

$$g_1 \in C^1(\mathbb{R}^3) \quad g_1(0,0,0) = 0 \quad \underline{\nabla g_1(0,0,0) = (1, 2, 3)^T} \quad g = (g_1, g_2)^T$$

$$g_1(x, y, z) = 0$$

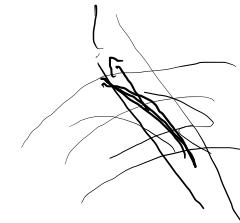
$$x + y + z = 0$$

$$g_2(x, y, z) = x + y + z$$

$$\nabla g_2 = (1, 1, 1)^T$$

$$\text{rank} \begin{pmatrix} \nabla g_1 \\ \nabla g_2 \end{pmatrix} = 2$$

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = 2$$



$$\omega \in \text{nuovo lungo quale } g_1 = 0 \\ g_2 = 0$$

$$c \dots$$

$$\omega = (a, b, c)^T$$

$$a = c$$

$$c = 1$$

$$b = -2c$$

$$c \approx c$$

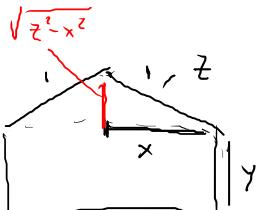
$$\omega \perp \nabla g_1 \quad \langle (a, b, c)^T, (1, 2, 3)^T \rangle = 0$$

$$\omega \perp \nabla g_2 \quad \langle (a, b, c)^T, (1, 1, 1)^T \rangle = 0$$

$$\begin{cases} a + 2b + 3c = 0 \\ a + b + c = 0 \end{cases} \quad \begin{aligned} -b - c + 2b + 3c &= 0 & b + 2c &= 0 \\ a &= -b - c & b &= -2c \end{aligned}$$

$$\omega \approx (1, -2, 1)^T \quad \| (1, -2, 1)^T \| = \sqrt{1+4+1} = \sqrt{6} \quad \begin{aligned} a^2 &= \cancel{-c} - b - c \\ &= 2c - c = c \end{aligned}$$

$$\omega = \frac{1}{\sqrt{6}} (1, -2, 1)^T$$



$P = \text{perimeter}$

$$P = 2x + 2y + 2z$$

$$\text{Area} = 2x \cdot y + x \cdot \sqrt{z^2 - x^2}$$

$$2y = P - 2x - 2z$$

$$2Px - 4x^2 - 4xz$$

$$f(x, z) = x \cdot (P - 2x - 2z) + x \sqrt{z^2 - x^2} \quad x > 0, z > 0$$

$$\nabla f(x, z) = \left(\begin{array}{c} P - 4x - 2z + \sqrt{z^2 - x^2} + \frac{x}{2\sqrt{z^2 - x^2}}(-2x), \\ -2x + x \cdot \frac{1}{x\sqrt{z^2 - x^2}} \cdot \frac{1}{2}z \end{array} \right)^T$$

$$\left(\begin{array}{c} P - 4x - 2z + \sqrt{z^2 - x^2} - \frac{x^2}{\sqrt{z^2 - x^2}}, \\ -2x + \frac{xz}{\sqrt{z^2 - x^2}} \end{array} \right)$$

$$\sqrt{z^2 - x^2} = \frac{1}{2}z$$

$$\frac{z}{\sqrt{3}}x \quad \frac{z}{\sqrt{3}} = \frac{z}{\frac{z}{\sqrt{3}}x} - \frac{\sqrt{3}}{x}$$

$$\frac{z}{\sqrt{z^2 - x^2}} = 2$$

$$z = 2\sqrt{z^2 - x^2}$$

$$z^2 - 4z^2 - 4x^2$$

$$3z^2 = 4x^2$$

$$P - 4x - \frac{4}{\sqrt{3}}x + \frac{1}{\sqrt{3}}x - x \cdot \frac{\sqrt{3}}{x} = 0$$

$$(-4 - \sqrt{3} - \sqrt{3})x + P = 0$$

$$x = \frac{P}{4 + 2\sqrt{3}}$$

$$\frac{2 - \sqrt{3}}{4 - 3} = \frac{P}{2}(2 - \sqrt{3})$$

$$B_{\text{opt}} = P(2 - \sqrt{3})$$

Integrali dipendenti da parametro

Tessore $f: [a,b] \times I \rightarrow \mathbb{R}$ I intervallo, f continua su $[a,b] \times I$.

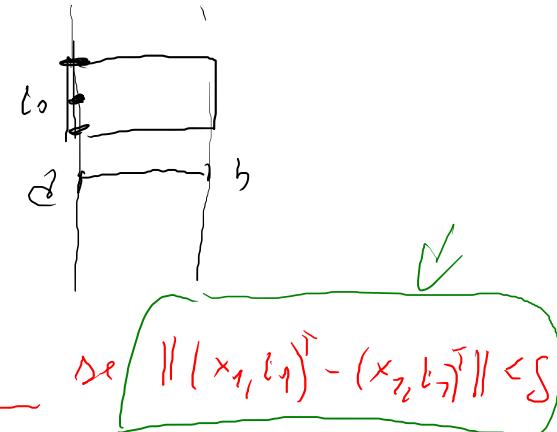
$$h(t) = \int_a^b f(x,t) dx. \quad \text{Allora } h \text{ è continua su } I.$$

Dim Sia $t_0 \in I$; sia $c \leq t_0 \leq d$ ($a, d \in I$)

e consideriamo f restreto a $[a,b] \times [c,d]$;

f è uniformemente continua su $[a,b] \times [c,d]$; quindi

$\forall \varepsilon > 0 \exists \delta > 0$ tale che $\forall (x_1, t_1)^T, (x_2, t_2)^T \in [a,b] \times [c,d]$

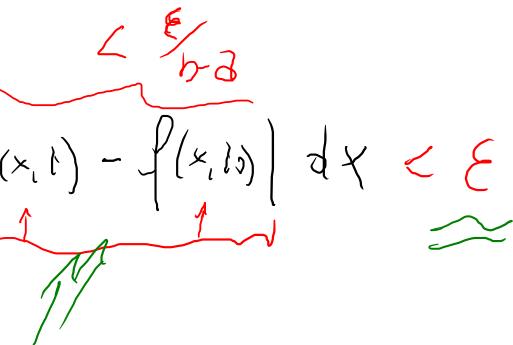


$$\text{se } \|(x_1, t_1)^T - (x_2, t_2)^T\| < \delta$$

$$|f(x_2, t_2) - f(x_1, t_1)| < \frac{\varepsilon}{b-a}. \quad \text{Allora}$$

$$|h(t) - h(t_0)| = \left| \int_a^b f(x, t) dx - \int_a^b f(x, t_0) dx \right| \leq \int_a^b |f(x, t) - f(x, t_0)| dx < \frac{\varepsilon}{b-a} \cdot (b-a) < \varepsilon$$

$$\|(x, t)^T - (x, t_0)^T\| = |t - t_0| < \delta$$



Derivabilitate

Teorema $f: [\alpha, b] \times] \rightarrow \mathbb{R}$ $f \in C^1([\alpha, b] \times], \mathbb{R})$, $h(t) = \int_a^b f(x, t) dx$

Aflare h este derivabile pe $]a, b]$ și în aceeași

$$h'(t) = \left(\frac{\partial}{\partial t} \right) \int_a^b f(x, t) dx = \int_a^b \frac{\partial}{\partial t} f(x, t) dx$$

$$\begin{aligned} \text{Dim} \quad h'(t_0) &= \lim_{t \rightarrow t_0} \frac{h(t) - h(t_0)}{t - t_0} = \lim_{t \rightarrow t_0} \frac{1}{t - t_0} \left[\int_a^b f(x, t) dx - \int_a^b f(x, t_0) dx \right] \\ &= \lim_{t \rightarrow t_0} \frac{1}{t - t_0} \int_a^b [f(x, t) - f(x, t_0)] dx \stackrel{\text{Lagrange}}{=} \lim_{t \rightarrow t_0} \int_a^b \frac{\partial f}{\partial t}(x, \xi) dx = \int_a^b \frac{\partial f}{\partial t}(x, t_0) dx \\ &\quad \left(\frac{\partial f}{\partial t} \text{ continuu} \right) \end{aligned}$$

~~$$\int_a^b \frac{\partial f}{\partial t}(x, \xi) (t - t_0) dx$$~~

⇒ puncte

$\frac{\partial f}{\partial t}(x, t)$ este continuu pe $[\alpha, b] \times]$

$\Rightarrow \int_a^b \frac{\partial f}{\partial t}(x, t) dx$ este continuu

$$\lim_{t \rightarrow t_0} \int_a^b \frac{\partial f}{\partial t}(x, t) dx = \int_a^b \frac{\partial f}{\partial t}(x, t_0) dx$$

$$EJ: h(t) = \int_0^t \sin(x^2 t) dx$$

$$h'(t) = \int_0^t \frac{\partial}{\partial t} \sin(x^2 t) dx =$$

$$= \int_0^t x^2 \cos(x^2 t) dx$$

$$g(t) = \int_0^t f(x) dx$$

$$g'(t) = f(t)$$

$$h(t) = \int_0^t f(x, \textcircled{t}) dx$$

$$h'(t) = ?$$

$$h(t) = \int_0^t f(x, t) dx$$

$$\varphi(u, v) = \underbrace{\int_0^u}_{\text{u}} \underbrace{f(x, v) dx}_{\text{x}}$$

$$h(t) = \varphi(t, t) = (\varphi \circ \gamma)(t)$$

$$\gamma(t) = (t, t)^T$$

$$\gamma'(t) = (1, 1)^T$$

$$h'(t) = \frac{d}{dt} (\varphi \circ \gamma)(t) = \langle \nabla \varphi(\gamma(t)), \gamma'(t) \rangle = \frac{\partial \varphi}{\partial u}(\gamma(t)) + \frac{\partial \varphi}{\partial v}(\gamma(t)) \cdot 1$$

$$\frac{\partial \varphi}{\partial u}(u, v) = \underbrace{f(u, v)}_{\text{u}} \quad \leftarrow$$

$$+ \int_0^t \frac{\partial f}{\partial t}(x, t) dx$$

$$\frac{\partial \varphi}{\partial v}(u, v) = \int_0^u \frac{\partial f}{\partial v}(x, v) dx$$

$$\frac{d}{dt} \left(\int_0^t f(x, t) dx \right) = f(t, t) + \int_0^t \frac{\partial f}{\partial t}(x, t) dx$$

$$h(t) = (\varphi \circ \gamma)(t)$$

$$h'(t) = \langle \nabla \varphi(t, t), (1, 1)^T \rangle = f(t, t) \cdot 1 + \int_0^t \frac{\partial f}{\partial t}(x, t) dx \cdot 1$$

$$\varphi(u, v) = \int_0^u f(x, v) dx$$

$$\gamma(t) = (t, t)^T$$

$$\frac{\partial \varphi}{\partial u} = f(u, v)$$

$$\frac{\partial \varphi}{\partial v} = \int_0^u \frac{\partial f}{\partial v}(x, v) dx$$

In generale se f, α, β sono

$$h(t) = \int_{\alpha(t)}^{\beta(t)} f(x, t) dx = \int_{z_0}^{\beta(t)} f(x, t) dx - \int_{z_0}^{\alpha(t)} f(x, t) dx$$

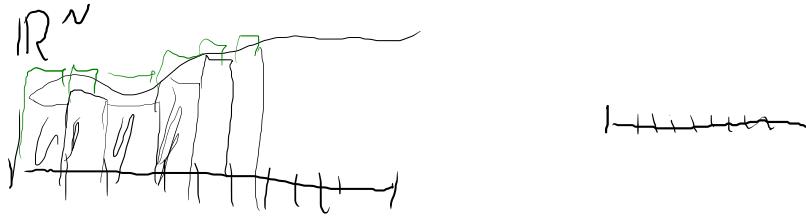
$$h'(t) = f(\beta(t), t) \cdot \beta'(t) - f(\alpha(t), t) \cdot \alpha'(t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(x, t) dx$$

Ese: Si vol calci $\nabla f(x, y)$

$$f(x, y) = \int_{x^2}^y \cos(t \cdot y) dt$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= -\cos(x^3 \cdot y) \cdot 2x + \int_{x^2}^y -\sin(tx \cdot y) \cdot ty dt \\ \frac{\partial f}{\partial y} &= \cos(xy^2) + \int_{x^2}^y -\sin(tx \cdot y) \cdot tx dt \end{aligned}$$

Integrale di Riemann in \mathbb{R}^n

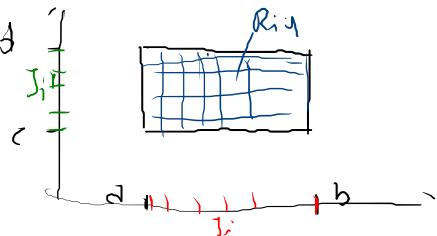


$$N=1 \quad \int_{[a,b]} f dm$$

$$N=2 \quad \text{Un 2-rettangolo in } \mathbb{R}^2 \text{ è } R = [a,b] \times [c,d]$$

Sia $R = [a,b] \times [c,d]$ un rettangolo; diciamo decomposizione di R una famiglia di sottorettagoli $\{R_{ij}, i=1, \dots, n; j=1, \dots, m\}$ del tipo $R_{ij} = I_i \times J_j$

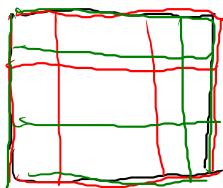
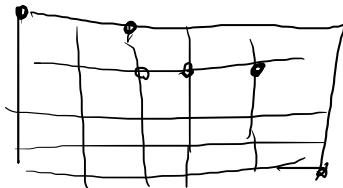
$$I_i = [x_{i-1}, x_i] \quad J_j = [y_{j-1}, y_j] \quad \text{e} \quad \begin{cases} I_i : i=1, \dots, n \\ J_j : j=1, \dots, m \end{cases} \text{ è una decomposizione di } [a,b] \times [c,d]$$



Siano S e S' decomposizioni di R ($\Delta(R)$)

dicono che S' è più fine di S se ogni nodo di S è un nodo di S'

Ogni punto del tipo $(x_i, y_j)^\top$ con $S = \{[x_0, y_0], [x_0, y_1], [x_0, y_2], \dots, [x_0, y_{m_0}], [x_1, y_0], [x_1, y_1], \dots, [x_n, y_{m_n}]\}$
si dice un nodo di S

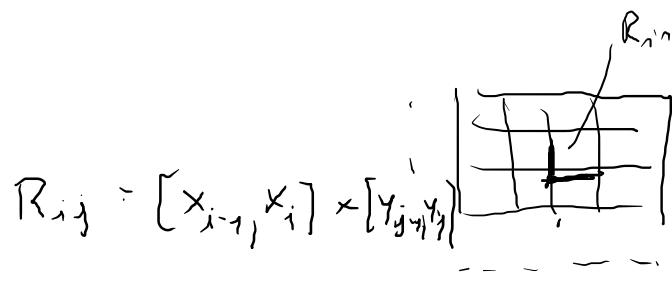


Si osservi che date due decomposizioni $S_1 + S_2$ di R non è detto che una sia più fine dell'altra, tuttavia esiste sempre S che è più fine di entrambe (i nodi di S sono l'unione dei nodi di $S_1 + S_2$).

Sia $f: \mathbb{R} \rightarrow \mathbb{R}$ limitata

Diciamo somma inferiore di f relativa a S il numero

$$\sigma(f, S) = \sum_{i=1}^n \sum_{j=1}^m \inf_{(x,y) \in R_{ij}} f(x,y) \cdot m(R_{ij})$$



$$S(f, S) = \sum_{i=1}^n \sum_{j=1}^m \sup_{(x,y) \in R_{ij}} f(x,y) \cdot m(R_{ij})$$

$$m(R_{ij}) = (x_i - x_{i-1}) \cdot (y_j - y_{j-1})$$

- Proprietà delle somme :
- $\forall S \in \Delta \quad \sigma(f, S) \leq S(f, S)$
 - se S' è più fine di S , $\sigma(f, S) \leq \sigma(f, S')$
 $S(f, S) > S(f, S')$
 - $\forall S_1, S_2 \in \Delta \quad \sigma(f, S_1) \leq S(f, S_2)$

$\mathcal{S}_1, \mathcal{S}_2$ sono i più famosi di entrambi

$$S(f_1, \mathcal{S}_1) \leq S(f_1, \mathcal{S}) \leq S(f_1, \mathcal{S}) \leq S(f_1, \mathcal{S}_2).$$

Consideriamo i soluzioni di \mathbb{R}

$$\sigma = \left\{ S(f_1, \mathcal{S}) : \mathcal{S} \in \Delta(\mathbb{R}) \right\} = \sum \left\{ S(f_1, \mathcal{S}) : \mathcal{S} \in \Delta(\mathbb{R}) \right\}$$

σ e Σ sono interni separati in \mathbb{R}

$$\sup \sigma = \underline{\int}_{\mathbb{R}} f dm \quad \text{integrale inferiore di } f \text{ su } \mathbb{R}$$

$$\inf \Sigma = \overline{\int}_{\mathbb{R}} f dm \quad \text{integrale superiore di } f \text{ su } \mathbb{R}$$

$$\underline{\int}_{\mathbb{R}} f dm \leq \overline{\int}_{\mathbb{R}} f dm$$

f si dice integrabile secondo Riemann su R se

$$\int_R^- f dm = \int_R^+ f dm$$

Questo avviene se σ e Σ sono insiem contigui, cioè

$\forall \varepsilon > 0$ esiste $S(f, \delta_1)$ e $S(f, \delta_2)$ tali che

$$S(f, \delta_2) - S(f, \delta_1) < \varepsilon$$

prendendo S più fine di $S_1 + S_2$ si ha

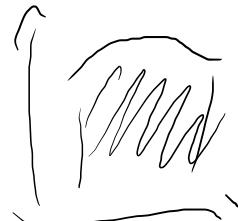
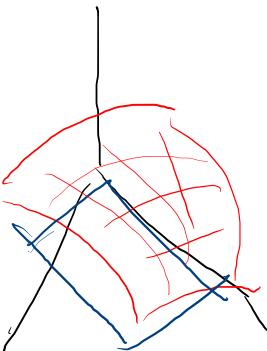
$$| S(f, \delta) - S(f, \omega) | < \varepsilon$$

Se f è integrabile si dico integrale di f il numero

$$\int_R^- f dm = \int_R^+ f dm = \int_R f dm \quad \left\{ \begin{array}{l} \text{è l'elemento rappresentante} \\ \text{tra } \sigma \text{ e } \Sigma \end{array} \right.$$

Se $f(x, y) \geq 0$

$\iint_R f dm$ rappresenta il
volume della regione



Δx

in \mathbb{R}^3 compresa tra i piani xy e il grafico $z = f$

$f(x, y)$

[il rettangolo di f]

$\iint_R f(u, t) du dt$

$$\iint_R f dm = \iint_R f(x, y) \overbrace{dx dy}^{\text{simbolo}}$$

mi è la misura che sto considerando

\mathbb{R}^3

\mathbb{R}^n

Esercizio 1

3-montagna



$$\text{Rijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$