11 Novembre

Lemma X spazio di B, $B: X \times X \to X$

Sia $\alpha < \frac{1}{4\|B\|}$. Allora $\forall\, x_0 \in D_X(0,\alpha)$

$\exists!\ x \in \overline{D_X(0,2\alpha)}$ t.c. $x = x_0 + B(x,x)$

$$\boxed{u = e^{\nu t \Delta} u_0 + B(u,u)} \quad *$$

$$\begin{cases} \partial_t u - \nu \Delta u = Q(u,u) \\ u|_{t=0} = u_0 \end{cases}$$

$$\begin{cases} \partial_t B(u,v) - \nu \Delta B(u,v) = Q(u,v) \\ B(u,v)|_{t=0} = 0 \end{cases}$$

$$X = L^4\left([0,T], \dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d)\right)$$

Teor Sia $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d)$. Allora $\exists\ T > 0$ ed una soluz di $u = e^{\nu t \Delta} u_0 + B(u,u)$ in $L^4([0,T], \dot{H}^{\frac{d-1}{2}})$. La soluzione è unica.

Soddisfa

$$u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1}), \quad \nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}-1})$$

Sia T_{u_0} il tempo di vita.

1) $\exists\ c > 0$ t.c. $|u_0|_{\dot{H}^{\frac{d}{2}-1}} \leq c\nu \Rightarrow T_{u_0} = \infty$

2) Se $T_{u_0} < +\infty$ allora

$$\int_0^{T_{u_0}} |u(t)|^4_{\dot{H}^{\frac{d-1}{2}}}\, dt = +\infty$$

3) Se $T_{u_0} < +\infty$ allora

$$\int_0^{T_{u_0}} |\nabla u|^2_{\dot{H}^{\frac{d}{2}-1}}\, dt = +\infty$$

Infine se u e v sono due soluzioni allora

$$|u(t) - v(t)|^2_{\dot{H}^{\frac{d}{2}-1}} + \nu \int_0^t |\nabla(u-v)|^2_{\dot{H}^{\frac{d}{2}-1}}\, dt' \leq$$

$$\leq |u_0 - v_0|^2_{\dot{H}^{\frac{d}{2}-1}}\, e^{C\nu^{-3}\int_0^t \left(|u|^4_{\dot{H}^{\frac{d-1}{2}}} + |v|^4_{\dot{H}^{\frac{d-1}{2}}}\right) dt'}$$

dove $C = C_d$.

Osservazioni

$$u \longrightarrow u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$$

u e u_λ sono insieme soluzioni di NS.

Lo scaling $f(x) \longrightarrow f_\lambda(x) = \lambda f(\lambda x)$ preserva la norma di $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$.

Si dice allora che $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ è critico per NS. Non esiste una funzione

$$T(\cdot) : [0,+\infty) \to (0,+\infty] \quad \text{t.c.}$$

$$T_{u_0} \geq T(|u_0|_{\dot{H}^{\frac{d}{2}-1}})$$

Sostituiremo più avanti $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ con $L^d(\mathbb{R}^d)$. Cioè sostituiremo $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ con $L^3(\mathbb{R}^3)$

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) \subset \cdots\cdots$$

$$f \to f_\lambda(x) = \lambda f(\lambda x)$$

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se $u_0 \in H^1$ $\exists$ $T = T(|u_0|_{H^1})$ t.c.

$$u \in L^\infty([0,T], H^1)$$

Questo non è vero in $\dot{H}^{\frac{d}{2}-1}$ perché

se u è definito $[0, T_{u_0})$ con $T_{u_0} < \infty$

con dato iniziale $u_0(x)$, allora

$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ questo vive per $\lambda^2 t < T_{u_0}$

in $\left[0, \frac{T_{u_0}}{\lambda^2}\right]$

$$T_{u_{0\lambda}} = \frac{T_{u_0}}{\lambda^2}$$

$$|u_0|_{\dot{H}^{\frac{d}{2}-1}} = |u_{0\lambda}|_{\dot{H}^{\frac{d}{2}-1}}$$

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che implica che se $T_{u_0} < \infty$ allora

$$|u|_{L^\infty([0,T_{u_0}), \dot{H}^{\frac{1}{2}})} = +\infty$$

$$|\nabla u|_{L^2([0,T_{u_0}), \dot{H}^{\frac{d}{2}-1})} = +\infty \quad \not\checkmark$$

$$|u|_{L^4([0,T_{u_0}), \dot{H}^{\frac{d-1}{2}})} = +\infty \quad \not\checkmark$$

$$\partial_t u - \nu \Delta u = Q(u,u) \qquad u \in C^0([0,T], \dot H^s)$$

$$u \in L^4([0,T], \dot H^{\frac{d-1}{2}})$$

$$Q(u,v) \in L^2([0,T], \dot H^{s-1})$$

Lemma

$$|Q(u,v)|_{\dot H^{\frac{d}{2}-2}} \le C\, |u|_{\dot H^{\frac{d-1}{2}}}\, |v|_{\dot H^{\frac{d-1}{2}}}$$

Dim

$$|Q(u,v)|_{\dot H^{-\frac12}} \le |\nabla u\, v|_{\dot H^{-\frac12}} + |u \nabla v|_{\dot H^{-\frac12}} \le$$

$$\dot H^{\frac12}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) \qquad \frac13 = \frac12 - \frac{\frac12}{3} = \frac12 - \frac16$$

$$L^{\frac32}(\mathbb{R}^3) \subset \dot H^{-\frac12}(\mathbb{R}^3)$$

$$\lesssim |\nabla u\, v|_{L^{\frac32}} + |u \nabla v|_{L^{\frac32}} \qquad \frac23 = \frac12 + \frac16$$

$$\le |\nabla u|_{L^2} |v|_{L^6} + |u|_{L^6} |\nabla v|_{L^2} \qquad \dot H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \quad \frac16 = \frac12 - \frac13$$

$$\lesssim |\nabla u|_{L^2} |\nabla v|_{L^2}$$

$$\frac{d-1}{2} = 1 \quad \text{per } d = 3.$$

$$Q(u,v) \in L^2([0,T], \dot H^{\frac{d}{2}-2})$$

Lemma

$$\|Q(u,v)\|_{L^2([0,T],\dot H^{\frac d2-2})} \le C\,\|u\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}\,\|v\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})}$$

<u>Dim</u>

$$\Big\|\,\|Q(u,v)\|_{\dot H^{\frac d2-2}}\Big\|_{L^2([0,T])} \lesssim \Big\|\,\|u\|_{\dot H^{\frac{d-1}{2}}}\,\|v\|_{\dot H^{\frac{d-1}{2}}}\Big\|_{L^2(0,T)}$$

$$\le \|u\|_{L^4((0,T),\dot H^{\frac{d-1}{2}})}\,\|v\|_{L^4((0,T),\dot H^{\frac{d-1}{2}})}.$$

$$(\partial_t-\nu\Delta)B(u,v)=Q(u,v)$$

$$B(u,v)|_{t=0}=0$$

$$B(u,v)\in \overset{p=\infty}{C^0([0,T],\dot H^{\frac d2-1})}\cap \overset{p=2}{L^2([0,T],\dot H^{\frac d2})}$$

$$L^p([0,T],\dot H^{\frac d2-1+\frac 2p})$$

$$L^4([0,T],\dot H^{\frac{d-1}{2}})$$

$$\|B(u,v)\|_{L^4([0,T],\dot H^{\frac{d-1}{2}})} \lesssim \frac{C'}{\nu^{\frac34}}\,\|Q(u,v)\|_{L^2([0,T],\dot H^{\frac d2-2})}$$

$$\le \frac{C}{\nu^{\frac34}}\,\|u\|_{L^4((0,T),\dot H^{\frac{d-1}{2}})}\,\|v\|_{L^4((0,T),\dot H^{\frac{d-1}{2}})}$$

$$B: X_T\times X_T\longrightarrow X_T \qquad X_T=$$

$$\|B\|\le \frac{C}{\nu^{\frac34}}$$

$$u = e^{\nu t \Delta} u_0 + B[u,u]$$

Se $\quad \| e^{\nu t \Delta} u_0 \|_{L^4([0,T], \dot H^{\frac{d-1}{2}})} < \frac{\nu^{\frac{3}{4}}}{4C} \leq \frac{1}{4\|B\|}$

allora $\exists$ ~~una~~ soluzione $u \in L^4([0,T], \dot H^{\frac{d-1}{2}})$

con norma $\quad \|u\|_{L^4([0,T], \dot H^{\frac{d-1}{2}})} < 2\| e^{\nu t \Delta} u_0 \|_{L^4([0,T]}$

$$< \frac{\nu^{\frac{3}{4}}}{2C}$$

1) $\quad \|u_0\|_{\dot H^{\frac{d}{2}-1}}$ piccolo

$$\| e^{t\nu\Delta} u_0 \|_{L^4([0,T], \dot H^{\frac{d-1}{2}})} \leq \| e^{t\nu\Delta} u_0 \|_{L^4(\mathbb{R}_+, \dot H^{\frac{d-1}{2}})}$$

$$\leq \nu^{-\frac{1}{4}} \|u_0\|_{\dot H^{\frac{d}{2}-1}} \overset{se}{<} \frac{\nu^{\frac{3}{4}}}{4C}, \text{ cioè se}$$

$$\|u_0\|_{\dot H^{\frac{d}{2}-1}} < \frac{\nu}{4C}$$

allora $\quad \| e^{t\nu\Delta} u_0 \|_{L^4(\mathbb{R}_+, \dot H^{\frac{d-1}{2}})} < \frac{\nu^{\frac{3}{4}}}{4C}$

$$\Rightarrow \quad u \in L^4([0,T], \dot H^{\frac{d-1}{2}}) \quad \forall T$$

$$\boxed{u \in L^4(\mathbb{R}_+, \dot H^{\frac{d-1}{2}})}$$

2) u_0 generico in $\dot{H}^{\frac{d}{2}-1}$

$$\left(\| e^{\nu \Delta t} u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} \right) \xrightarrow{T \to 0} 0$$

e quindi $\exists$ T sufficientemente piccolo t.c.

$$\leq \frac{\nu^{\frac{3}{4}}}{4C} \implies u \in L^4([0,T], \dot{H}^{\frac{d-1}{2}})$$

$$u_0 = P_\rho u_0 + (1 - P_\rho) u \qquad \widehat{P_\rho u_0} = \chi_{[0,\rho]}(|\xi|) \hat{u}_0(\xi)$$

$\exists$ ρ sufficientemente grande t.c.

$$\| (1-P_\rho) u \|_{\dot{H}^{\frac{d}{2}-1}} < \frac{\nu}{8C}$$

$$\left(\int_{|\xi| > \rho} |\xi|^{d-2} |\hat{u}(\xi)|^2 \right)^{\frac{1}{2}} < \frac{\nu}{8C}$$

$$\| e^{\nu t \Delta} u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} \leq \| e^{\nu t \Delta} P_\rho u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} + \underbrace{\| e^{\nu t \Delta} (1-P_\rho) u_0 \|_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})}}_{< \frac{\nu^{\frac{3}{4}}}{8C}}$$

$$\| e^{\nu t \Delta} \chi_{[0,\rho]}(\sqrt{-\Delta}) u_0 \|_{L^4((0,T), \dot{H}^{\frac{d-1}{2}})}$$

$$\leq \| e^{\nu t \Delta} \chi_{[0,\rho]}(\sqrt{-\Delta}) \sqrt{\rho} \cancel{\frac{(-\Delta)^{\frac{1}{4}}}{\sqrt{\rho}}} u_0 \|_{L^4((0,T), \dot{H}^{\frac{d}{2}-1})}$$

$$\leq \sqrt{\rho} \, \| e^{\nu t \Delta} \cancel{\chi_{[0,\rho]}(\sqrt{-\Delta})} u_0 \|_{L^4((0,T), \dot{H}^{\frac{d}{2}-1})}$$

$$\leq \sqrt{\rho} \, T^{\frac{1}{4}} \| \cancel{e^{\nu t \Delta}} u_0 \|_{L^\infty(\mathbb{R}_+, \dot{H}^{\frac{d}{2}-1})} \leq \sqrt{\rho} \, T^{\frac{1}{4}} \| u_0 \|_{\dot{H}^{\frac{d}{2}-1}}$$

$$\left|e^{\nu t\Delta}\dot{\chi}_{[0,\rho]}(\sqrt{-\Delta})\,u_0\right|_{L^4((0,T),\dot H^{\frac{d-1}{2}})} \leq \rho^{\frac{1}{2}}\,T^{\frac{1}{4}}\,|u_0|_{\dot H^{\frac{d}{2}-1}} < \frac{\nu^{\frac{3}{4}}}{8C}$$

$$\left|e^{\nu t\Delta}\dot{\chi}_{[\rho,+\infty)}(\sqrt{-\Delta})\,u_0\right|_{L^4((0,T),\dot H^{\frac{d-1}{2}})} \leq \frac{\nu^{\frac{3}{4}}}{8C}$$

Così si avrà

$$\left|e^{\nu t\Delta}u_0\right|_{L^4((0,T),\dot H^{\frac{d-1}{2}})} \leq \frac{\nu^{\frac{3}{4}}}{4C}$$

$$\Longrightarrow \quad u \in L^4((0,T),\dot H^{\frac{d-1}{2}})$$

$$T < \left(\frac{\nu^{\frac{3}{4}}}{8\rho^{\frac{1}{2}}C\,|u_0|_{\dot H^{\frac{d}{2}-1}}}\right)^4$$

$$\begin{cases} (\partial_t - \nu\Delta)u = Q(u,u) \\ u_{|t=0} = u_0 \end{cases}$$

$$u \in C^0([0,T],\dot H^{\frac{d}{2}-1})$$

$$\nabla u \in L^2([0,T],\dot H^{\frac{d}{2}-1})$$

Siano u, v siano due soluzioni e sia $w = u - v$.

$$\begin{cases} \partial_t w - \nu \Delta w = Q(u,u) - Q(v,v) = Q(\overset{w}{\widetilde{u-v}}, u+v) \\ w(0) = u_0 - v_0 \end{cases}$$

$$\|w(t)\|^2_{\dot H^{\frac{d}{2}-1}} + 2\nu \int_0^t \|\nabla w\|^2_{\dot H^{\frac{d}{2}-1}} dt' =$$

$$= \|w(0)\|^2_{\dot H^{\frac{d}{2}-1}} + 2 \int_0^t \langle Q(w, u+v), w\rangle_{\dot H^{\frac{d}{2}-1}} dt'$$

$$\langle Q(a,b), c\rangle_{\dot H^{\frac{d}{2}-1}} \le C \|a\|_{\dot H^{\frac{d-1}{2}}} \|b\|_{\dot H^{\frac{d-1}{2}}} \|c\|_{\dot H^{\frac{d}{2}}}$$

$$\langle Q(a,b), c\rangle_{\dot H^{\frac{d}{2}-1}} \le \|Q(a,b)\|_{\dot H^{\frac{d}{2}-2}} \|\cancel{\nabla} c\|_{\dot H^{\frac{d}{2}\cancel{-1}}}$$

$$\le C \|u\|_{\dot H^{\frac{d-2}{2}}} \|b\|_{\dot H^{\frac{d-1}{2}}} \|c\|_{\dot H^{\frac{d}{2}}}$$

$$\Delta_W = |w(t)|^2_{\dot H^{\frac d2-1}} + 2\nu \int_0^t |\nabla w|^2_{\dot H^{\frac d2-1}}\, dt' =$$

$$= |w(0)|^2_{\dot H^{\frac d2-1}} + 2\int_0^t \langle Q(w, u+v), w\rangle_{\dot H^{\frac d2-1}}\, dt'$$

$$\Delta_W \le |w_0|^2_{\dot H^{\frac d2-1}} + 2C\int_0^t |w|_{\dot H^{\frac{d-1}{2}}} |\nabla w|_{\dot H^{\frac d2-1}} N \, dt'$$

$$N = |u|_{\dot H^{\frac{d-1}{2}}} + |v|_{\dot H^{\frac{d-1}{2}}}$$

$$|w|_{\dot H^{\frac{d-1}{2}}} \le |w|^{\frac12}_{\dot H^{\frac d2-1}} \, |\nabla w|^{\frac12}_{\dot H^{\frac d2-1}}$$

$$\Delta_w \le |w_0|^2_{\dot H^{\frac d2-1}} + 2C\int_0^t \left(|w|^{\frac12}_{\dot H^{\frac d2-1}} N\right)\left(|\nabla w|^{\frac32}_{\dot H^{\frac d2-1}}\right)$$

$$ab \le \frac{a^4}{4} + \frac34 b^{\frac43}$$

$$\Delta_w \le \underbrace{|w_0|^2_{\dot H^{\frac d2-1}} + C_\nu\int_0^t |w|^2_{\dot H^{\frac d2-1}} N^4}\ \cancel{+ \nu\int_0^t |\nabla w|^2_{\dot H^{\frac d2-1}}\, dt}$$

$$\underbrace{|\overset{u-v}{w}(t)|^2_{\dot H^{\frac d2-1}} + \cancel{2}\nu\int_0^t |\nabla w|^{(u-v)2}_{\dot H^{\frac d2-1}}\, dt}_{X(t)} \le$$

$$X(t) \le |w_0|^2_{\dot H^{\frac d2-1}} + C_\nu\int_0^t N^4 \underbrace{|w|^2_{\dot H^{\frac d2-1}}}_{\le X}\, dt'$$

$$X(t) \le |w_0|^2_{\dot H^{\frac d2-1}} + C_\nu\int_0^t N^4(t') X(t')\, dt'$$

$$X(t') \le |w_0|^2_{\dot H^{\frac d2-1}}\, e^{C_\nu \int_0^t N^4(t')\, dt}$$

$$N^4 \sim |u|^4_{\dot H^{\frac{d-1}{2}}} + |v|^4_{\dot H^{\frac{d-1}{2}}}$$