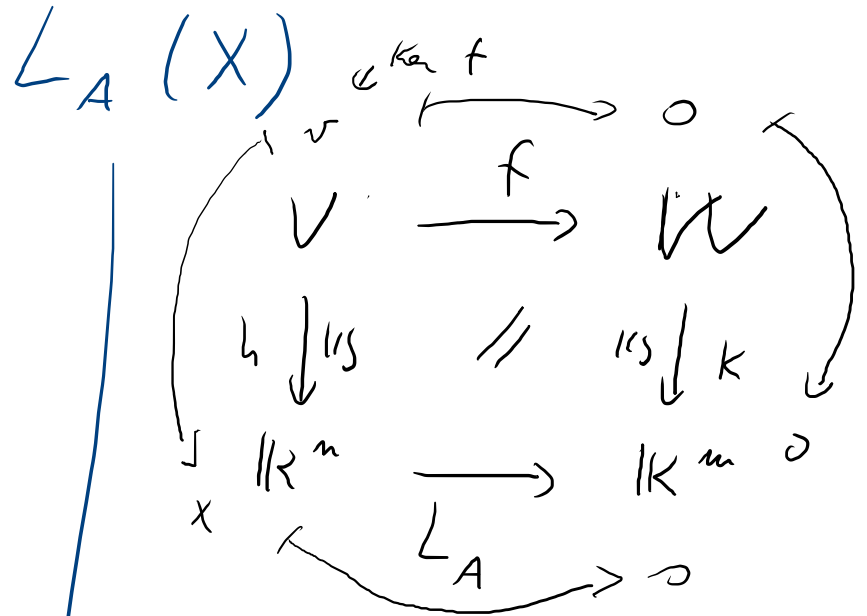
$$v = \sum_i x_i v_i$$

$$k(w) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$w = \sum_i y_i w_i$$

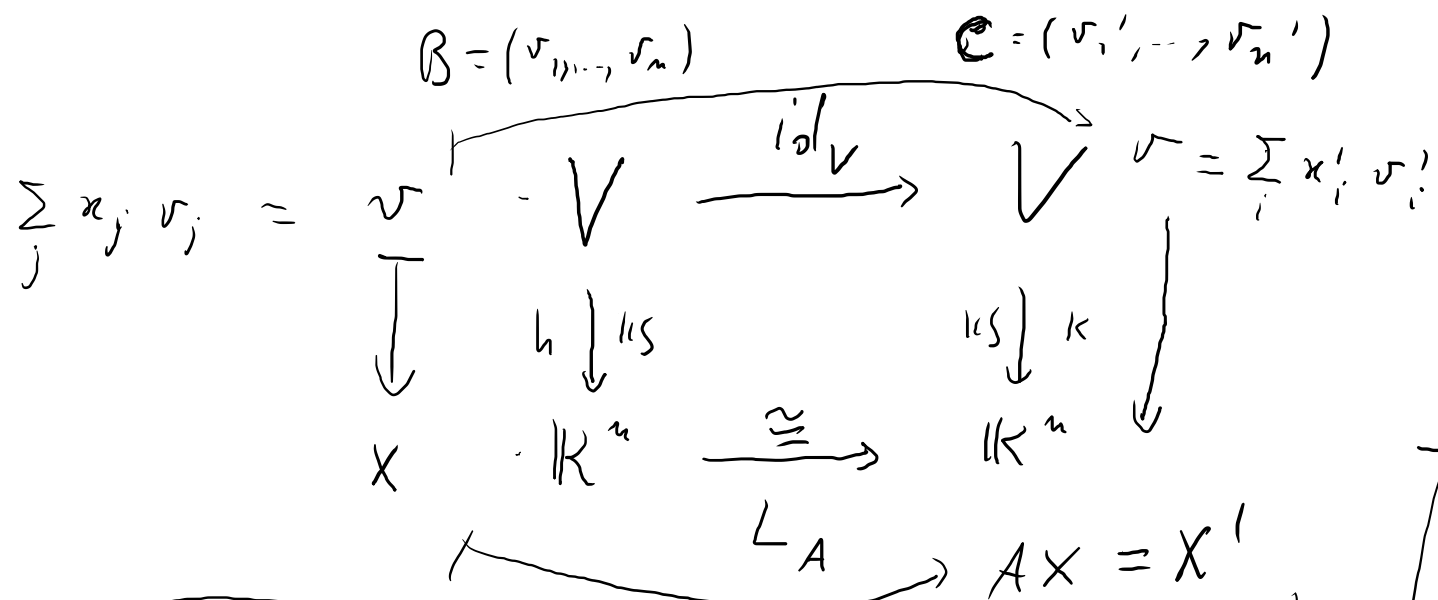
$$\rightarrow h(\ker f) = \ker L_A$$

$$k(\operatorname{im} f) = \operatorname{im} L_A \quad (*)$$



$$k \circ f = L_A \circ h$$

$$f = k^{-1} \circ L_A \circ h$$

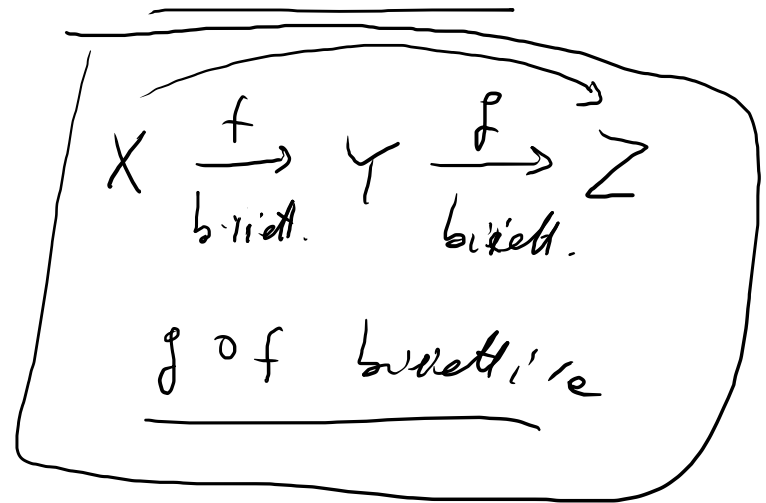


$$L_A = K \circ h^{-1} \quad \text{ISO}$$

$$A = M_B^e(\text{id}_V) \quad \text{invertible}$$

motivo del cambio de base

$$X' = AX$$



matrice del cambiamento di base
 da (t_1, t_2) a (e_1, e_2)

$$A = \begin{matrix} & t_1 & t_2 \\ \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} & \in M_2(\mathbb{R}) & \text{invertibile} \end{matrix}$$

\mathbb{R}^2 (e_1, e_2) base canonica

(t_1, t_2) base

$$t_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\mathbb{R}^2 \xrightarrow{\text{id}} \mathbb{R}^2$$

$(t_1, t_2) \qquad (e_1, e_2)$