9 Novembre

<u>Teor</u> Sia u una soluzione in dim 3 del Teor di Leray t.c.

$$(6.19) \quad u \in L^r((0,T), L^s(\mathbb{R}^3)) \quad \text{con} \quad \frac{2}{r} + \frac{3}{s} = 1 \qquad s > 3$$

Allora $u \in C^\infty((0,T] \times \mathbb{R}^3, \mathbb{R}^3)$ e $\forall\, \varepsilon > 0$ è una soluz. regolare come nel Teor 6.1 in $[\varepsilon, T]$.

Inoltre se v è una soluzione con medesimo dato iniziale, allora $u = v$ in $[0,T]$.

<u>Dim</u> Assumiamo $u_0 \in V$. Allora $\exists\, T^* > 0$ t.c. $u \in L^\infty([0,T_1], V)$ $\forall\, 0 < T_1 < T^*$ e $u \in C^\infty((0,T^*) \times \mathbb{R}^3, \mathbb{R}^3)$. Dimostreremo che $T < T^*$.

Supponiamo $T \geq T^* \Rightarrow \lim\limits_{t \to T^*} |\nabla u(t)|_{L^2} = +\infty$

$$\frac{d}{dt}|\nabla u|^2_{L^2} + 2\nu|\Delta u| = 2\langle \operatorname{div}(u\otimes u), \Delta u\rangle$$

$$\partial_t u - \nu\Delta u = -\mathbb{P}\operatorname{div}(u\otimes u) \qquad \text{in } \mathcal{D}'((0,T^*), L^2_x)$$

$$-\langle\cdot, \Delta u\rangle_{L^2}$$

$$\leq c\,|u|_{L^s}\,|\nabla u|_{L^{\frac{2s}{s-2}}}\,|\Delta u|_{L^2}$$

$$1 = \frac{1}{s} + \frac{s-2}{2s} + \frac{1}{2} = \cancel{\frac{1}{s}} + \frac{1}{2} - \cancel{\frac{1}{s}} + \frac{1}{2} = 1$$

$$s > 3 \iff \frac{2s}{s-2} \in [2,6) \qquad \frac{s-2}{2s} = \frac{\frac{s-3}{s}}{2} + \frac{\frac{3}{s}}{6}$$

$$\leq c\,|u|_{L^s}\,|\nabla u|_{L^2}^{\frac{s-3}{s}}\,|\nabla u|_{L^6}^{\frac{3}{s}}\,|\Delta u|_{L^2}$$

$$\lesssim |u|_{L^s}\,|\nabla u|_{L^2}^{\frac{s-3}{s}}\,|\Delta u|_{L^2}^{\frac{s+3}{s}}$$

$$\leq C\left(|u|_{L^s}\,|\nabla u|_{L^2}^{\frac{s-3}{s}}\right)^{\frac{2s}{s-3}} + \nu\left(|\Delta u|_{L^2}^{\frac{s+3}{s}}\right)^{\frac{2s}{s+3}}$$

$$1 = \frac{s-3}{2s} + \frac{s+3}{2s} \qquad \lambda a \frac{b}{\lambda} \leq \frac{\lambda^p a^p}{p} + \frac{b^q}{q\lambda^q}$$

$$\frac{d}{dt}|\nabla u|_{L^2}^2 + \cancel{2}\nu |\Delta u|_{L^2}^2 \le C\, |u|_{L^s}^{\frac{2s}{s-3}} |\nabla u|_{L^2}^2 + \cancel{\nu |\Delta u|_{L^2}^2}$$

$$\frac{d}{dt}|\nabla u|_{L^2}^2 \le C\, |u|_{L^s}^{\frac{2s}{s-3}} |\nabla u|_{L^2}^2 = C\,|u|_{L^s}^{r} |\nabla u|_{L^2}^2$$

Nell' intervallo $[0,T]$

$$\frac{\cancel{2}(s-3)}{\cancel{2}s} + \frac{3}{s} \simeq 1$$

$$= \frac{s-\cancel{3}}{s} + \cancel{\frac{3}{s}} = 1$$

$$+\infty \xleftarrow{t \to T^*} |\nabla u(t)|_{L^2}^2 \le e^{C\int_0^t |u(t')|_{L^s}^r \, dt'} |\nabla u(0)|_{L^2}^2$$

$$\le e^{C\,|u|^r_{L^r((0,T),L^s)}} |\nabla u(0)|_{L^2}^2$$

non è possibile che $T^* \le T$

$u_0 \in V$.

Nel caso generale so che esiste $t_n \searrow 0$ t.c.

$$u(t_n) \in V \implies u \in C^\infty((t_n, T] \times \mathbb{R}^3)$$

$$\implies u \in C^\infty((0,T] \times \mathbb{R}^3)$$

Siano u e v due soluzioni con stesso u_0

Affermiamo che possono ~~esse~~ essere prese come funzioni test in $[0,T]$ nella def. di soluzione debole.

$$\int_0^t \left(\nu\langle\nabla v,\nabla u\rangle - \langle v,\partial_t u\rangle + \langle \mathrm{div}(v\otimes v),u\rangle\right)dt' = |u_0|_{L^2}^2 - \langle v(t),u(t)\rangle$$

$$\int_0^t \left(\nu\langle\nabla v,\nabla u\rangle - \langle \partial_t v,u\rangle + \langle \mathrm{div}(u\otimes u),v\rangle\right)dt' = |u_0|_{L^2}^2 - \langle v(t),u(t)\rangle$$

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$$\int_0^t \left(\nu\langle\nabla v,\nabla u\rangle + \langle v,\partial_t u\rangle + \langle \mathrm{div}(u\otimes u),v\rangle\right)dt = 0$$

$u,v \in C^\infty((0,T],L^2)$. Per $w = v-u$

$$\frac{1}{2}|w(t)|_{L^2}^2 + \int_0^t \left(\nu|\nabla w|^2 - \langle \mathrm{div}(w\otimes w),u\rangle\right]dt'$$

$$= \frac{1}{2}|u(t)|_{L^2}^2 + \nu\int_0^t |\nabla u|_{L^2}^2 - \frac{1}{2}|u_0|_{L^2}^2 \ (=0)$$

$$+ \frac{1}{2}|v(t)|_{L^2}^2 + \nu\int_0^t |\nabla v|_{L^2}^2 - \frac{1}{2}|v_0|_{L^2}^2 \ (=0)$$

$$|w(t)|_{L^2}^2 + \cancel{2}\nu\int_0^t |\nabla w|_{L^2}^2 dt' \le 2\int_0^t \langle \mathrm{div}(w\otimes w),u\rangle\, dt'$$

$$\le 2\int_0^t |u|_{L^2}|w|_{L^{\frac{2s}{s-2}}}|\nabla w|_{L^2}\,dt'$$

$$\le C\int_0^t |u|_{L^s}^r |w|_{L^2}^2 + \cancel{\nu\int_0^t |\nabla w|_{L^2}^2}\, dt'$$

$$|w(t)|_{L^2}^2 \le C\int_0^t |u|_{L^s}^r |w|_{L^2}^2\, dt' \qquad [0,T]$$

$$\int_0^t |u|_{L^s}^r\, dt' \le |u|_{L^r((0,T),L^s_x)}$$

$$\Longrightarrow \quad w(t)\equiv 0 \quad \text{in} \quad [0,T]$$

Resta da dimostrare che v che soddisfa Serrin in $[0,T]$ è una funzione test. Sia u una soluzione debole. Supponiamo che $v \in C^\infty([\varepsilon, T] \times \mathbb{R}^3)$ e che si può scrivere in (ε, T)

$$\int_\varepsilon^t \langle u, \partial_t v\rangle dt' = \langle u(\varepsilon), v(\varepsilon)\rangle - \langle u(t), v(t)\rangle +$$
$$+ \nu \int_\varepsilon^t \langle \nabla v, \nabla u\rangle dt' + \int_\varepsilon^t \langle \operatorname{div}(u\otimes u), v\rangle dt'$$

$\varepsilon \to 0^+$ $\quad \langle u(\varepsilon), v(\varepsilon)\rangle \longrightarrow \langle u_0, v_0\rangle$

$$u(\varepsilon) \xrightarrow{\varepsilon\to 0^+} u_0$$
$$v(\varepsilon) \xrightarrow[\varepsilon\to 0^+]{} v_0$$

in $L^2(\mathbb{R}^3)$. Vogliamo dimostrare

$$\int_0^t \langle u, \partial_t v\rangle dt' = \langle u_0, v_0\rangle - \langle u(t), v(t)\rangle$$
$$+ \nu \int_0^t \langle \nabla v, \nabla u\rangle dt' + \int_0^t \langle \operatorname{div}(u\otimes u), v\rangle dt$$

$$\nabla u, \nabla v \in L^2(\mathbb{R}_+, L^2) \Rightarrow \int_\varepsilon^t \langle \nabla v, \nabla u\rangle dt' \to \int_0^t \langle \nabla v, \nabla u\rangle dt'$$

$$\lim_{\varepsilon\to 0} \int_\varepsilon^t \langle \operatorname{div}(u\otimes u), v\rangle dt' = \int_0^t \langle \operatorname{div}(u\otimes u), v\rangle dt'$$

$$I \doteq \int_0^t |\langle \operatorname{div}(u\otimes u), v\rangle|\, dt' < +\infty$$

$$r = \frac{2s}{s-3}$$
$$r' = \frac{2s}{s+3}$$

$$I \lesssim \int_0^t |v|_{L^s} |u|_{L^2}^{\frac{s-3}{s}} |\nabla u|_{L^2}^{\frac{s+3}{s}} dt' \leq$$
$$\lesssim \int_0^t |v|_{L^s}^r + \int_0^t \left(|u|_{L^2}^{\frac{s-3}{s}} |\nabla u|_{L^2}^{\frac{s+3}{s}}\right)^{r'}$$

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$$I \lesssim \int_0^t |v|_{L^s} \, |u|_{L^2}^{\frac{s-3}{s}} \, |\nabla u|_{L^2}^{\frac{s+3}{s}} \, dt' \lesssim$$

$$\lesssim \int_0^t |v|_{L^s}^r + \int_0^t \left(|u|_{L^2}^{\frac{s-3}{s}} \, |\nabla u|_{L^2}^{\frac{s+3}{s}} \right)^{\frac{2s}{s+3}}$$

$$\lesssim \int_0^T |v|_{L^s_x}^r \, dt' + |u|_{L^\infty(\mathbb{R}_+, L^2_x)}^{\frac{2(s-3)}{s+3}} \int_0^\infty |\nabla u|_{L^2}^2 \, dt'$$

$$< \infty \, . \qquad \square$$

Soluzioni Mild.

Equazione del calore

Sia $T > 0$ e sia $f : [0,T] \to \dot{H}^{s-1}(\mathbb{R}^d, \mathbb{R}^d)$

$f = \mathbb{P} f$

$$\begin{cases} u_t - \nu \Delta u = f \\ \nabla \cdot u = 0 \\ u|_{t=0} = u_0 \in \mathbb{P} \dot{H}^s \end{cases}$$

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Def Per $s < \frac{d}{2}$ sia $f \in L^2([0,T], \dot{H}^{s-1})$

Allora u e' una soluz debole in $[0,T]$ se

$u \in L^\infty([0,T], \dot{H}^s)$, $\nabla u \in L^2([0,T], \dot{H}^s)$,

$u \in C^0([0,T], \dot{H}^s_w)$ e se

$\forall \quad \psi \in C_c^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ si ha $\psi(t) \in \dot{H}^{-1}$

$$\langle u(t), \psi(t) \rangle = \langle u_0, \psi \rangle + \int_0^t \left(\nu \langle u, \Delta \psi \rangle + \langle u, \partial_t \psi \rangle + \langle f, \psi \rangle \right) dt'$$

$\dot{H}^1 \times \dot{H}^{-1} \longrightarrow \mathbb{R}$ $\qquad L^2_{tx}([0,T] \times \mathbb{R}^3)$

Teor Esiste esattamente una soluzione debole in $[0,T]$

Si ha $u \in C^0([0,T], \dot{H}^s)$

e si ha

$$|u(t)|^2_{\dot{H}^s} + 2\nu \int_0^t |\nabla u|^2_{\dot{H}^s}\, dt' = |u_0|^2_{\dot{H}^s} + 2\int_0^t \langle f, u\rangle_{\dot{H}^s}\, dt'$$

$$\int_0^T \langle \cdot, \cdot \rangle_{L^2} dt' : L^2([0,T], \dot{H}^{s-1}) \times L^2([0,T], \dot{H}^{s+1})$$

$$\nabla u \in L^2([0,T], \dot{H}^s)$$

$$\longrightarrow \mathbb{R}$$

$$u(t) = e^{\nu t \Delta} u_0 + \int_0^t e^{\nu(t-t')\Delta} f(t')\, dt'$$

Teor 2 Sia f come prima e sia $u \in C^0([0,T], \dot{H}^s)$
$\nabla u \in L^2([0,T], \dot{H}^s)$ la soluzione.

Allora $\left(u \in L^\infty([0,T], \dot{H}^s) \cap L^2([0,T], \dot{H}^{s+1}) \right)$

$$u \in L^p\left([0,T], \dot{H}^{s+\frac{2}{p}}\right)$$

$2 \leq p \leq \infty$ $\qquad |u(t)|_{\dot{H}^{s+\frac{2}{p}}} \in L^p(0,T)$

e inoltre

$$\|u\|_{L^p((0,T),\dot H^{s+\frac{2}{p}})} \leq \nu^{-\frac{1}{p}} \left(\|u_0\|_{\dot H^s} + \nu^{-\frac{1}{2}} \|f\|_{L^2([0,T],\dot H^{s-1})} \right)$$

$$\boxed{V(t) = \left(\int_{\mathbb{R}^d} |\xi|^{2s} \sup_{0 \leq t' \leq t} |\hat{u}(t',\xi)|^2 \, d\xi \right) \leq \|u_0\|_{\dot H^s} + \frac{1}{(2\nu)^{\frac{1}{2}}} \|f\|_{L^2([0,T],\dot H^{s-1})}}$$

$$\|u\|_{L^\infty([0,T],\dot H^s)} \leq \quad \text{"}$$

$$\leq \|u\|_{\dot H^s L^\infty[0,T]} \leq \quad \ldots \text{"}$$

$$\begin{cases} \partial_t u - \nu \Delta u = Q(u,u) \\ \nabla \cdot u = 0 \\ u_{|t=0} = u_0 \end{cases} \qquad Q(u,u) = -\mathbb{P}\,\mathrm{div}(u \otimes u)$$

$$Q(u,v) \qquad u = v$$

Si vuole interpretare come equazione del calore

Supponendo che u e v siano opportune $B(u,v)$

$$\begin{cases} \partial_t B(u,v) - \nu \Delta B(u,v) = Q(u,v) \\ B(u,v)_{|t=0} = 0 \end{cases}$$

$$\boxed{u = e^{t\nu\Delta} u_0 + B(u,u)}$$ Kato–Fujita

$$e^{t\nu\Delta} u_0 \in X$$

$$B : X \times X \longrightarrow X$$

Lemma Sia X di Banach $B : X^2 \to X$ continua bilineare. Sia $\alpha < \frac{1}{4|B|}$

$\left(|B| = \sup_{|x|=|y|=1} |B(x,y)| \right)$. Allora se $x_0 \in X$

$x_0 \in D_X(0,\alpha)$ esiste ed è unica in $\overline{D_X(0,2\alpha)}$

la soluzione x di $x = x_0 + B(x,x)$.

$$u = e^{\gamma t \Delta} u_0 + B(u,u)$$

$$\uparrow$$

$$X$$

Dim

$$x \to x_0 + B(x,x) \qquad \overline{D(0,2\alpha)} \circlearrowleft$$

$$|x_0 + B(x,x)| \le |x_0| + |B(x,x)| < \alpha + |B|\,|x|^2 \le \alpha + 4|B|\alpha^2 =$$

$$= \alpha\,(1 + \underbrace{4|B|\alpha}_{<1}) < 2\alpha$$

$$|B(x,x) - B(y,y)| = |B(x+y, x-y)| \le |B|\,|x+y|\,|x-y|$$

$$\le |B|\,(|x|+|y|)\,|x-y| \le \underbrace{4\alpha\,|B|}_{<1}\,|x-y|$$

Conclusione. Esiste un unico punto fisso

$x = x_0 + B(x,x)$ in $\overline{D(0,2\alpha)}$.