10 Novembre

$$\underset{x_0}{\cdot} I \xrightarrow{f} \underset{y_0}{\cdot} J \xrightarrow{g} \mathbb{R}$$

<u>Teor</u> Date $f : I \to J$, $g : J \to \mathbb{R}$, I, J intervalli in $\mathbb{R}$, $x_0 \in I$, $y_0 \in J$ con $y_0 = f(x_0)$. Allora se esistono $f'(x_0)$ e $g'(y_0)$ si ha

$$\big(g(f(x))\big)'(x_0) = f'(x_0)\, g'(y_0)$$

$x = h + x_0$

$$\lim_{h \to 0} \frac{g(f(x_0+h)) - g(f(x_0))}{h}$$

$$\lim_{h \to 0} \frac{g(f(x_0+h)) - g(f(x_0))}{h}$$

$$= \lim_{h \to 0} \frac{g(f(x_0+h)) - g(f(x_0))}{f(x_0+h) - f(x_0)} \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \quad f'(x_0)$$

$$= g'(y_0) \quad f'(x_0) = \big(g(f(x))\big)'\Big|_{x=x_0}$$

Il precedente ragionamento è valido solo se $f(x)-f(x_0) \neq 0$ quando $x \neq x_0$

$(\Rightarrow \quad f(x_0+h) \neq f(x_0)$ se $h \neq 0)$

Definiamo una nuova funzione $G(y)$

$$G(y) = \begin{cases} \dfrac{g(y)-g(y_0)}{y-y_0} & \text{per } y \neq y_0 \\ g'(y_0) & \text{per } y = y_0 \end{cases}$$

1) Osservo $\lim_{y \to y_0} G(y) = \lim_{y \to y_0} \dfrac{g(y)-g(y_0)}{y-y_0} = \underset{\displaystyle G(y_0)}{g'(y_0)}$

Quindi $G(y)$ è continua in y_0.

2) La seconda osservazione è che

$$\frac{g(f(x_0+h))-g(f(x_0))}{h} = G(f(x_0+h))\,\frac{f(x_0+h)-f(x_0)}{h}$$

Ricordare prima la non rigorosa "uguaglianza"

$$\frac{g(f(x_0+h))-g(f(x_0))}{h} = \frac{g(f(x_0+h))-g(f(x_0))}{f(x_0+h)-f(x_0)}\;\frac{f(x_0+h)-f(x_0)}{h}$$

$$G(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} & \text{per } y \neq y_0 \\ g'(y_0) & \text{per } y = y_0 \end{cases}$$

$$\overbrace{\frac{g(f(x_0+h)) - g(f(x_0))}{h}}^{L} = \overbrace{G(f(x_0+h)) \, \frac{f(x_0+h) - f(x_0)}{h}}^{R}$$

1) Se $f(x_0+h) \neq f(x_0) = y_0$ allora

$$R = G(f(x_0+h)) \, \frac{f(x_0+h) - f(x_0)}{h} =$$

$$= \frac{g(f(x_0+h)) - g(f(x_0))}{\cancel{f(x_0+h) - f(x_0)}} \, \frac{\cancel{f(x_0+h) - f(x_0)}}{h}$$

$$= L$$

2) Se $f(x_0+h) = f(x_0) = y_0$

$$R = G(f(x_0+h)) \, \frac{f(x_0+h) - f(x_0)}{h}$$

$$= G(y_0) \, \frac{f(x_0+h) - f(x_0)}{h} = g'(y_0) \, 0 = 0$$

$$L = \frac{g(f(x_0+h)) - g(f(x_0))}{h} = 0$$

Conclusione :

$$\frac{g(f(x+h)) - g(f(x_0))}{h} = G(f(x_0+h)) \; \frac{f(x_0+h) - f(x_0)}{h}$$

$$\lim_{h \to 0} \frac{g(f(x+h)) - g(f(x_0))}{h} =$$

$$= \lim_{h \to 0} G(f(x_0+h)) \quad \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$y = f(x_0+h)$$

$$= \lim_{y \to y_0} G(y) \quad \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= G(y_0) \; f'(x_0) = g'(y_0) \; f'(x_0) .$$

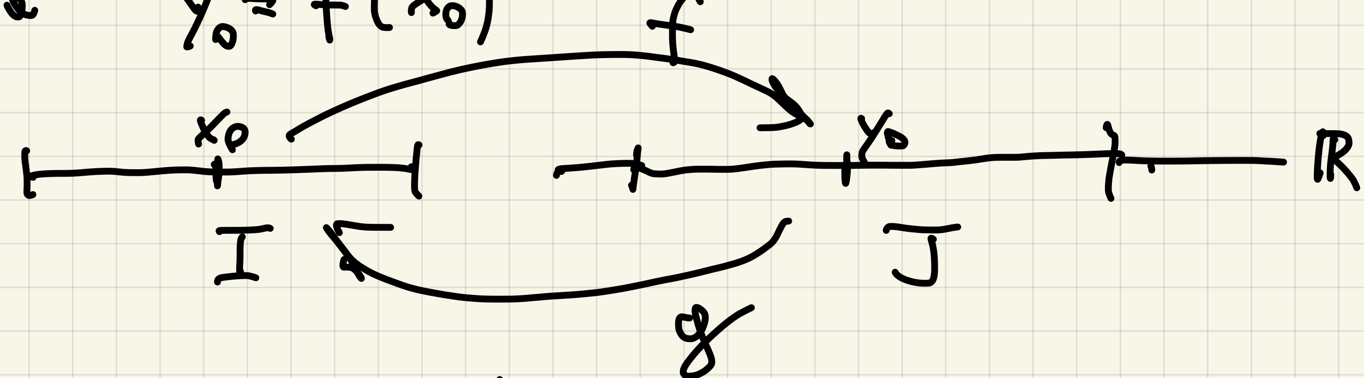
$$\frac{\Delta g(f)}{\Delta x} = \frac{\Delta g(f)}{\Delta f} \; \frac{\Delta f}{\Delta x}$$

Teor (derivata funzione inversa)

Sia $f : I \to \mathbb{R}$, $f \in C^0(I)$ e strettamente monotona. Sia $J = f(I)$

Sia $x_0 \in I$ t.c. $f'(x_0)$ esiste e si ha $f'(x_0) \neq 0$. Consideriamo la funzione inversa $g : J \to I$ e

sia $y_0 = f(x_0)$



Allora esiste $g'(y_0)$ ed e' data da

$$g'(y_0) = \frac{1}{f'(x_0)}$$

Dim $\quad g'(y_0) = \frac{1}{f'(x_0)}$

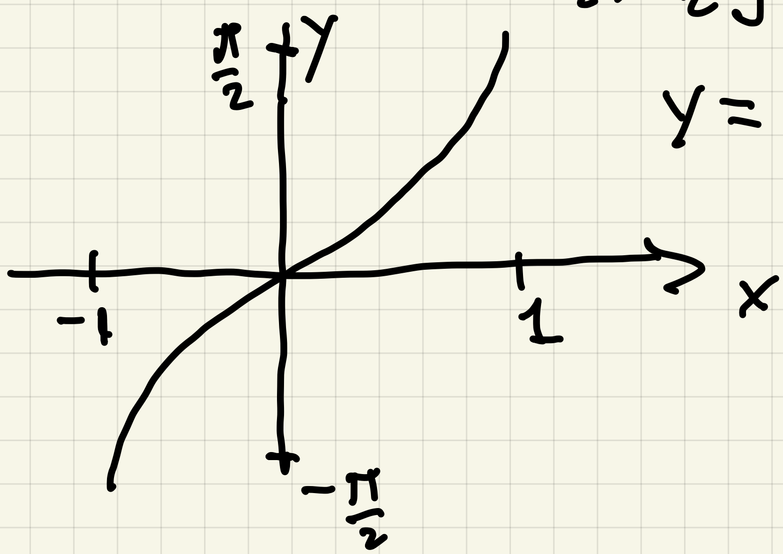
$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} \qquad \begin{array}{l} y = f(x) \\ x = g(y) \end{array}$$

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

$$g'(y_0) = \frac{1}{f'(x_0)}$$

$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$

$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

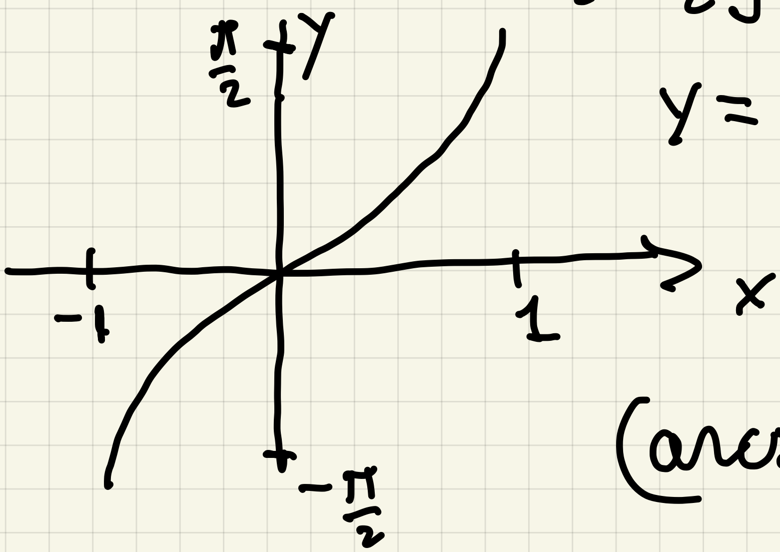


$y = \arcsin x$

$$g'(y_0) = \frac{1}{f'(x_0)}$$

$$\sin: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

$$\arcsin: [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$y = \arcsin x$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$g'(y) = \frac{1}{f'(x)}$$

$x = \sin y$

$$(\arcsin x)' = \frac{1}{(\sin(y))'} = \frac{1}{\cos(y)} =$$

$$= \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

$$\left(\tan(x)\right)' = \left(\frac{\sin(x)}{\cos(x)}\right)' =$$

$$= \frac{\left(\sin(x)\right)' \cos(x) - \sin(x)\left(\cos(x)\right)'}{\cos^2(x)}$$

$$= \frac{\cos^2(x) - \sin(x)\left(-\sin(x)\right)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

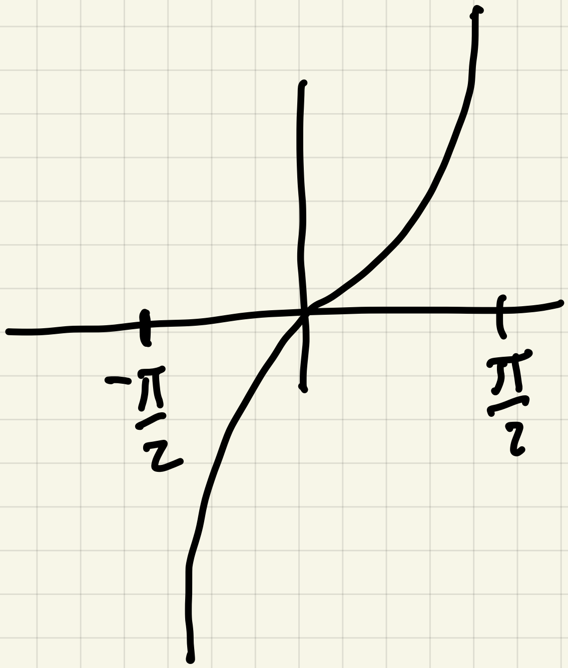
$$= 1 + \tan^2(x) = \left(\tan x\right)'$$

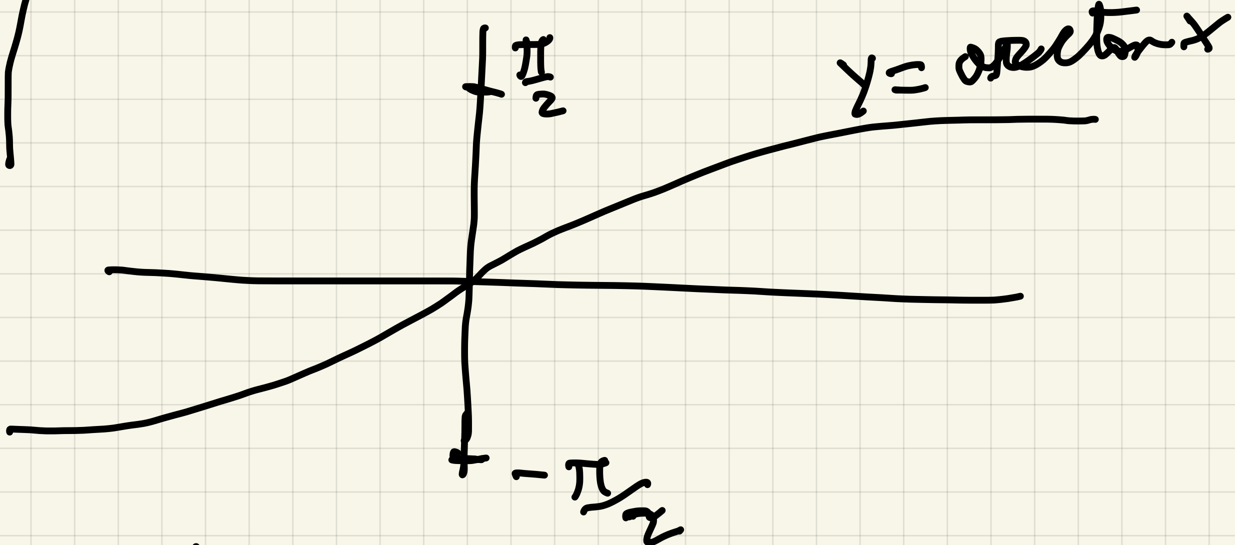
$$\sec(x) := \frac{1}{\cos(x)}$$

$$(\tan(x))' = 1 + \tan^2(x)$$

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

$$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$





$$(\arctan x)' = \frac{1}{1 + x^2}$$

$$g'(x) = \frac{1}{f'(y)}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

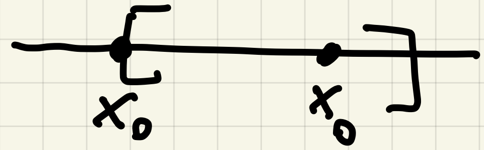
$x = \tan y$

$$(\arctan x)' = \frac{1}{(\tan y)'}$$

$$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Def 1) (Derivata destra). Sia $f : I \to \mathbb{R}$

$x_0 \in I$. Se



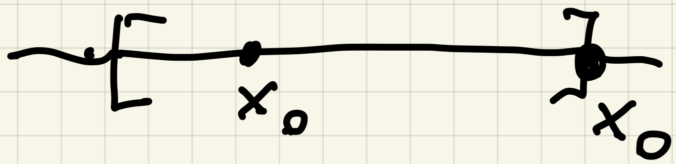
$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$ esiste ed è

finito viene denotato con $f'_d(x_0)$

ed è chiamato derivata destra di f

nel punto x_0

2) (Der. sinistra) se



$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$ esiste ed è

finito viene denotato con $f'_s(x_0)$

ed è chiamato derivata sinistra . . .

<u>Osservazione</u> Se x_0 è un punto interno di I, $f: I \to \mathbb{R}$, sono equivalenti

1) Esiste $f'(x_0)$

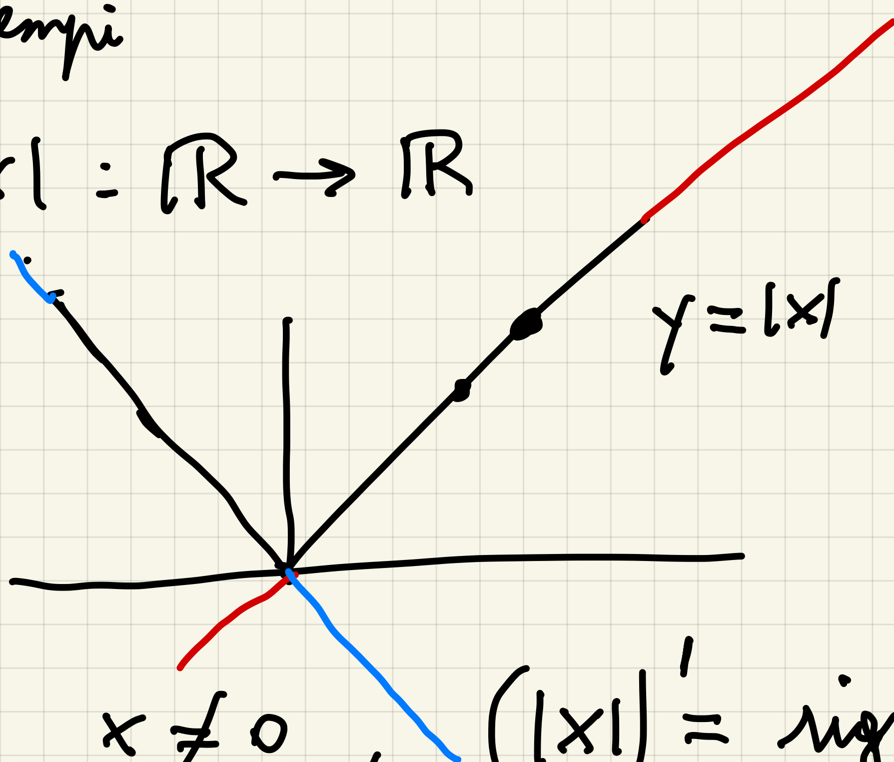
2) Esistono $f'_d(x_0)$ e $f'_s(x_0)$ e sono uguali.

Quando 1) 2) sono vere si ha

$$f'(x_0) = f'_d(x_0) = f'_s(x_0) .$$

Esempi

1) $|x| : \mathbb{R} \to \mathbb{R}$



Per $x \neq 0$, $(|x|)' = \text{sign}(x) = \dfrac{x}{|x|}$

Per $x = 0$

$$(|x|)'_d(0) = \lim_{x \to 0^+} \frac{|x| - |0|}{x} = \lim_{x \to 0^+} \frac{|x|}{x}$$

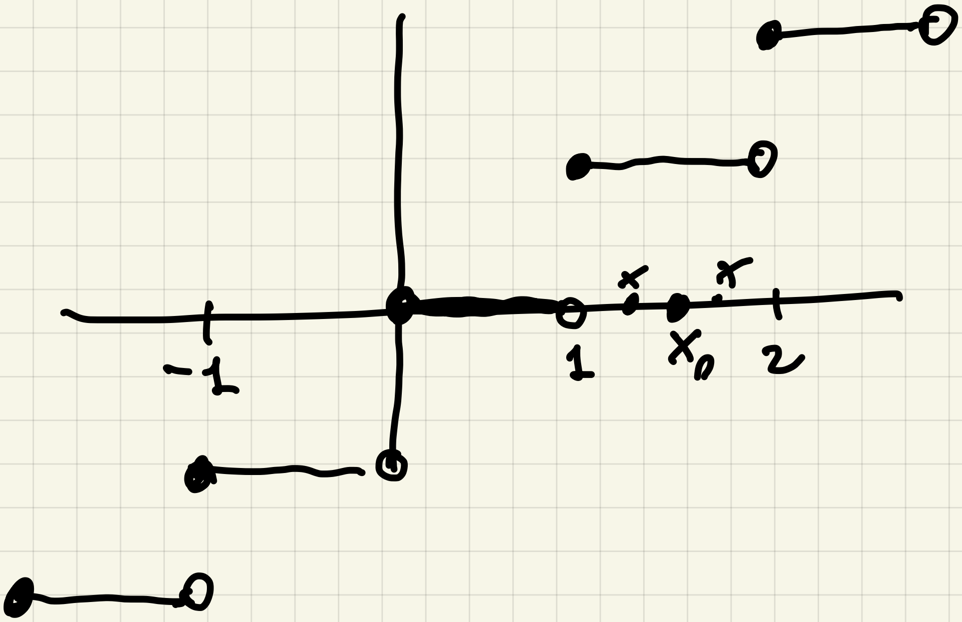
$$= \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$(|x|)'_s(0) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x}$$

$$= \lim_{x \to 0^-} -1 = -1$$

2) $[x] : \mathbb{R} \to \mathbb{Z}$

$[x] \le x < [x] + 1$



Se $n < x_0 < n+1$ dove $n \in \mathbb{Z}$

$$([x])'(x_0) = \lim_{x \to x_0} \frac{[x]-[x_0]}{x-x_0} = \lim_{x \to x_0} 0$$
$$= 0$$

Sia $n \in \mathbb{Z}$

$$([x])'_d(n) = \lim_{x \to n^+} \frac{[x]-n}{x-n} =$$

$$= \lim_{x \to n^+} \frac{[x]-n}{x-n}\Big|_{(n,n+1)} = \lim_{x \to n^+} \frac{\overbrace{n-n}^{0}}{x-n}$$

$$= \lim_{x \to n^-} 0 = 0$$

Funzioni Iperboliche

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh(x) = \frac{sh(x)}{ch(x)}$$

$$z = r(\cos\vartheta + i\sin\vartheta) =$$

$$= r\, e^{i\vartheta}$$

$$\boxed{ch^2(x) - sh^2(x) = 1 \quad \forall x \in \mathbb{R}}$$

$$\mathrm{ch}^2(x) - \mathrm{sh}^2(x) = 1$$

$$\left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2 =$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}}{4}$$

$$= \frac{2}{4} - \frac{-2}{4} = 1$$