$$\begin{tikzcd} X \arrow[r, "f", "\text{biiett.}"'] \arrow[rr, bend right=50, "g \circ f"'] & Y \arrow[l, bend right=50, "f^{-1}"'] \arrow[r, "g", "\text{biiett.}"'] & Z \arrow[l, bend right=50, "g^{-1}"'] \end{tikzcd}$$

$g \circ f$ biiettiva

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

$(v_1,\dots,v_n) = \mathcal{B}$ $\qquad\qquad$ $\mathcal{B}' = (v_1',\dots,v_n')$ $\qquad\qquad$ $L_A = k' \circ k^{-1}$

$$
\begin{tikzcd}
v_i \arrow[d, mapsto, "k"'] & V \arrow[r, "id_V"] \arrow[d, "k"', "\cong"] \arrow[dr, phantom, "//"] & V \arrow[d, "k'", "\cong"'] & v_i' \arrow[d, mapsto, "k'"] \\
e_i & \mathbb{K}^n \arrow[r, "\cong", "L_A"'] \arrow[u, bend left] & \mathbb{K}^n \arrow[l, bend left, "L_A^{-1}"] & e_i
\end{tikzcd}
$$

$A \in M_n(\mathbb{K})$ invertibile

matrice del cambiamento di base dalla base $\mathcal{B}$ alla base $\mathcal{B}'$

$$V \ni v = x_1 v_1 + \dots + x_n v_n = x_1' v_1' + \dots + x_n' v_n'$$

$$k(v) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = X \qquad\qquad k'(v) = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = X'$$

$$X' = L_A(X) = A \cdot X$$

A^{-1} matrice del camb. di base da $\mathcal{B}' \longrightarrow \mathcal{B}$

$$GL_n(\mathbb{K}) \stackrel{\text{def}}{=} \{ A \in M_n(\mathbb{K}) \mid A \text{ invertibile} \}$$

<u>gruppo lineare</u>

$$A, B \in GL_n(\mathbb{K}) \Rightarrow AB \in GL_n(\mathbb{K}), \quad A^{-1} \in GL_n(\mathbb{K})$$

AB è la matrice di $\underline{L_A \circ L_B : \mathbb{K}^n \to \mathbb{K}^n} = L_{AB}$

$$L_{AB}^{-1} = (L_A \circ L_B)^{-1} = L_B^{-1} \circ L_A^{-1} = L_{B^{-1}} \circ L_{A^{-1}} = L_{B^{-1}A^{-1}}$$

$$L_{AB}^{-1} = L_{(AB)^{-1}}$$

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

$$(A_1 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}$$

$$
\begin{tikzcd}
(v_1', \ldots, v_n') = \mathcal{B}' & \mathbb{K}^n \arrow[rr, "L_{A'}"] \arrow[dr, bend right, "(k')^{-1}"'] \arrow[rr, bend left] & & \mathbb{K}^m & \mathcal{C}' = (w_1', \ldots, w_m') \\
& & V \arrow[r, "f"] \arrow[ul, "k'", "\cong"'] \arrow[dl, "k"', "\cong"] \arrow[dl, bend right] & W \arrow[u, "h'"', "\cong"] \arrow[d, "h", "\cong"'] \arrow[u, bend right] & A' = M^{\mathcal{C}'}_{\mathcal{B}'}(f) \\
(v_1, \ldots, v_n) = \mathcal{B} & \mathbb{K}^n \arrow[rr, "L_A"'] \arrow[rr, bend right] & & \mathbb{K}^m \arrow[u, bend right] & \mathcal{C} = (w_1, \ldots, w_m)
\end{tikzcd}
$$

$$\boxed{A' = TAS}$$

$$A = M^{\mathcal{C}}_{\mathcal{B}}(f)$$

$$k \circ (k')^{-1} = L_S \qquad S \in GL_n(\mathbb{K})$$

$$h' \circ h^{-1} = L_T \qquad T \in GL_m(\mathbb{K})$$

$$L_{A'} = \underbrace{h' \circ h^{-1}} \circ \underbrace{L_A} \circ \underbrace{k \circ (k')^{-1}} = L_T \circ L_A \circ L_S$$

Prop. Siano V, W spazi vett. su $\mathbb{K}$, $n = \dim V$, $m = \dim W$ e siano $A, A' \in M_{m,n}(\mathbb{K})$. Allora A e A' rappresentano la stessa applicazione lineare $f : V \longrightarrow W$ $\Longleftrightarrow$ $\exists\, S \in GL_n(\mathbb{K})$ e $T \in GL_m(\mathbb{K})$ t.c.

$$\boxed{A' = T A S}$$

$f : V \longrightarrow W$ lineare , $\underline{rg(f) = dim(Im\, f)}$

$Im\, f \subset W$ sottospazio vett.

$\underline{rg\, f \leq dim\, W}$

$(v_1, \dots, v_n)$ base di $V \Rightarrow \boxed{\{f(v_1), \dots, f(v_n)\}}$ generano $\underline{Im\, f}$

$dim\, V = rg(f) + null(f)$

$\Longrightarrow \underline{rg(f) \leq dim\, V}$

$rg\, f \leq min(dim\, V, dim\, W)$

rango per colonne

$A \in M_{m,n}(\mathbb{K})$ $\qquad$ $rg_c(A) \overset{def}{=} rg(L_A)$ $= \max \#$ di colonne di A lin. indip.

$L_A : \mathbb{K}^n \longrightarrow \mathbb{K}^m$

$$\boxed{rg_c(A) \leq \min(m,n)}$$

$\|$

$rg(L_A)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$L_A(e_1)$ $\quad L_A(e_2)$ $\quad L_A(e_n)$

generatori di Im L_A

$$\boxed{rg_r(A) \overset{def}{=} rg_c({}^tA)}$$

$$rg_r(A) \leq \min(m,n)$$

$$A \in GL_n(\mathbb{K}) \iff \boxed{rg_c\, A = n}$$

$$\Downarrow$$

$$L_A : \mathbb{K}^n \xrightarrow[\cong]{} \mathbb{K}^n \quad \text{suriettiva} \implies \text{~~iso~~ isomorfismo}$$

$$\dim(\mathbb{K}^n) = \text{null}(L_A) + rg(L_A)$$

$$\dim(\mathbb{K}^n) = n$$

$$n = \text{null}(L_A) + n$$

$$\implies \text{null}(L_A) = 0$$

$$\mathrm{rg}_r(A) = \mathrm{rg}_c(A) =: \mathrm{rg}(A)$$

Prop Se $A \in M_{m,n}(\mathbb{K})$, $B \in M_{n,r}(\mathbb{K})$ allora

$$^t(AB) = {}^tB \cdot {}^tA$$

$^tB \in M_{r,m}(\mathbb{K})$, $^tA \in M_{n,m}(\mathbb{K})$

$$\left({}^t(AB)\right)_{ij} = (AB)_{ji} = \underline{A^{(j)}} \cdot B_{(i)} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$\| \qquad\qquad \|$$

$$\underbrace{\begin{pmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \end{pmatrix}}_{A^{(j)}} \cdot \underbrace{\begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{pmatrix}}_{B_{(i)}}$$

$$\|$$

$$\underline{\left({}^tB\ {}^tA\right)_{ij}} = \left({}^tB\right)^{(i)} \cdot \left({}^tA\right)_{(j)} = {}^t\left(B_{(i)}\right) \cdot {}^t\left(A^{(j)}\right)$$

$$= \begin{pmatrix} b_{1i} & b_{2i} & \cdots & b_{ni} \end{pmatrix} \cdot \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{pmatrix} = \sum_{k=1}^{n} b_{ki} a_{jk}$$

$$\Downarrow$$

$$\boxed{{}^t(AB) = {}^tB\ {}^tA}$$

Prop. Se $A \in M_{m,n}(\mathbb{K}) \implies rg_r(A) = rg_c(A) =: rg(A)$

Dim

$L_A : \mathbb{K}^n \longrightarrow \mathbb{K}^m$; $L_{{}^tA} : \mathbb{K}^m \longrightarrow \mathbb{K}^n$

${}^tA \in M_{n,m}(\mathbb{K})$

$rg_c(A) = \boxed{rg(L_A) = k}$

$rg_r(A) = rg_c({}^tA) = rg(L_{{}^tA})$

$\ker L_A \subset \mathbb{K}^n$ $\quad \ker L_A = \{ x \in \mathbb{K}^n \mid L_A(x) = 0 \}$

$\exists\, \mathcal{B} = (v_1, \dots, v_n)$ base di $\mathbb{K}^n$ t.c. $\{ v_{k+1}, \dots, v_n \}$ siano base di $\ker L_A$

$$A v_{k+1} = A v_{k+2} = \dots = A v_n = 0$$

$$\left(\overset{w_1}{\overset{\shortparallel}{A v_1}}, \overset{w_2}{\overset{\shortparallel}{A v_2}}, \dots, \overset{w_k}{\overset{\shortparallel}{A v_k}}\right) \quad \boxed{\text{base per } \operatorname{Im} L_A}$$

$$0 = \lambda_1 A v_1 + \dots + \lambda_k A v_k = A(\lambda_1 v_1 + \dots + \lambda_k v_k)$$

$$\Longrightarrow \lambda_1 v_1 + \dots + \lambda_k v_k \in \ker L_A \Longrightarrow$$

$$\lambda_1 v_1 + \dots + \lambda_k v_k = \lambda_{k+1} v_{k+1} + \dots + \lambda_n v_n$$

per certi $\lambda_{k+1}, \dots, \lambda_n \in \mathbb{K}$

$$\lambda_1 v_1 + \dots + \lambda_k v_k - \lambda_{k+1} v_{k+1} - \dots - \lambda_n v_n = 0$$

$$\Longrightarrow \lambda_i = 0 \quad \forall i$$

$\mathbb{K}^m \ni \underbrace{w_1 = A v_1, \dots, w_k = A v_k}$ lin. indip.

(in quanto base di Im L_A)

$\rightsquigarrow$ li estendono a base di $\mathbb{K}^m$

aggiungendo $\underbrace{w_{k+1}, \dots, w_m}$

certi vettori

$$\mathcal{B} = (v_1, \dots, v_k, \underbrace{v_{k+1}, \dots, v_n}_{\downarrow\, 0})$$

$\mathcal{C} = (w_1, \dots, w_m)$ base di $\mathbb{K}^m$

$$\underbrace{B = M^{\mathcal{C}}_{\mathcal{B}}(L_A)} = \begin{array}{c} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & 0 & & 0 \\ \vdots & 0 & \cdots & \vdots & \vdots & & \vdots \\ & \vdots & & 1 & \vdots & & \vdots \\ & \vdots & & 0 & & & \vdots \\ 0 & & & \vdots & & & \\ & 0 & & 0 & 0 & & 0 \end{pmatrix} \\ (w_1) \quad (w_2) \;\cdots\; (w_k) \end{array} = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$A = T B S \qquad T \in GL_m(\mathbb{K}) \, , \quad S \in GL_n(\mathbb{K})$$

$$rg_c A = k = rg_c(B) = rg(L_A)$$

$${}^tA = {}^t(TBS) = {}^tS \, {}^tB \, {}^tT \qquad {}^tS \in GL_n(\mathbb{K}) \, , \quad {}^tT \in GL_m(\mathbb{K})$$

$$B = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) \in M_{m,n}(\mathbb{K}) \qquad {}^tB = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) \in M_{n,m}(\mathbb{K})$$

$$rg_c \, {}^tB = k = rg_c \, {}^tA$$

$$ {}^tA = {}^tS\,{}^tB\,{}^tT \implies \underbrace{\mathrm{rg}_c\,{}^tA}_{\substack{\| \\ \mathrm{rg}_r(A)}} = \mathrm{rg}_c\,{}^tB = k = \mathrm{rg}_c\,A $$