Image Processing for Physicists

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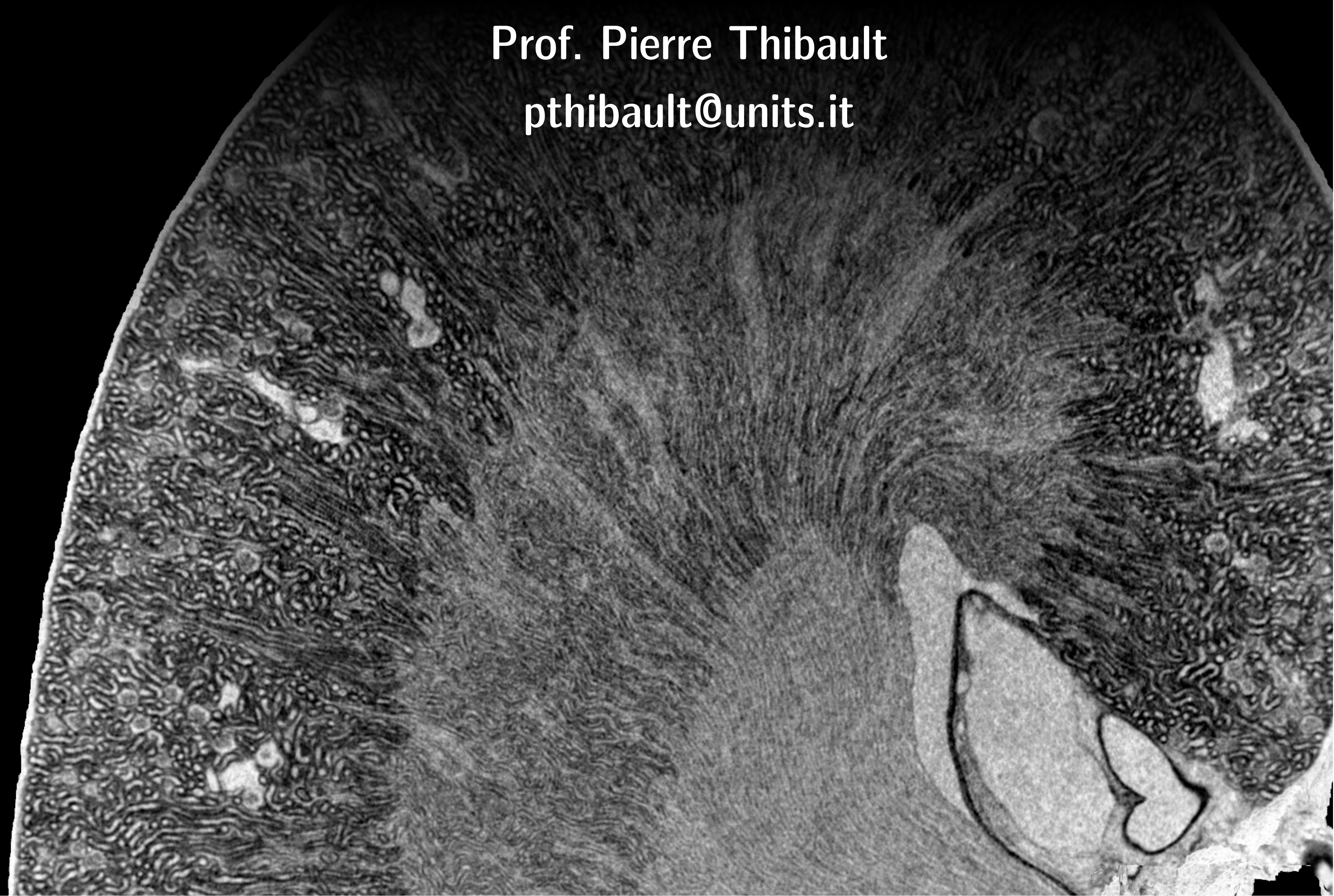


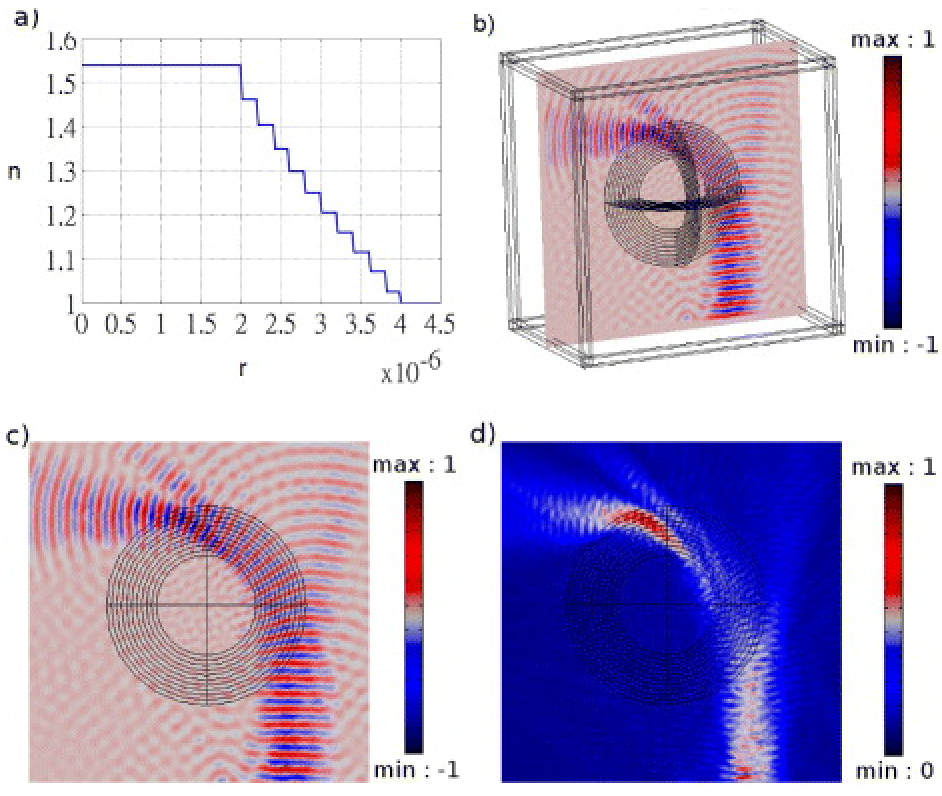
Overview

- Propagation modelization
- Wave propagation:
  - Near-field regime
  - Far-field regime

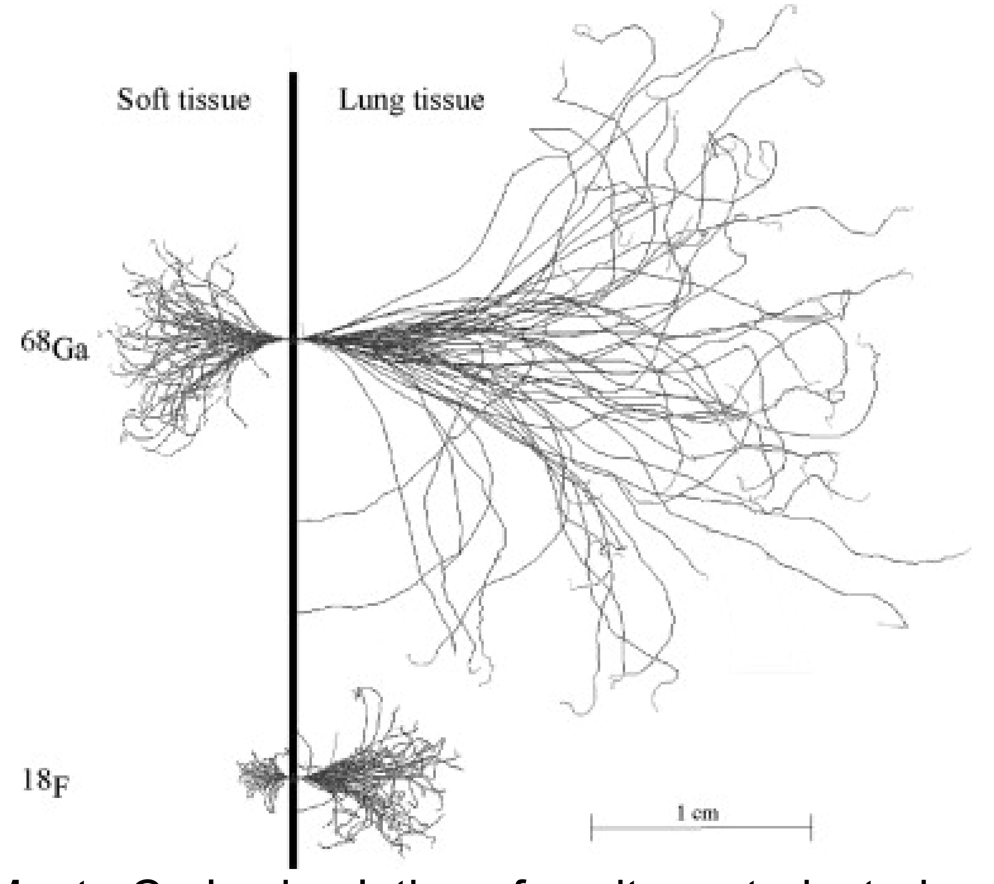
Propagation modeling

- Motivations:

  1. Validation



Finite element simulation of an electro-magnetic field in a dielectric



Monte Carlo simulation of positrons trajectories resulting from ^{68}Ga and ^{18}F decay.

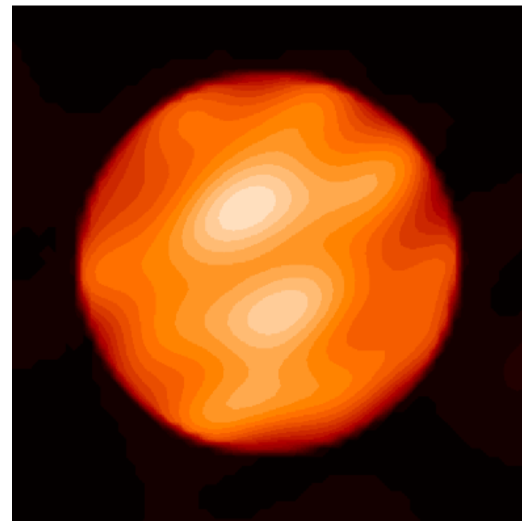
sources: T.M. Chang *et al.* New J. Phys. (2012)
A. Sanchez-Crespo, Appl. Rad. Isotopes (2012)

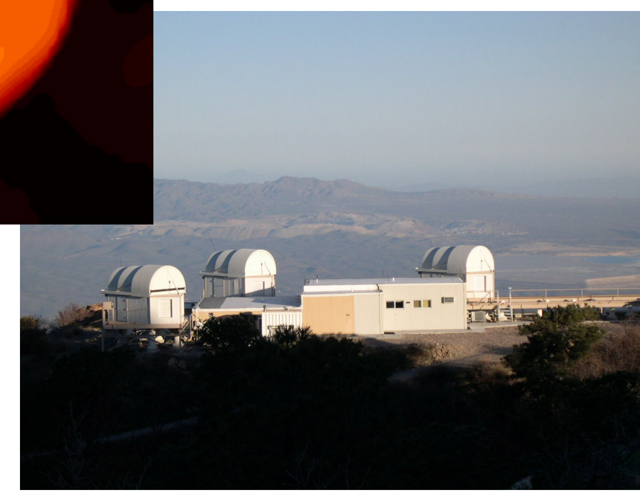
Propagation modeling

- Motivations:

  2. Inversion

Image reconstruction from sound wave propagation (ultrasonography)





The surface of Betelgeuse reconstructed from interferometric data (IOTA)

sources: wikipedia
Haubois *et al. Astronom. & Astrophys.* (2009)

Propagation modeling

- Particles
  - Model particle tracks (rays) through different media
  - Model may include: refraction, force fields, particle decay and interactions
  - Not included: diffraction
- Wave
  - Model the interaction of a field with a medium
  - Can be very complicated $\rightarrow$ approximations are needed

Propagation modeling

Starting point: Helmholtz equation

- for EM field: neglect polarization (scalar wave approximation)
- for electron wave, assume high energy electrons

Maxwell eq. → $\nabla^2 \psi + \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \psi = 0$

(n: index of refraction (spatially dependent))

ψ: complex-valued field (scalar: no polarization)

n: index of refraction

c: speed of light

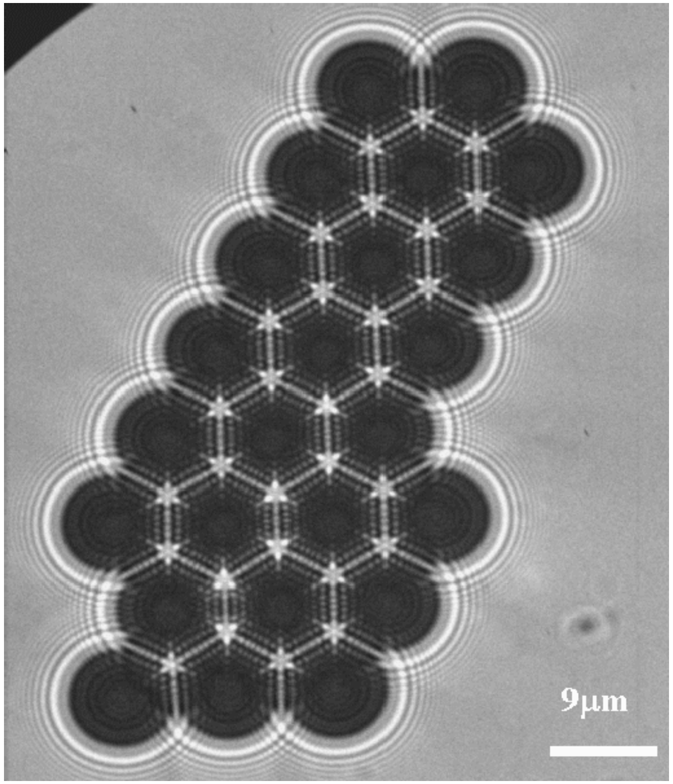
constant n → plane wave solution $\psi = \psi_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$k^2 = \frac{n^2 \omega^2}{c^2}$ "dispersion relation"

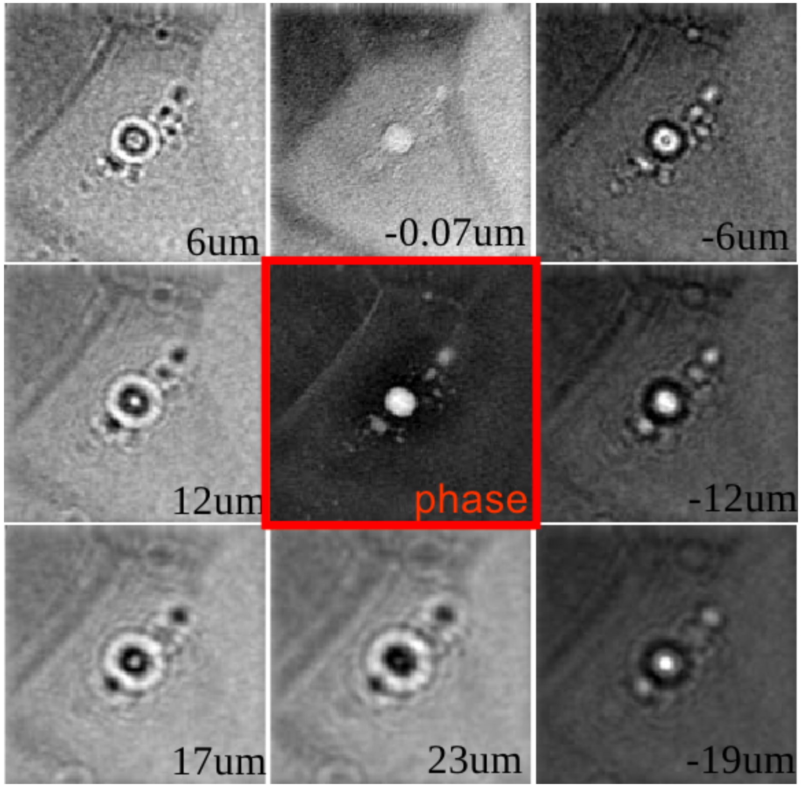
Propagation modeling

- Useful to:
  - better understand optical systems
  - understand diffraction, holography, phase contrast, interferometry, ...

X-ray hologram



TEM through-focus series



sources: Mayo *et al.* Opt. Express (2003)
http://www.christophtkoch.com/Vorlesung/

The physics of propagation

In free space $(n=1)$ General solution

$$\psi(\vec{r},t) = \sum_{\omega} \sum_{\vec{k}} A_{\omega \vec{k}} \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\hookrightarrow |\vec{k}|^2 = \frac{\omega^2}{c^2}$$

$$|\vec{k}| = \frac{2\pi}{\lambda}$$

$$\text{“}u = \frac{1}{p}\text{”}$$

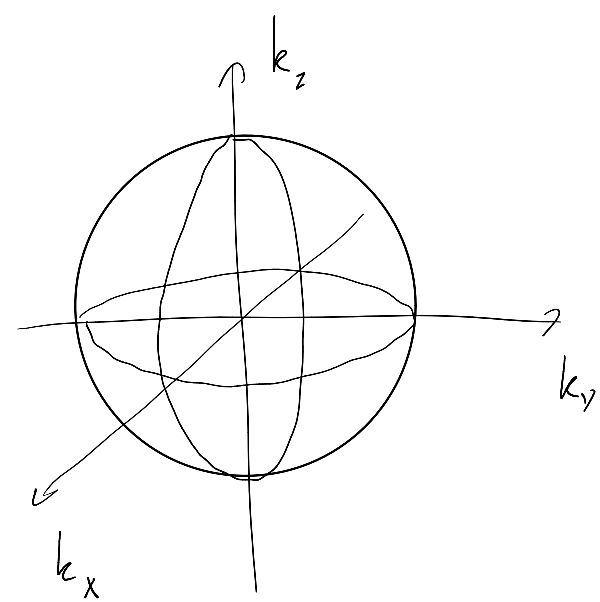
Commonly: fix ω and solve monochromatic case

$$\longrightarrow \psi(\vec{r}) = \sum_{\vec{k}} A_{\vec{k}} \, e^{i\vec{k}\cdot\vec{r}} \quad \longleftarrow \text{is this a Fourier transform?}$$

$$\hookrightarrow |\vec{k}| = \frac{\omega}{c}$$

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

"Ewald" sphere

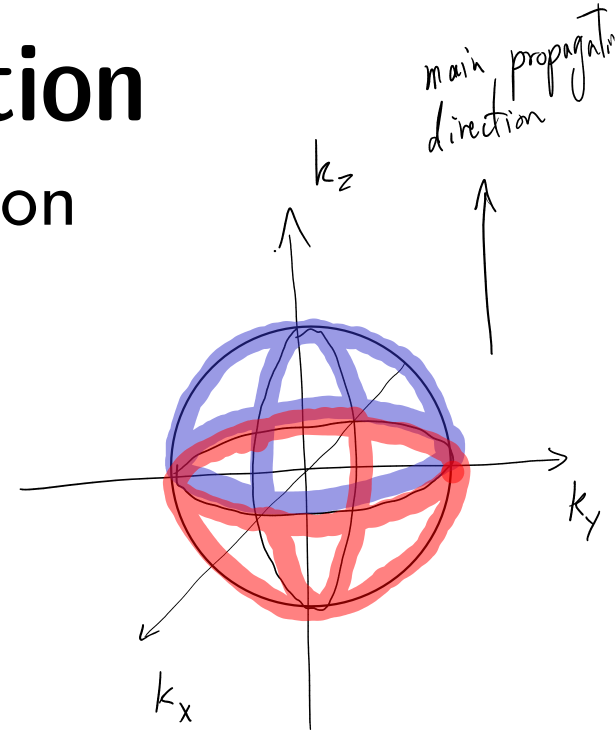


The physics of propagation

"Angular spectrum" representation

$$\frac{\omega^2}{c^2} = k^2 = k_z^2 + k_x^2 + k_y^2$$

$$k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2}$$



$$\psi(\vec{r}) = \sum_{k_x k_y} A^{+}_{k_x k_y} e^{i(k_x x + k_y y + \sqrt{k^2 - k_x^2 - k_y^2}\, z)}$$

$$+ A^{-}_{k_x k_y} e^{i(k_x x + k_y y - \sqrt{k^2 - k_x^2 - k_y^2}\, z)}$$

← forward propagation

$$\psi(\vec{r}_\perp ; z) = \sum_{\vec{k}_\perp} A_{\vec{k}_\perp} \exp(i \vec{k}_\perp \cdot \vec{r}_\perp) \exp(i \sqrt{k^2 - k_\perp^2}\, z)$$

2D Fourier transform

Forward propagation

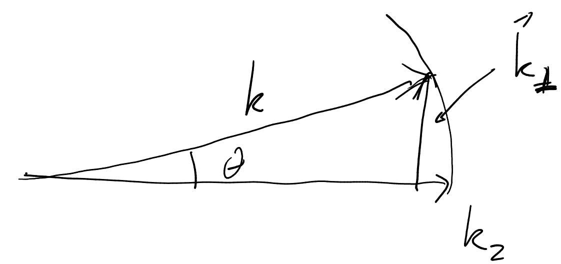
Angular spectrum

$k_x = $ "$k \sin\theta$"

$z=0$: $\psi(\vec{r}_\perp; z=0) = \sum_{\vec{k}_\perp} A_{\vec{k}_\perp} \exp(i\vec{k}_\perp \cdot \vec{r}_\perp)$

$\Rightarrow A_{\vec{k}_\perp} = \mathcal{F}\{\psi(\vec{r}_\perp; z=0)\}$

Often: $|\vec{k}_\perp| \ll k$ (small angle diffraction)



$$\sqrt{k^2 - k_\perp^2} = k\sqrt{1 - \frac{k_\perp^2}{k^2}} \simeq k\left(1 - \frac{k_\perp^2}{2k^2}\right)$$

"paraxial approximation"

$$= k - \frac{k_\perp^2}{2k}$$

parabolic approximation of a sphere

$$\exp\left(i\sqrt{k^2 - k_\perp^2}\, z\right) \simeq \exp(ikz)\exp\left(\frac{-izk_\perp^2}{2k}\right)$$

"Fresnel propagator"

Forward propagation

$$\psi(r_\perp; z) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{\psi(r_\perp; z=0)\} \exp\left(-2\pi i \lambda z \vec{u}^2\right) \right\}$$

The trick for numerical implementations:

F.T. $e^{i2\pi x u}$ D.F.T. $e^{i2\pi \frac{n \cdot k}{N}}$ ← integers

$x = n \cdot \Delta x$

$u = k \cdot \Delta u$ → $\boxed{\Delta x \Delta u = \frac{1}{N}}$

↑ sampling in real and Fourier space.

e.g. Δx, N are known

↳ $\Delta u = \frac{1}{\Delta x N}$

$$\exp(-\pi i \lambda z u^2) \rightarrow \exp\left(-\pi i \lambda z \cdot k^2 \frac{1}{\Delta x^2 N^2}\right)$$

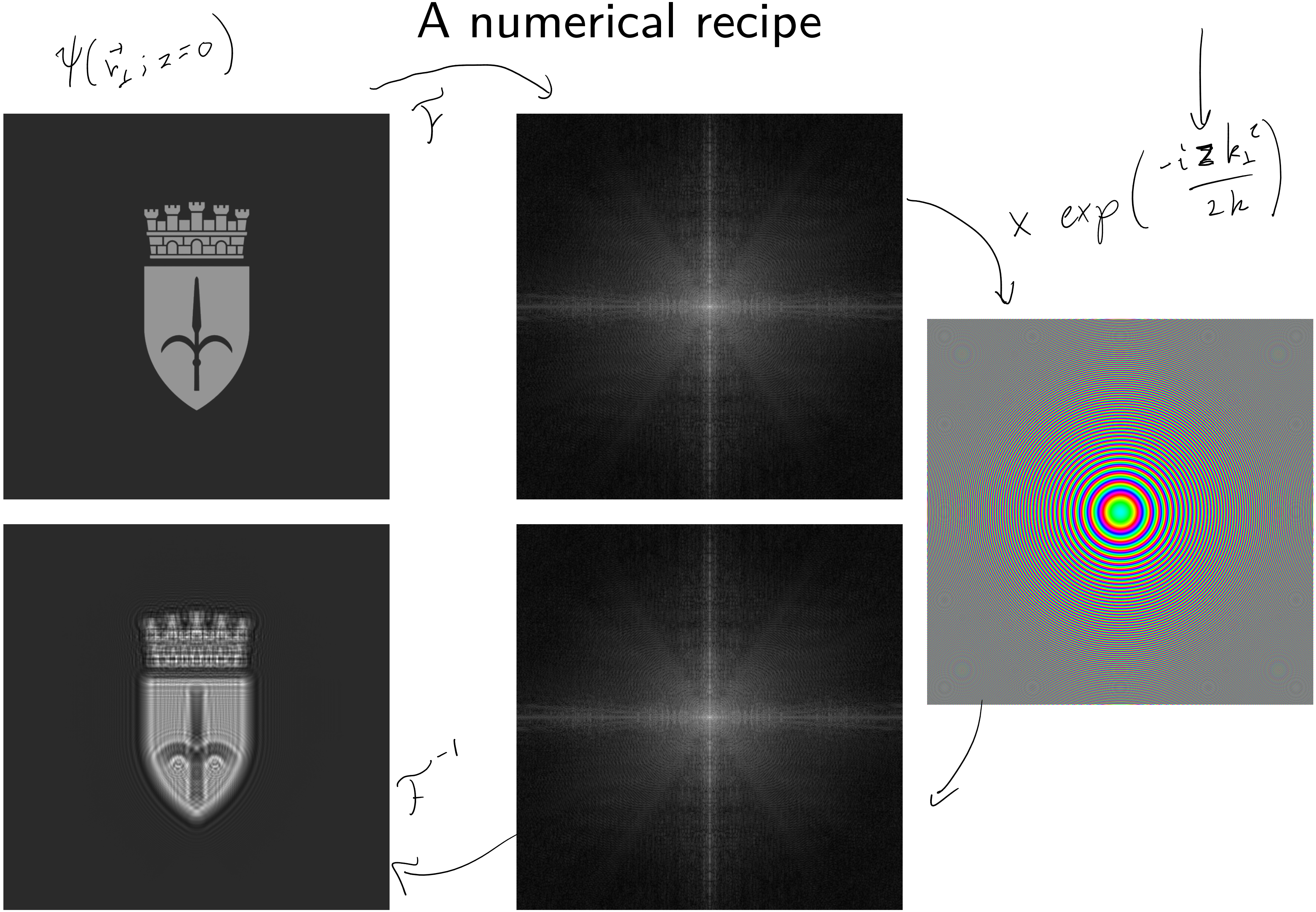
↑ not $\frac{2\pi}{\lambda}$ here!

$$\exp\left(-\pi i \frac{\lambda}{\Delta x} \frac{z}{\Delta x} \left(\frac{k}{N}\right)^2\right)$$

→ $\frac{k}{N}$ output of numpy.fft.fftfreq

Forward propagation

A numerical recipe



Near field, far field

$$\psi(r_\perp; z) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{\psi(r_\perp; z=0)\} \exp\left(-\frac{i z k_\perp^2}{2k}\right)\right\}$$

$k_\perp^2 \to 4\pi^2\vec{u}^2$, $\quad k \to \frac{2\pi}{\lambda}$

$\vec{k}_\perp = \vec{u}\cdot 2\pi$

$e^{i 2\pi x u}$

e^{ikx}

$$\psi(r_\perp; z) = \mathcal{F}^{-1}\left\{ \mathcal{F}\{\psi(r_\perp; z=0)\} \exp\left(-\pi i \lambda z \vec{u}^2\right)\right\}$$

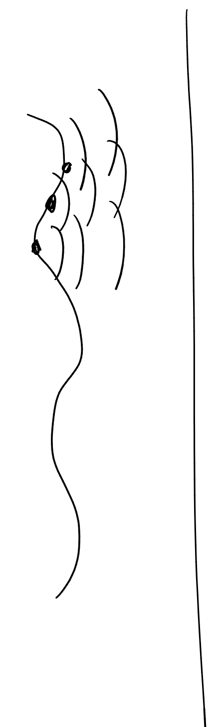
"$\mathcal{F}^{-1}\{F \cdot G\}$" $= f * g$

$$= P_z(\vec{r}_\perp) * \psi(\vec{r}_\perp; z=0)$$

$P_z \leftarrow \mathcal{F}^{-1}\left\{\exp\left(-\pi i \lambda z \vec{u}^2\right)\right\}$

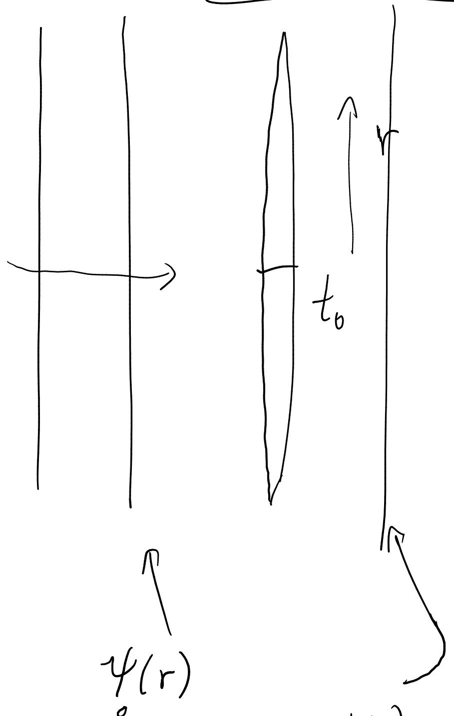
notation $(\vec{r}_\perp \to \vec{r})$

$$\psi(\vec{r}, z) \approx \frac{-i2\pi}{\lambda z}\int d^2r'\, \psi(r', z=0) \exp\left(\frac{ik(r-r')^2}{2z}\right)$$



"Fresnel-Huygens integral"

Back focal plane of a lens



$t(r) = t_0 - \alpha r^2$ (model for thin lens thickness profile)

Passed the lens:

$$\phi(r) = \frac{2\pi}{\lambda}(n-1)\, t(r)$$

$$\frac{1}{2f} = (n-1)\alpha$$

$$= k(n-1)t_0 - k(n-1)\alpha r^2$$

$$\psi(r) = \psi_0 e^{i\phi(r)}$$

$$\psi(r;z) = \frac{-ik}{z}\int d^2r' \underbrace{\psi_0 e^{-ik(n-1)\alpha r'^2}}_{\text{exit wave}} e^{\frac{ik(\vec{r}-\vec{r}')^2}{2z}}$$

(A lens acts as a Fourier transform operator)

$$= -\frac{ik}{z}\int d^2r' \psi_0 \exp\left(ik\left[-\frac{1}{2f}r'^2 + \frac{r^2}{2z} + \frac{r'^2}{2z} - \frac{\vec{r}\cdot\vec{r}'}{z}\right]\right)$$

$$= \frac{-ik}{z} e^{\frac{ikr^2}{2z}} \int d^2r' \psi_0 \exp\Big(\underbrace{\frac{ik}{z}\vec{r}}_{\text{"}\vec{u}\text{"}}\cdot\vec{r}'\Big) \exp\left(\frac{ikr'^2}{2}\left(\frac{1}{z} - \frac{1}{f}\right)\right)$$

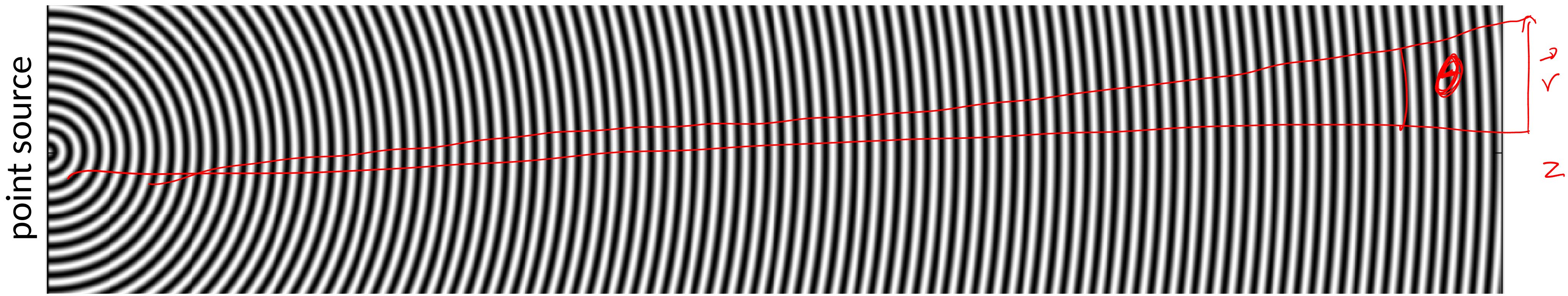
$$\psi(r; z=f) = \frac{ik}{z} e^{\frac{ikr^2}{2z}} \mathcal{F}\{\psi_0(r)\}\left(\vec{u} = \frac{k\vec{r}}{z}\right)$$

Plane waves, point sources

"$z \sin\theta$"

Far-field propagation (Fraunhofer regime):

$$\psi(r; z\to\infty) = \frac{ik}{z} e^{\frac{ikr^2}{2z}} \mathcal{F}\{\psi_0(r)\}\left(\vec{u} = \frac{k\vec{r}}{z}\right)$$



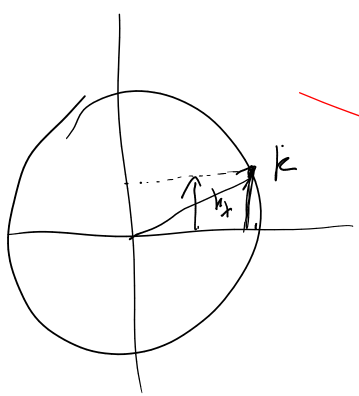
circular waves
evanescent waves
contact region

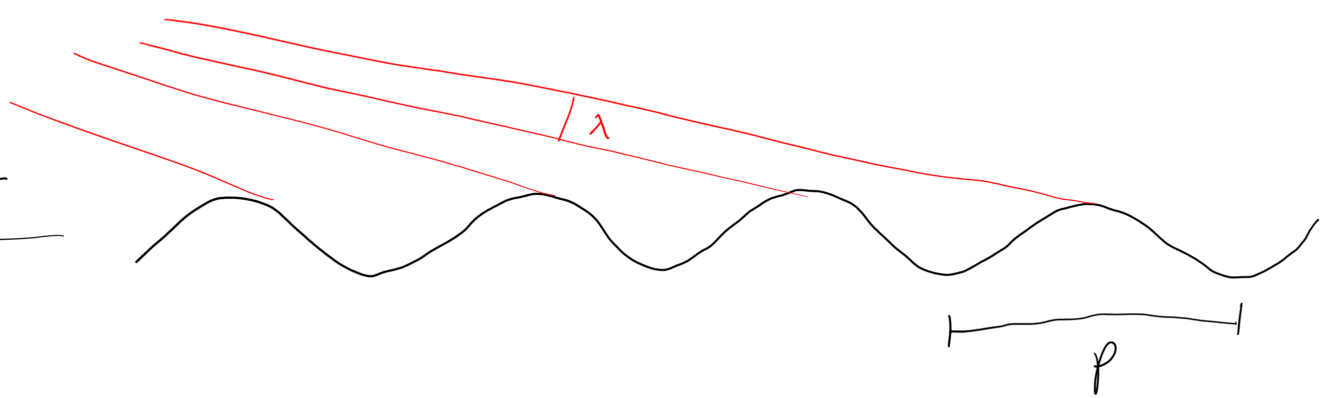
parabolic waves
near field
Fresnel region

carrier wave λ

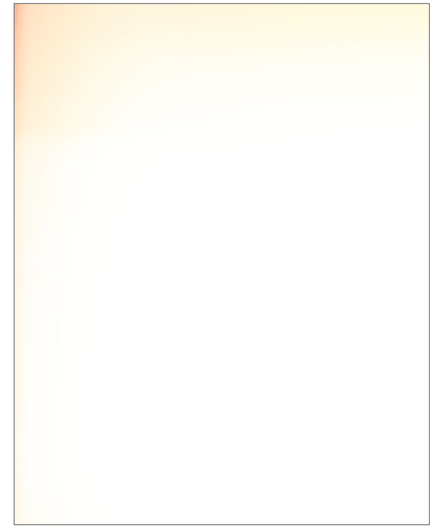
plane waves
far field
Fraunhofer region

oscillation of given spatial frequency in initial image





Why optical elements?

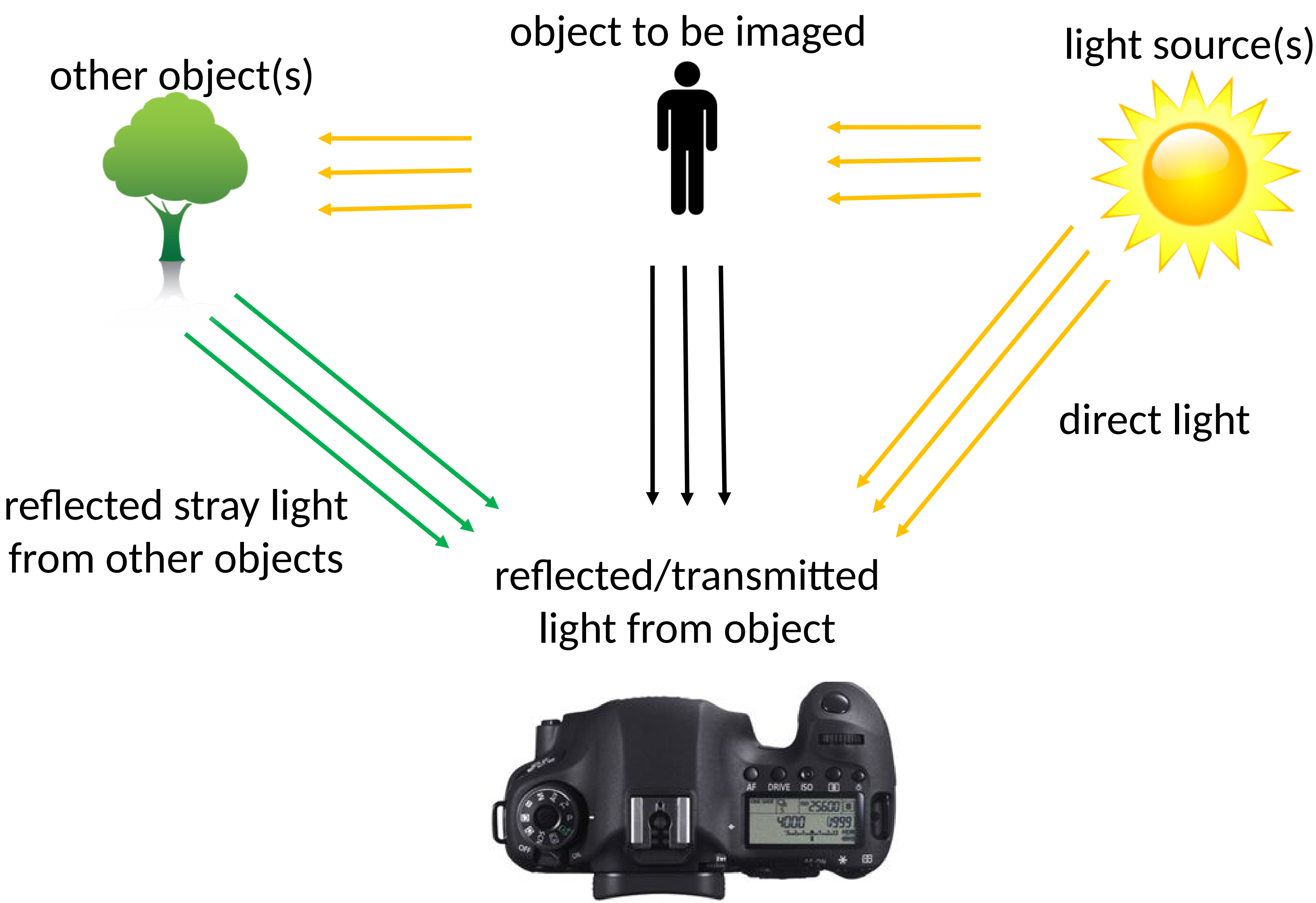


with objective lens

without objective lens

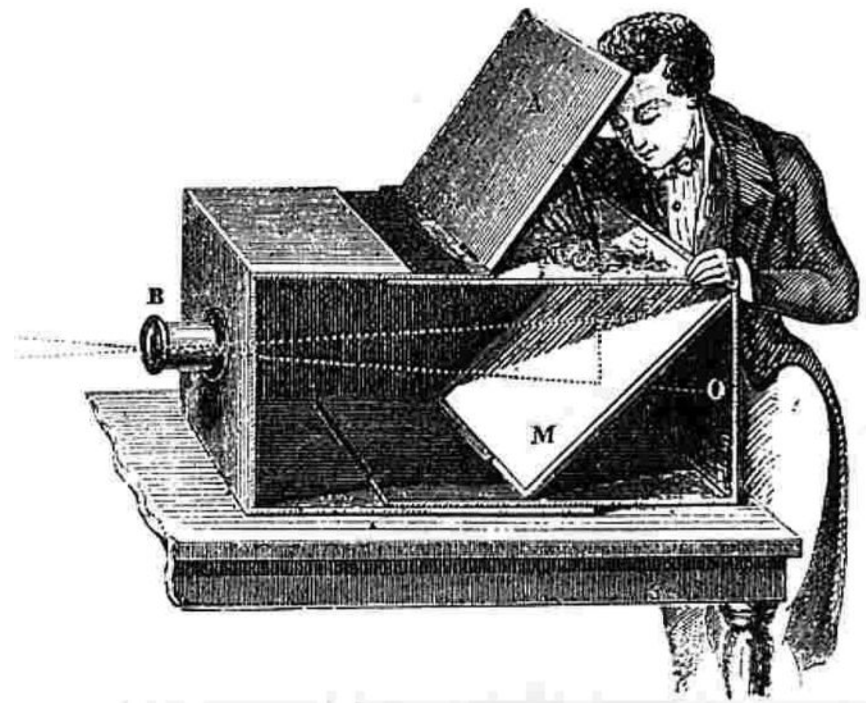
Why optical elements?

- Information from many sources overlaps in detector plane
- Need models to understand image forming systems



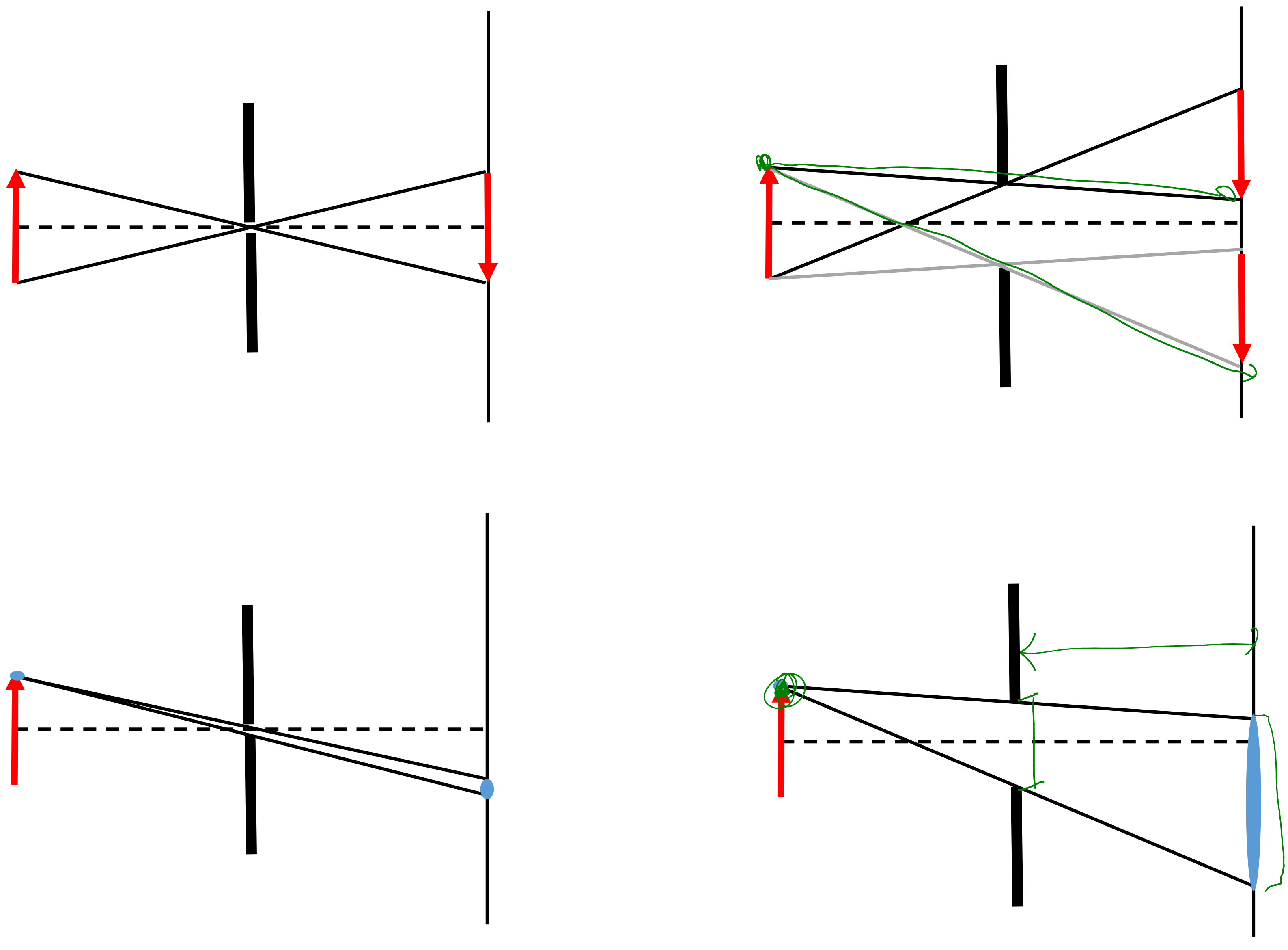
Pinhole camera model

camera obscura



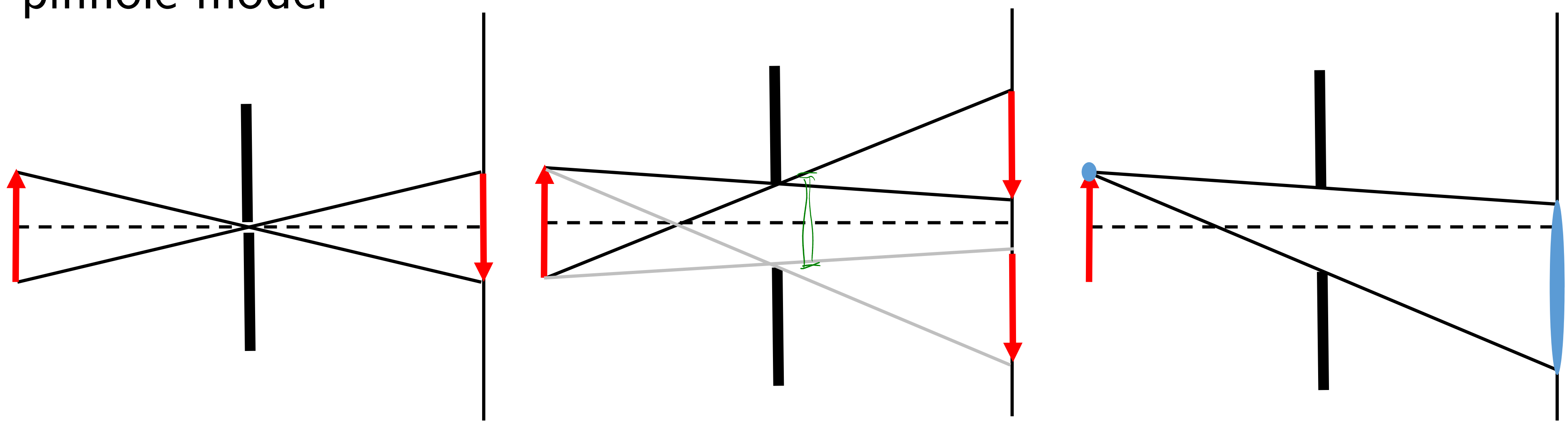
Pinhole camera model

PSF determined by aperture width



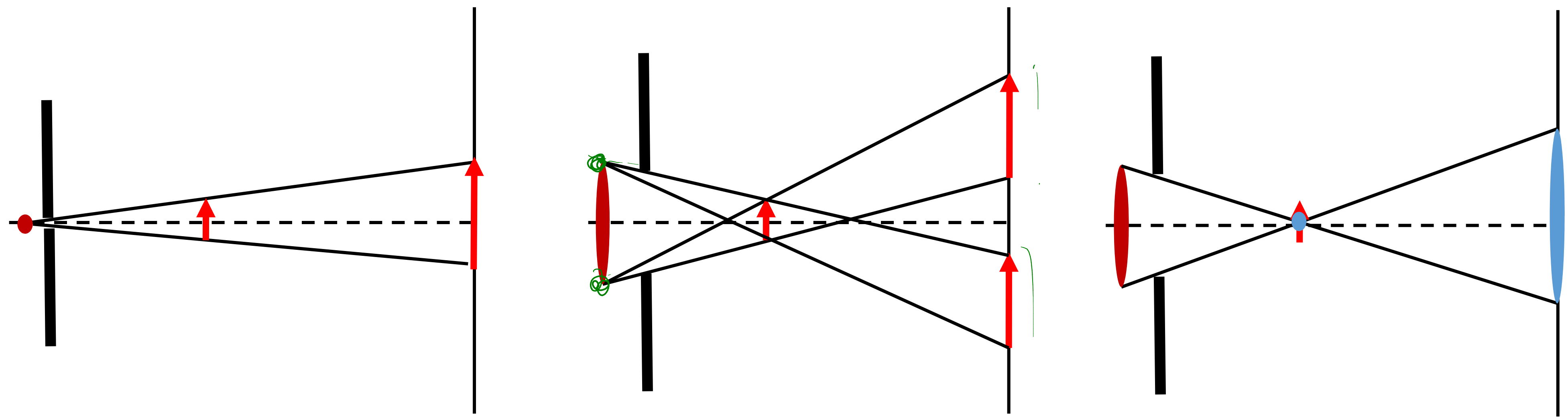
Projection model

pinhole model

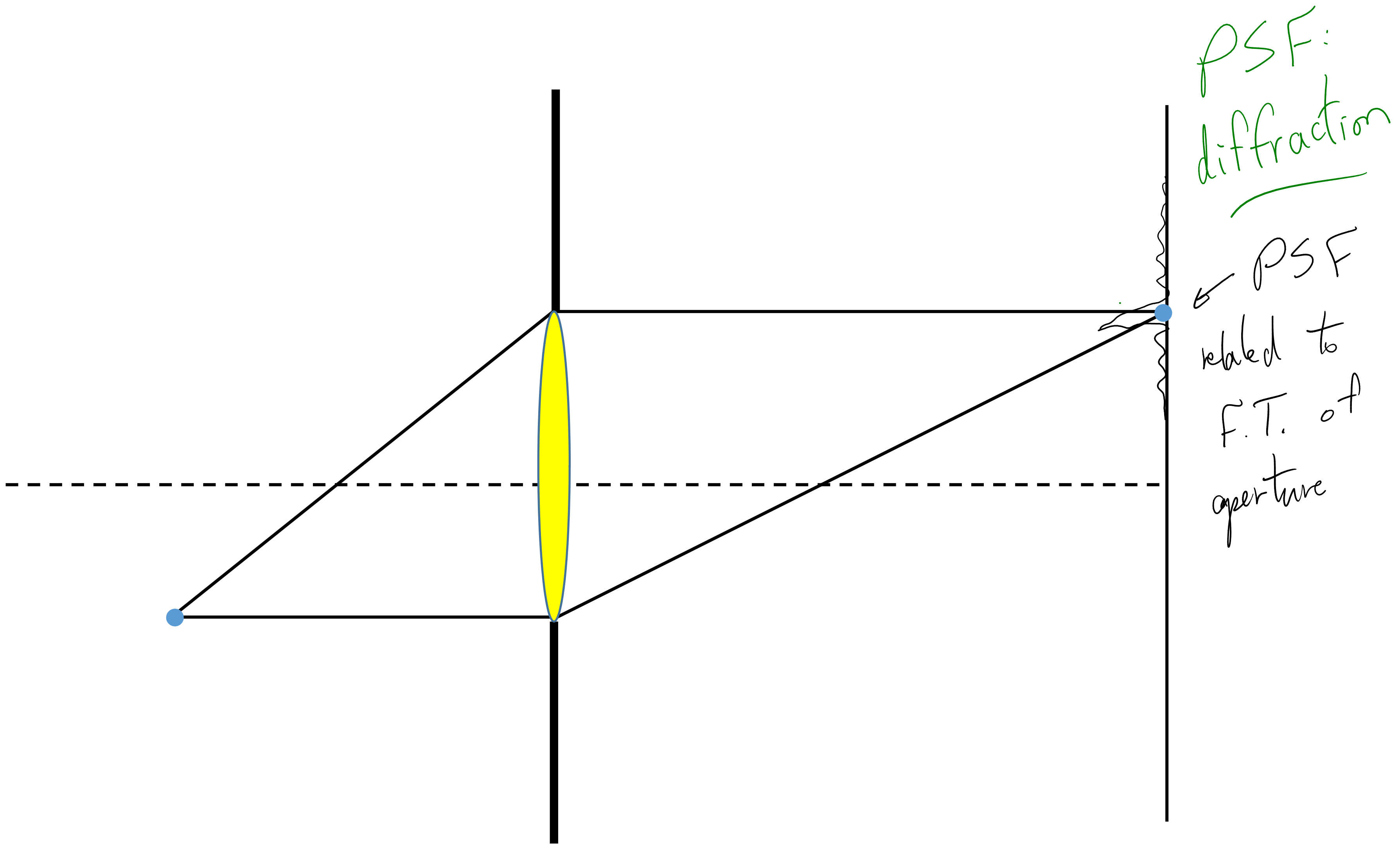


projection model

PSF ⟷ source size

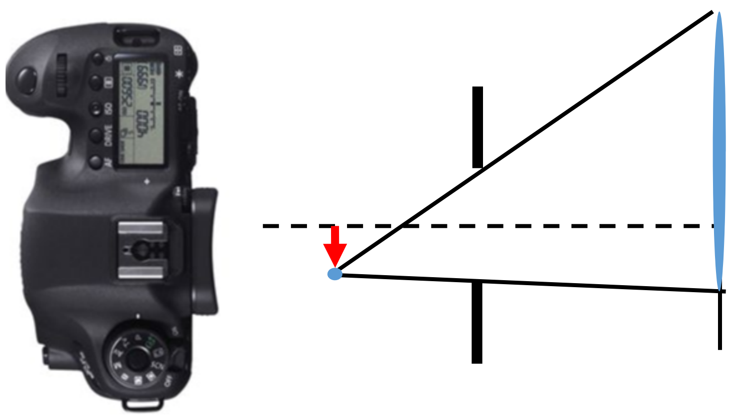


Lens camera model

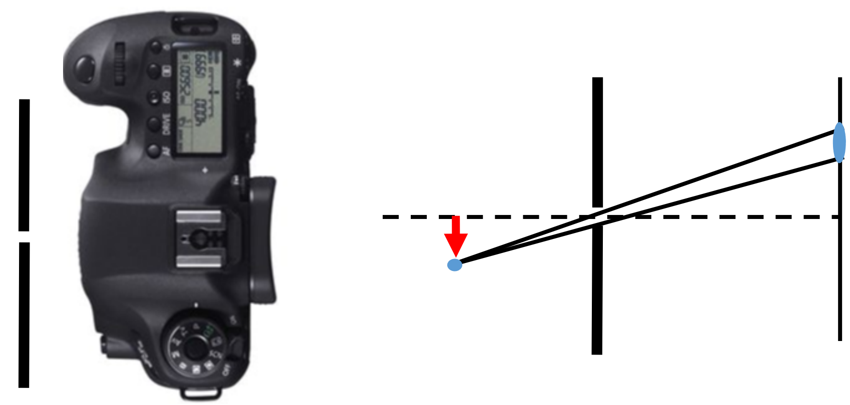


Lens camera model

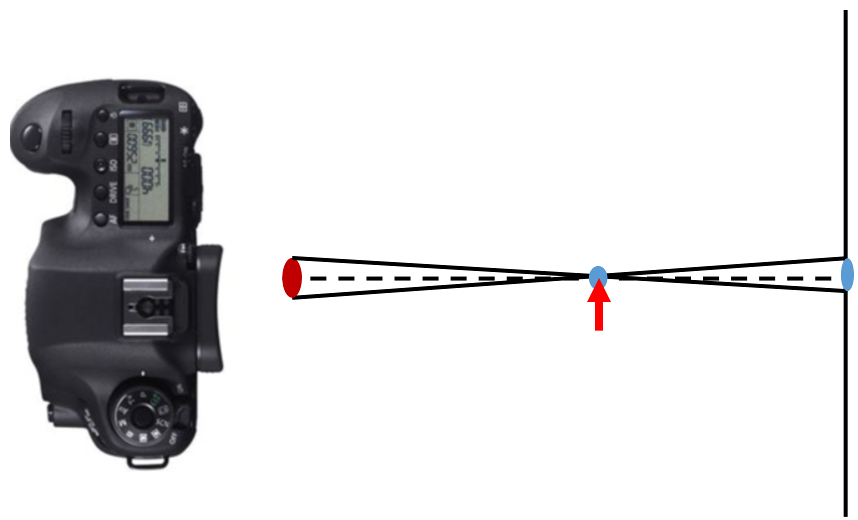
lensless model



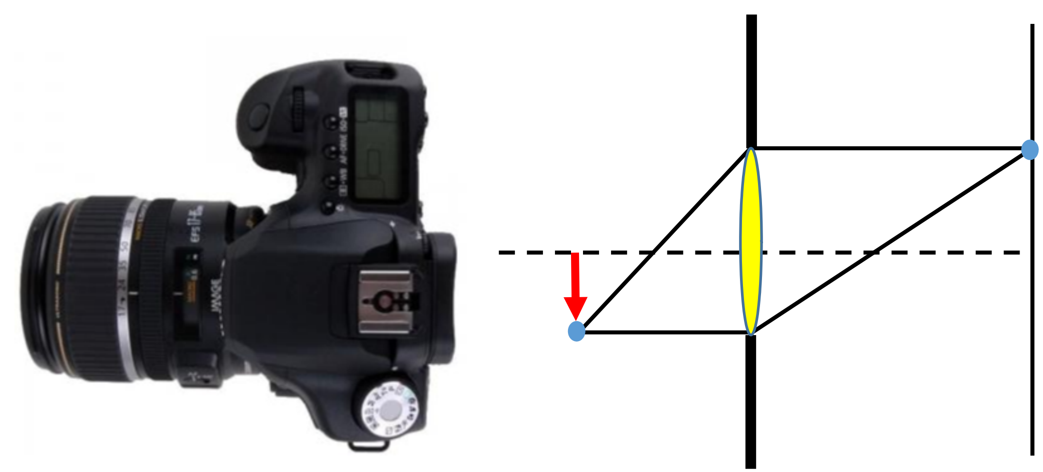
pinhole camera model



projection model

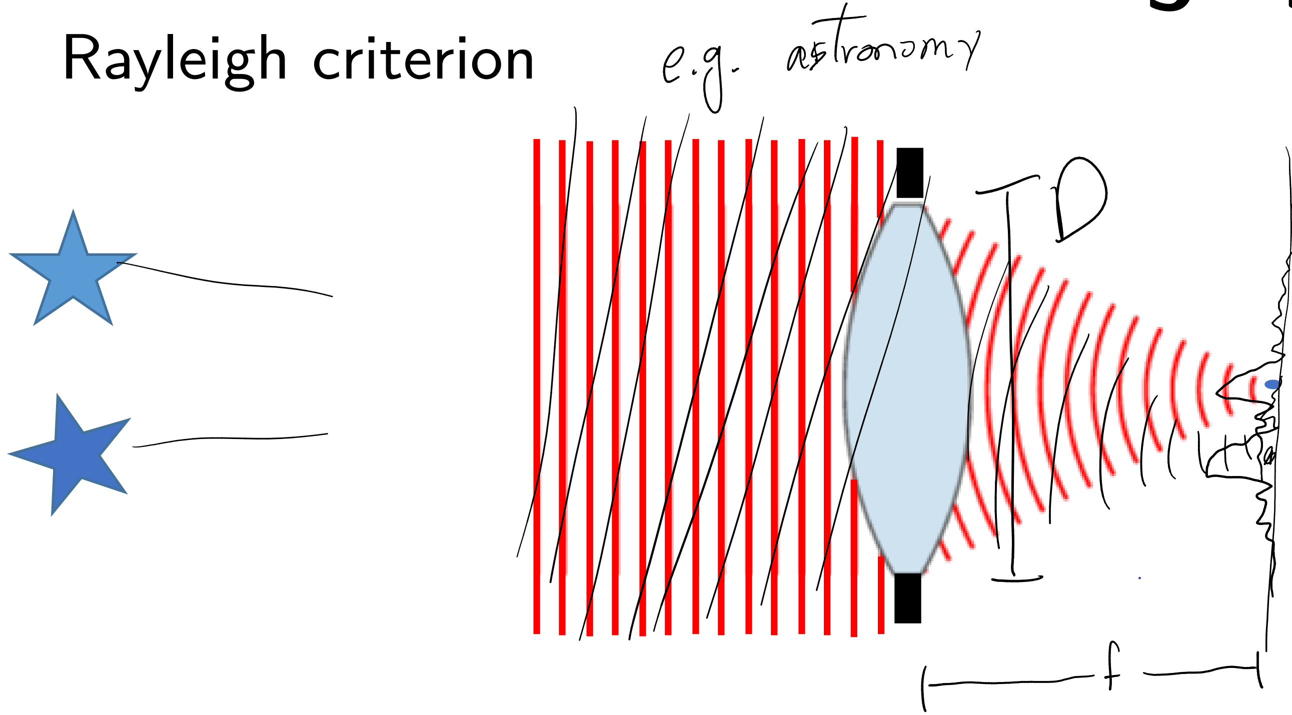


lens camera model



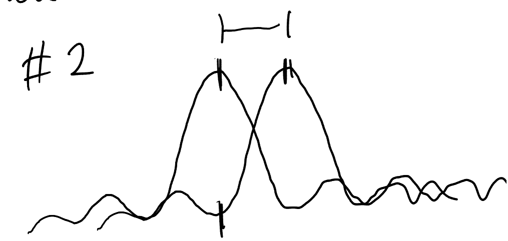
Diffraction-limited imaging systems

- Rayleigh criterion



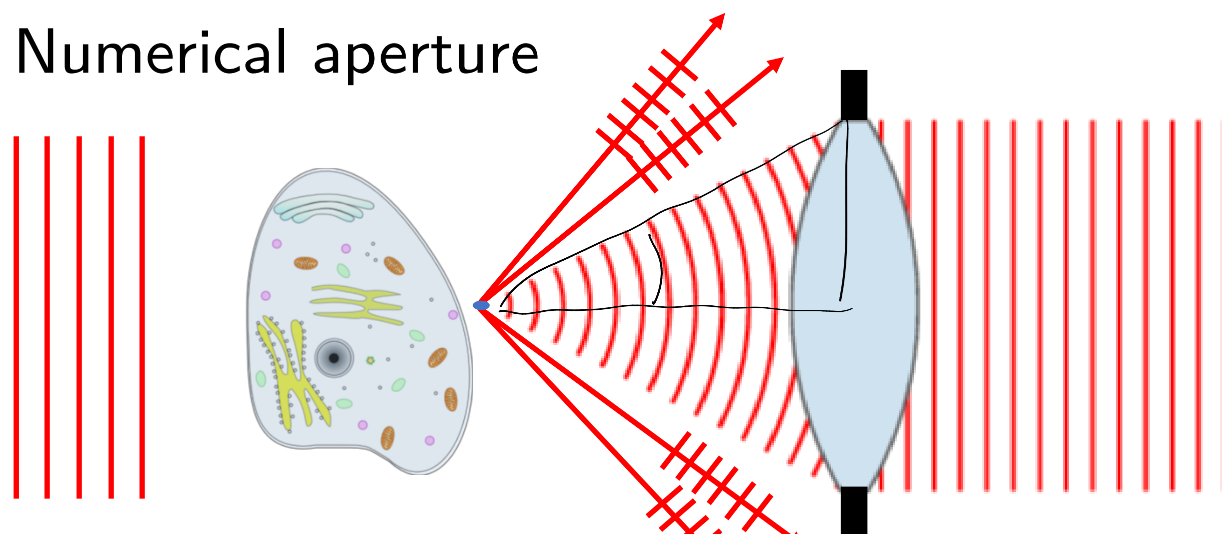
PSF: Airy "disc"

Rayleigh criterion: two points are resolvable if maximum of #1 aligned with 1st minimum of #2



$$d_{min} = \frac{f \cdot \lambda}{D} \cdot 1.22$$

- Numerical aperture



small NA: $\sin\theta \approx \frac{D}{2f}$

NA: numerical aperture

$$NA = n \cdot \sin\theta$$

$$d_{min} = 1.22 \frac{\lambda}{2NA}$$

Scanning systems

Transmission

- **S**canning **T**ransmission **E**lectron **M**icroscopy
- **S**canning **T**ransmission **X**-ray **M**icroscopy
- …

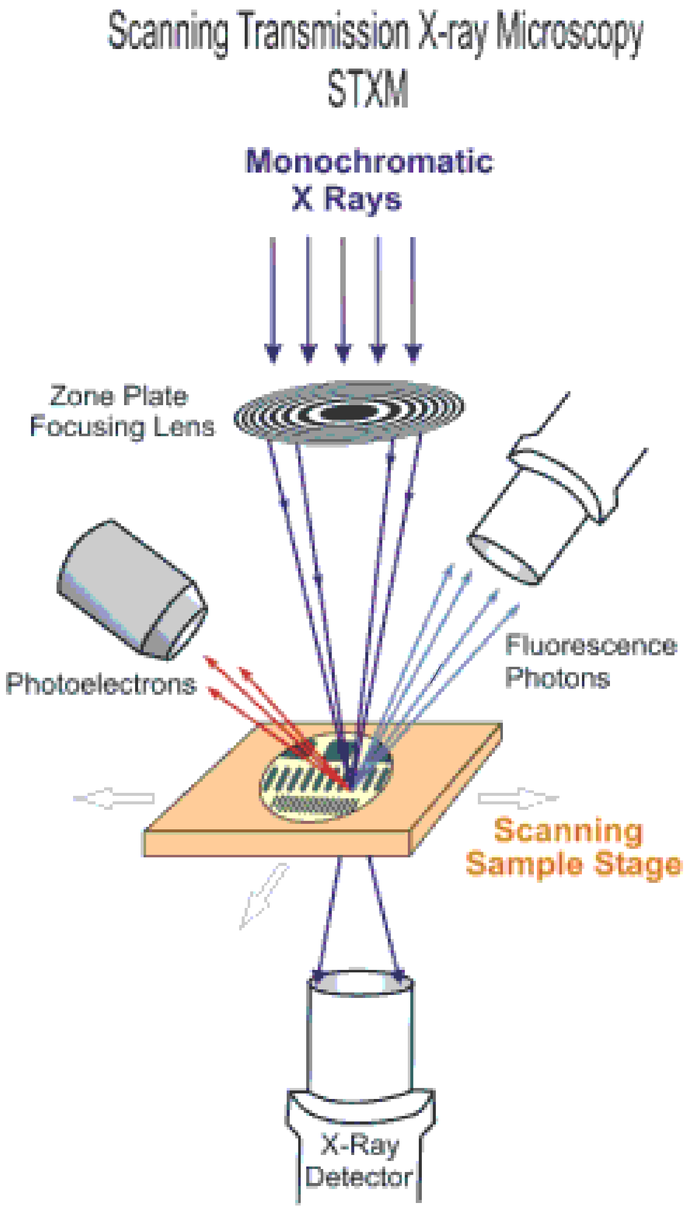
Indirect (reflection, scattering, fluorescence, …)

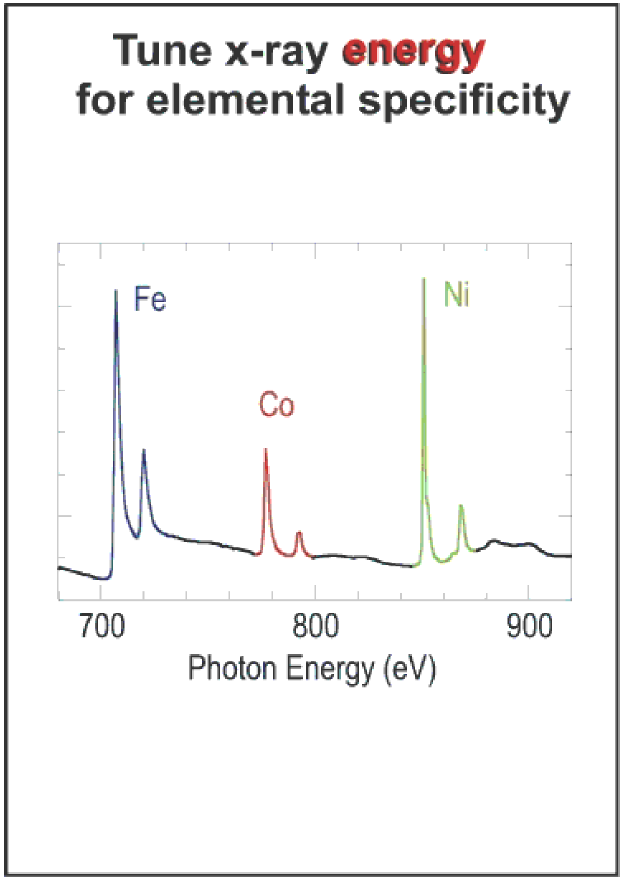
- **L**aser **S**canning **C**onfocal **M**icropsopy
- **S**canning **E**lectron **M**icroscopy
- **X**-ray **F**luorescence **M**icroscopy
- **P**hoto**E**mission **E**lectron **M**icroscopy
- ...

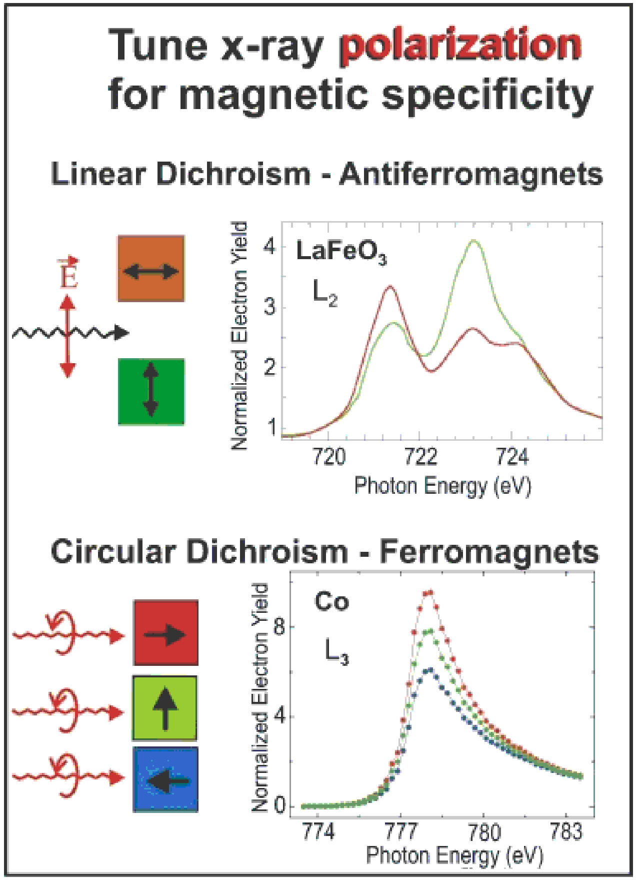
Physical probe

- **A**tomic **F**orce **M**icroscopy
- **S**canning **T**unneling **M**icroscopy
- ...

Scanning transmission X-ray microscopy

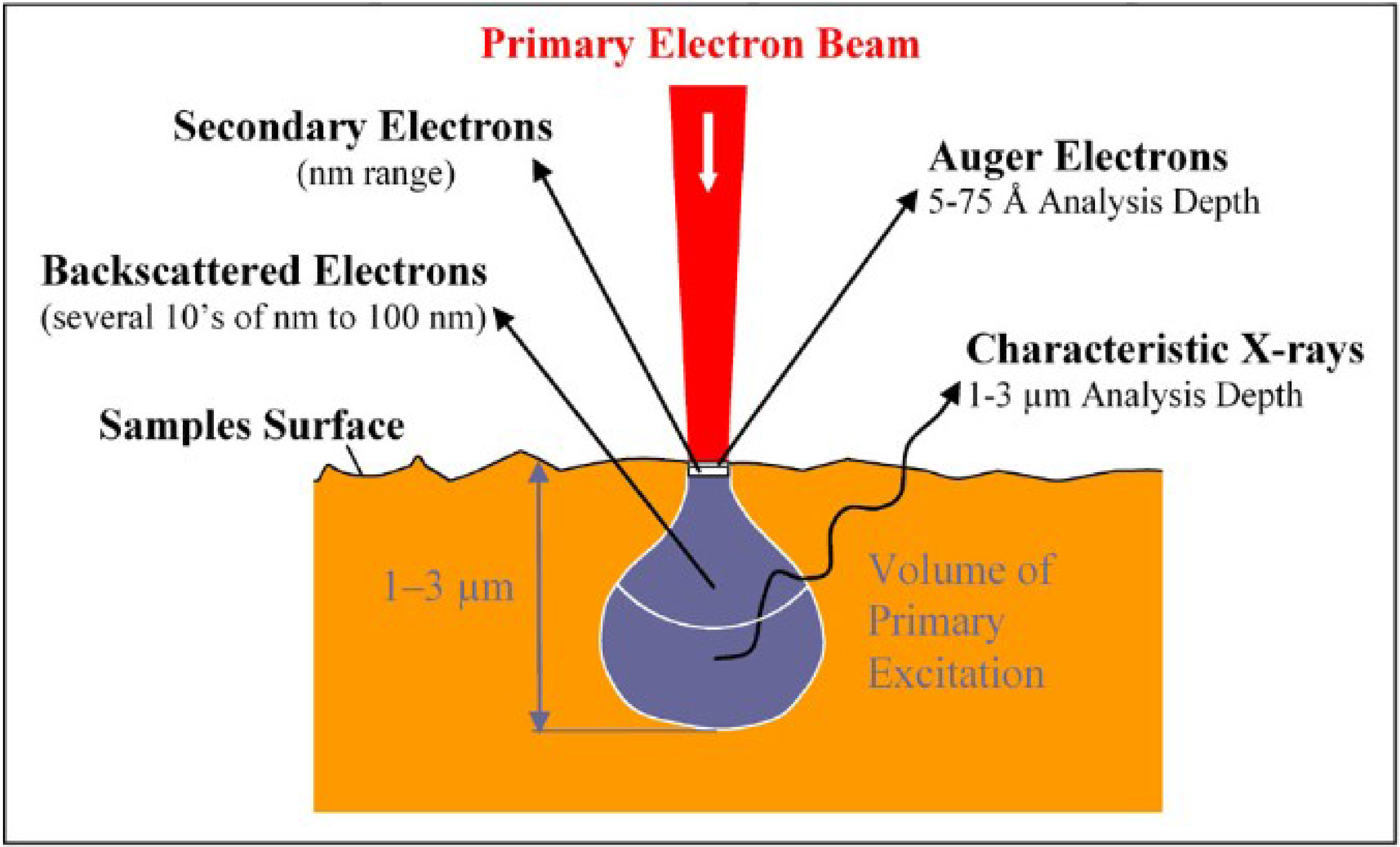




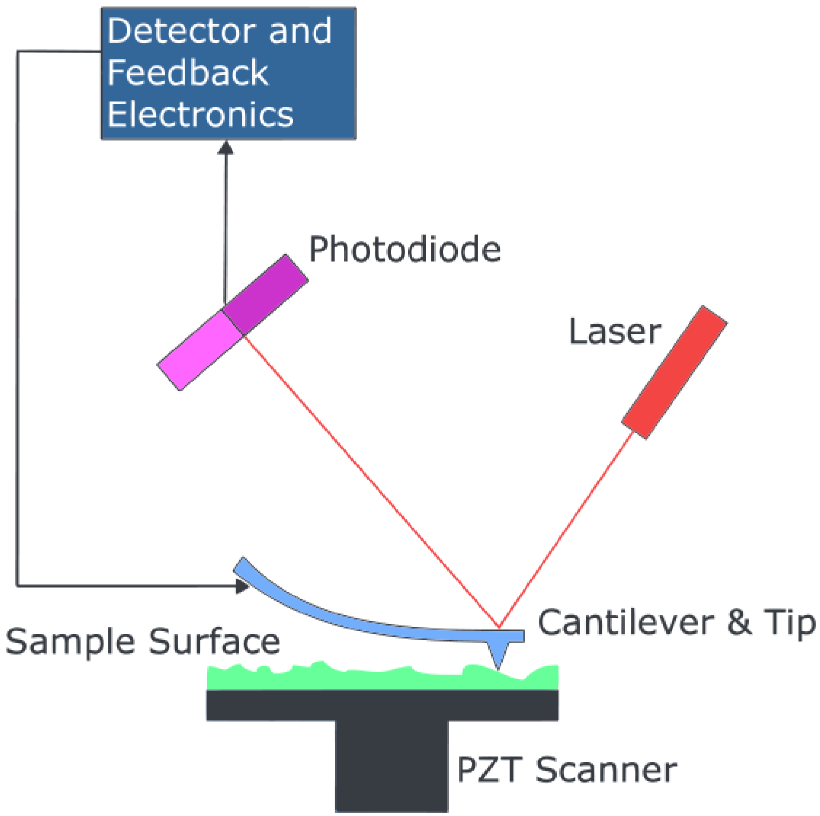


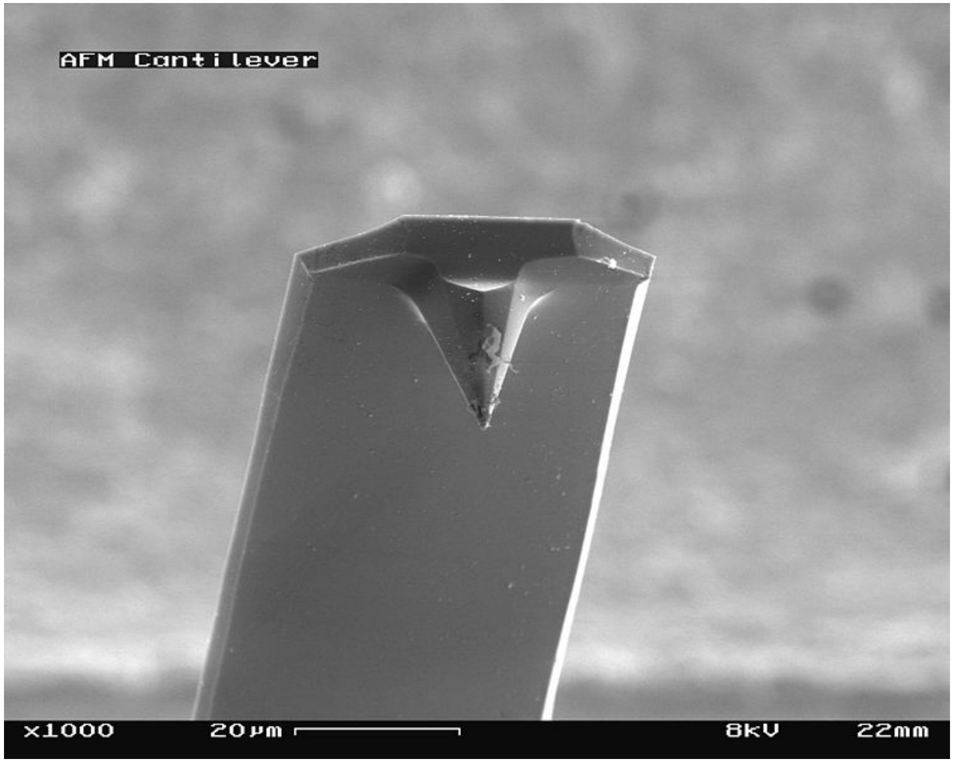
source: http://www-ssrl.slac.stanford.edu

Scanning electron microscopy



Atomic force microscopy

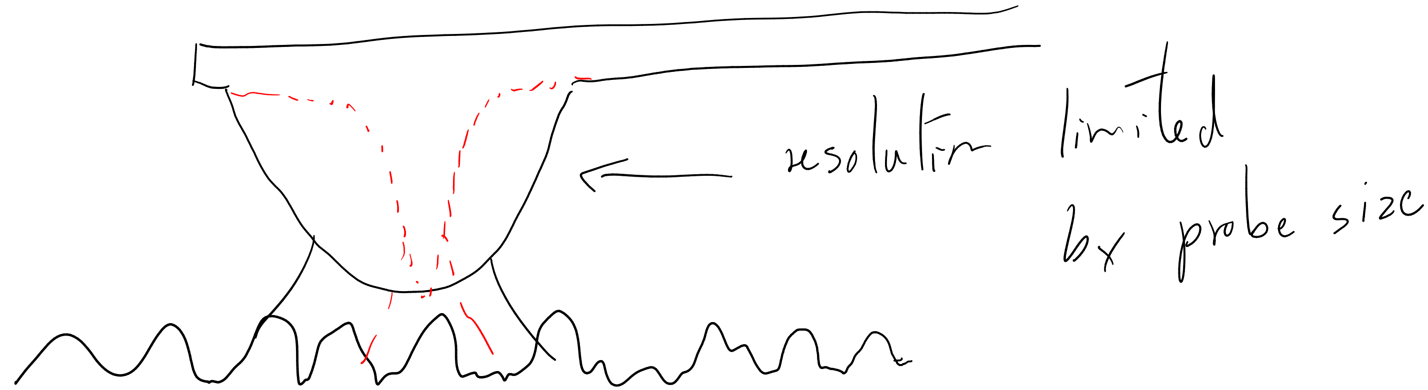




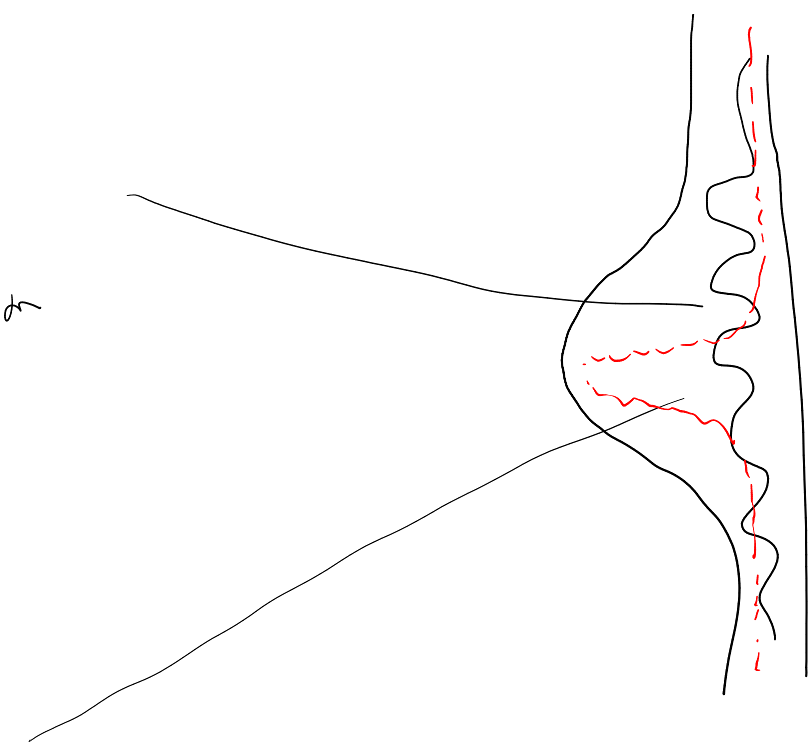
Resolution in scanning systems

Resolution mainly limited by probe size

e.g. AFM



e.g. transmission



Scanning vs. full field systems

Transmission probe: the reciprocity theorem