12 Novembre

$$\sinh(x) = sh(x) = \frac{e^x - e^{-x}}{2}$$

$$ch(x) = \frac{e^x + e^{-x}}{2}$$

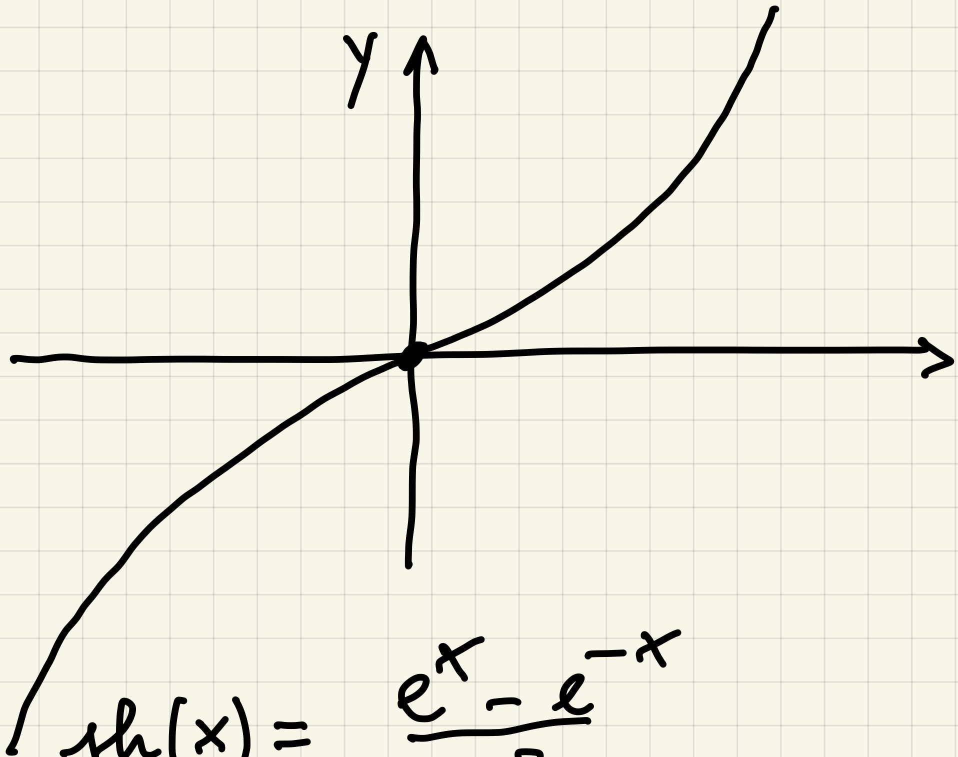
$$\lim_{x \to +\infty} sh(x) = \lim_{x \to +\infty} \frac{e^x - e^{-x}}{2}$$

$$= \lim_{x \to +\infty} \frac{e^x}{2} \underbrace{(1 - e^{-2x})}_{\downarrow \, 1} = \lim_{x \to +\infty} \frac{e^x}{2}$$

$$= +\infty$$

$$sh(0) = \frac{e^0 - e^0}{2} = 0$$

$$sh(-x) = -sh(x)$$



$$sh(x) = \frac{e^x - e^{-x}}{2}$$

per $x > 0$ $\quad e^x - e^{-x} > 0$

perché $\quad e^x > 1 > e^{-x}$

$$ch(x) = \frac{e^x + e^{-x}}{2} \geq 1 \iff \underbrace{e^x + e^{-x} - 2}_{} \geq 0$$

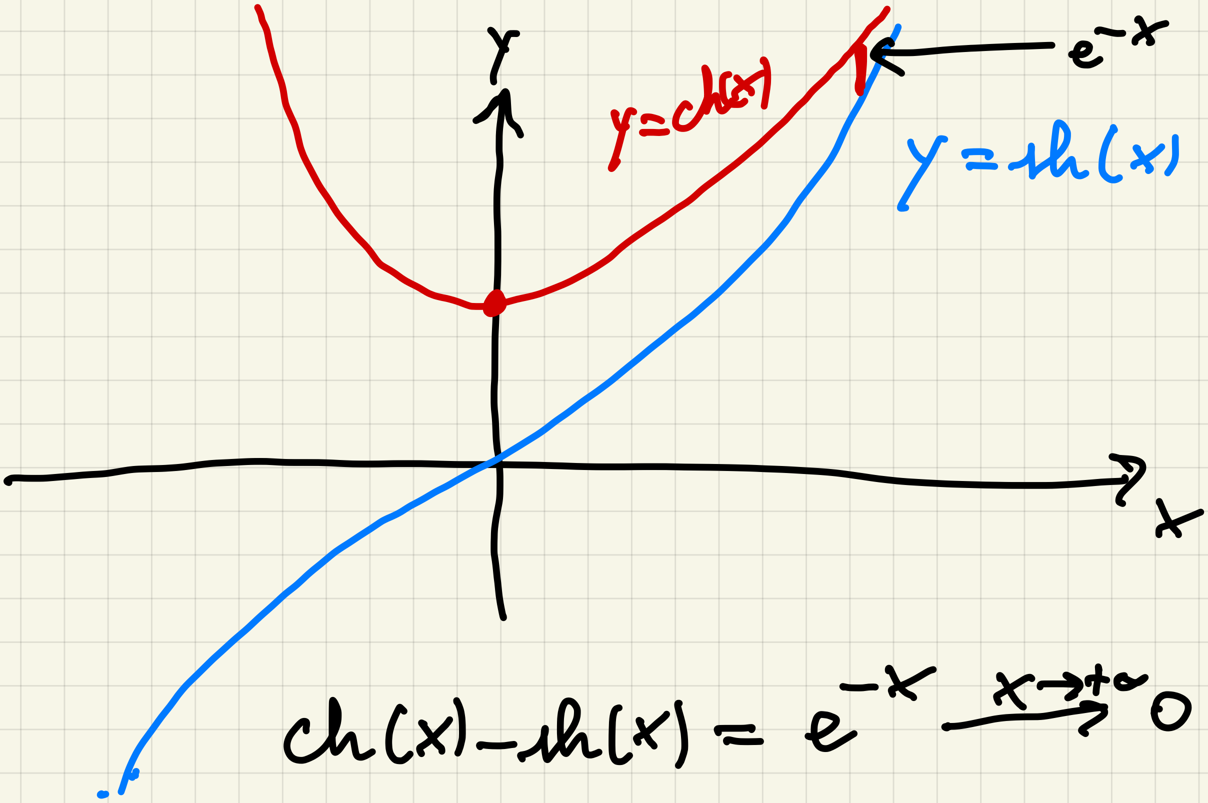
$$(e^{\frac{x}{2}} - e^{-\frac{x}{2}})^2 \geq 0$$

$\forall x \neq 0 \quad ch(x) > 0$

perché $\quad e^{\frac{x}{2}} - e^{-\frac{x}{2}} \neq 0$

$$\forall x \qquad ch(x) = \frac{e^x + e^{-x}}{2} > \frac{e^x - e^{-x}}{2} = sh(x)$$

per $x >> 1$ $\quad ch(x) = \frac{e^x}{2}(1 + e^{-2x}) \approx \frac{e^x}{2}$



$$ch(x) - sh(x) = e^{-x} \xrightarrow{x \to +\infty} 0$$

$$\lim_{x\to+\infty} th(x) = \lim_{x\to+\infty} \frac{sh(x)}{ch(x)} =$$

$$= \lim_{x\to+\infty} \frac{\frac{e^x - e^{-x}}{\cancel{2}}}{\frac{e^x + e^{-x}}{\cancel{2}}} = \lim_{x\to+\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \lim_{x\to+\infty} \frac{\cancel{e^x}(1 - e^{-2x})}{\cancel{e^x}(1 + e^{-2x})} = 1$$

$$1 - th(x) = 1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$

$$= \frac{e^x + e^{-x}}{e^x + e^{-x}} - \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \frac{(\cancel{e^x} + e^{-x}) - (\cancel{e^x} - e^{-x})}{e^x + e^{-x}} = \frac{2e^{-x}}{e^x + e^{-x}}$$

$$1-\operatorname{th}(x) = 1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$

$$= \frac{e^x + e^{-x}}{e^x + e^{-x}} - \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \frac{(\cancel{e^x} + e^{-x}) - (\cancel{e^x} - e^{-x})}{e^x + e^{-x}} = \frac{2e^{-x}}{e^x + e^{-x}}$$

$$= \frac{2\, e^{-x}}{e^x(1+e^{-2x})} = 2\, e^{-2x}\, \frac{1}{1+e^{-2x}}$$

Notiamo che

$$\lim_{x\to+\infty} \left(\frac{1}{1+e^{-2x}} - 1 \right) = 0$$

Denoteremo con $o(1)$ una funzione $f(x)$ t.c.

$$\lim_{x\to x_0} f(x) = 0$$

e scriviamo $f(x) = o(1)$

$$1-\operatorname{th}(x) = 2\, e^{-2x}\, \frac{1}{1+e^{-2x}} =$$

$$= 2\, e^{-2x}\,(1 + o(1)) \quad \text{per } x\to+\infty$$

$$-1-\operatorname{th}(x) = 2\, e^{2x}\,(1+o(1)) \quad \text{per } x\to-\infty$$

$$\left(sh(x)\right)' = ch(x)$$

$$\left(ch(x)\right)' = sh(x)$$

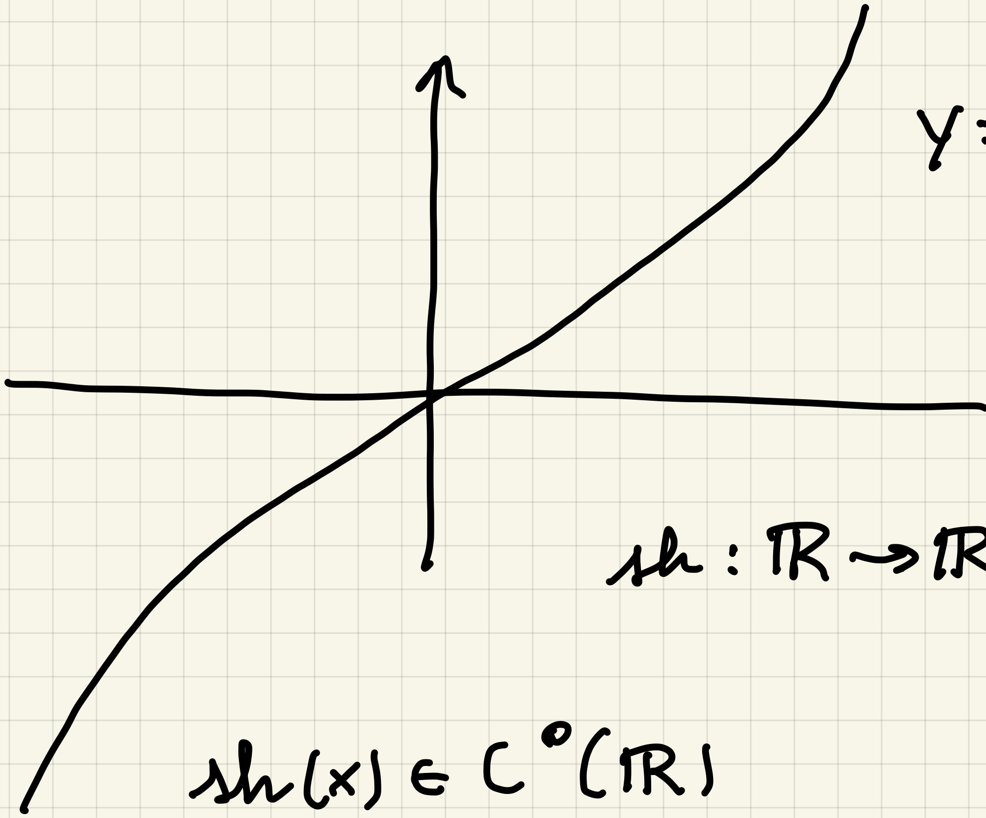
$$\left(th(x)\right)' = \frac{1}{ch^2(x)} = \frac{ch^2(x) - sh^2(x)}{ch^2(x)} =$$

$$= 1 - th^2(x)$$

$$\left(ch(x)\right)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{1}{2}\left((e^x)' + (e^{-x})'\right) =$$

$$= \frac{1}{2} e^x + \frac{1}{2} e^{-x} (-x)' =$$

$$= \frac{1}{2} e^x - \frac{1}{2} e^{-x} = sh(x)$$



$sh : \mathbb{R} \to \mathbb{R}$ biettiva.

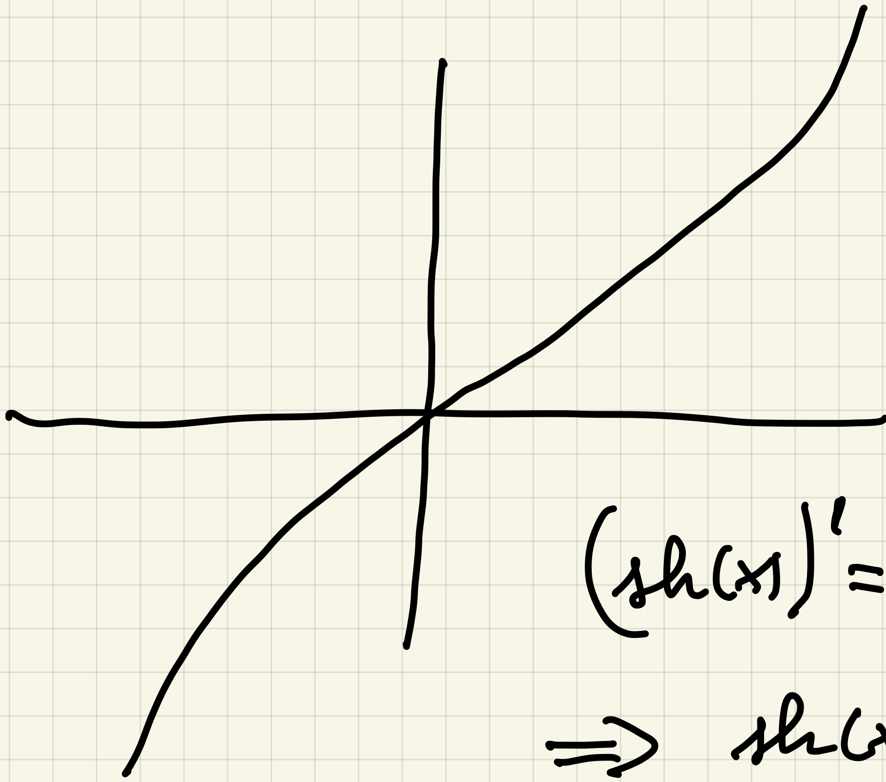
$$sh(x) \in C^0(\mathbb{R})$$

$$\Rightarrow \underline{sh(\mathbb{R})}$$

$$\lim_{x \to +\infty} sh(x) = +\infty \Rightarrow \sup sh(\mathbb{R}) = +\infty$$

$$\lim_{x \to -\infty} sh(x) = -\infty \Rightarrow \inf sh(\mathbb{R}) = -\infty$$

$$\Rightarrow sh(\mathbb{R}) = (-\infty, +\infty)$$



$y = sh(x)$

$(sh(x))' = ch(x) \geq 1$

$\Longrightarrow$ $sh(x)$ e' strett. crescente

$$y = \frac{e^x - e^{-x}}{2} \qquad \cdot 2$$

$$2y = e^x - e^{-x} \qquad \cdot e^x$$

$$2e^x y = e^{2x} - 1$$

$$e^{2x} - 2ye^x - 1 = 0$$

$$(e^x)^2 - 2ye^x - 1 = 0$$

$$(e^x)^2 - 2y\,e^x - 1 = 0$$

$$(e^x)_\pm = y \pm \sqrt{y^2+1}$$

$$e^x \begin{cases} y + \sqrt{y^2+1} & ? \\ \cancel{y - \sqrt{y^2+1}} < 0 \end{cases}$$

$$e^x = y + \sqrt{y^2+1}$$

$$x = \lg\left(y + \sqrt{y^2+1}\right)$$

$$\left(\lg\left(x + \sqrt{x^2+1}\right)\right)' =$$

$$= \lg'\left(x + \sqrt{x^2+1}\right) \quad \left(x + \sqrt{x^2+1}\right)'$$

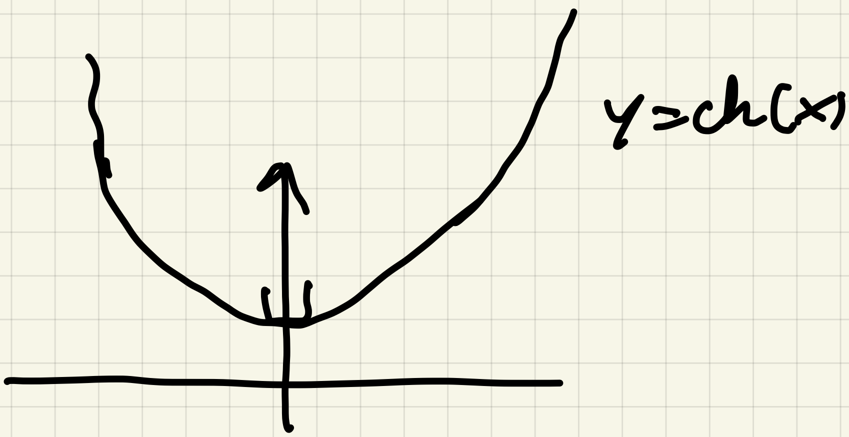
$$= \frac{1}{x + \sqrt{x^2+1}} \quad \left(1 + \frac{1}{\cancel{2}}\left(x^2+1\right)^{-\frac{1}{2}}\ \cancel{2}x\right)$$

$$= \frac{1}{x + \sqrt{x^2+1}} \quad \left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}} \left(1 + \frac{x}{\sqrt{x^2+1}}\right) =$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+1}}} \cdot \frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}} \leq 1$$

$\forall x$.



$ch : \mathbb{R} \to [1, +\infty)$

$ch : [0, +\infty) \to [1, +\infty)$ e' biettiva

$ch(0) = 1$

$\lim_{x \to +\infty} ch(x) = +\infty \Rightarrow ch([0,+\infty)) = [1,+\infty)$

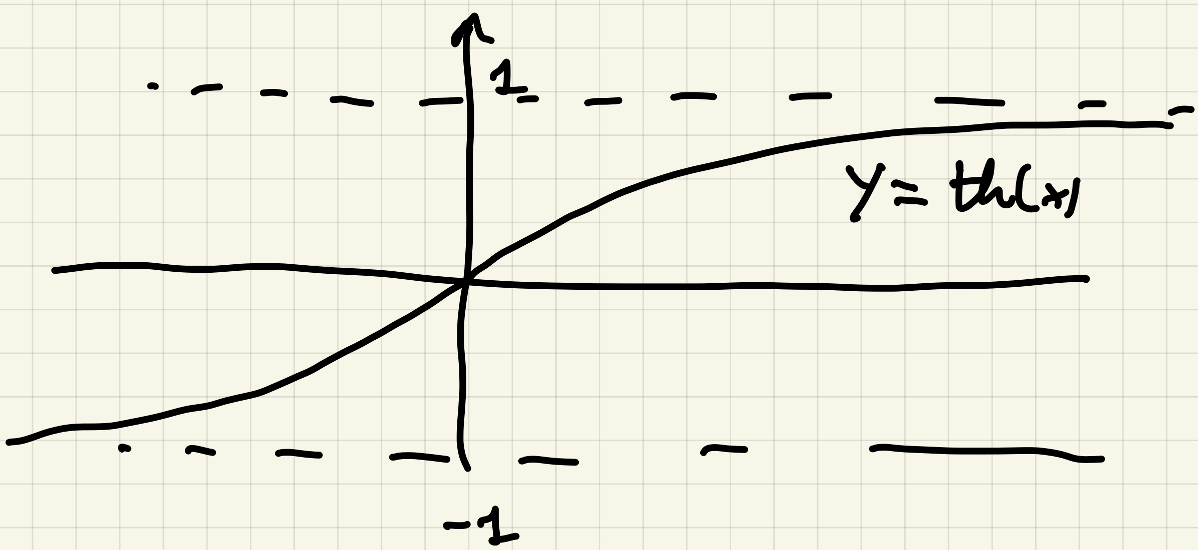
$ch(x) \geq 1$

$ch'(x) = sh(x) \geq 0$ e

$ch'(x) > 0$ per $x > 0$.

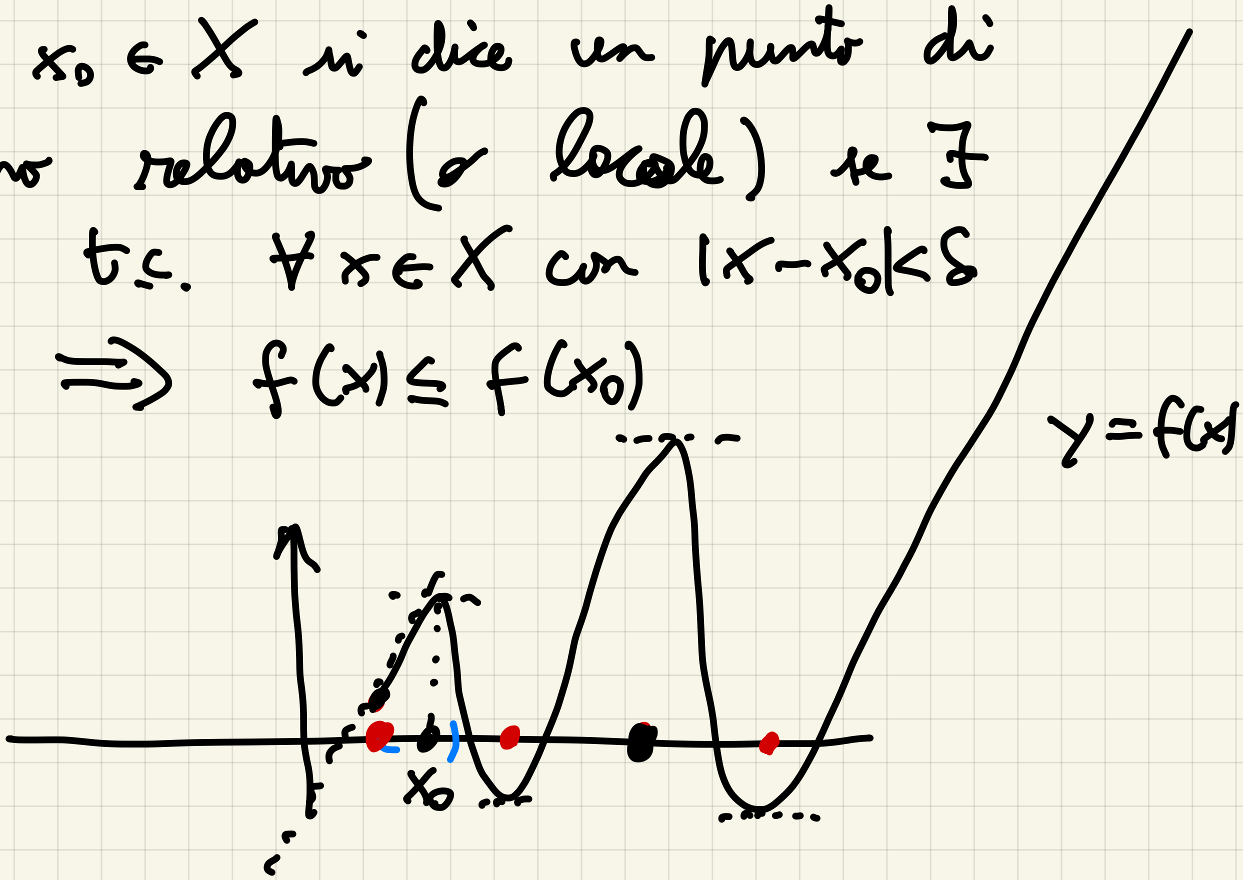
$ch : [0,+\infty) \to [1,+\infty)$ e' biettiva.

$th: \mathbb{R} \to (-1, 1)$



Def Sia $f: X \to \mathbb{R}$ $X \subseteq \mathbb{R}$. Un punto $x_0 \in X$ si dice un punto di massimo relativo (o locale) se $\exists$ $\delta > 0$ t.c. $\forall x \in X$ con $|x - x_0| < \delta$

$$\implies f(x) \leq f(x_0)$$

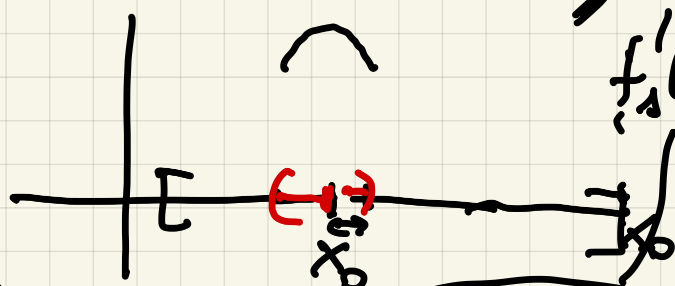


Def Un punto dove $f'(x_0)=0$ si dice un punto critico di f.

Teor (Fermat) Sia $f \in C^0(I)$, I un intervallo, e sia x_0 un punto interno di I dove esiste $f'(x_0)$. Allora se x_0 è un estremo locale allora x_0 è un punto critico, cioè

$$f'(x_0)=0.$$

Dim Sia qui x_0 un punto di max locale



$$f'(x_0) = \overbrace{\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}}^{f'_d(x_0)} = \overbrace{\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}}^{f'_s(x_0)}$$

Esiste $\delta>0$ t.c. $|x-x_0|<\delta \Rightarrow f(x)\le f(x_0)$

Per il limite sinistro è sufficiente considerare

$x_0-\delta < x < x_0$ e $\dfrac{f(x)-f(x_0)}{x-x_0} \ge 0 \Rightarrow f'_s(x_0) \ge 0$

Esiste $\delta > 0$ t.c. $|x - x_0| < \delta \Rightarrow f(x) \leq f(x_0)$

Per il limite sinistro è sufficiente considerare

$x_0 - \delta < x < x_0$ e $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0 \Rightarrow f'_s(x_0) \geq 0$

Per il limite destro è suff. considerare

$x_0 < x < x_0 + \delta$ e $\dfrac{f(x) - f(x_0)}{x - x_0} \leq 0 \Rightarrow f'_d(x_0) \leq 0$.

Per ipotesi $f'(x_0) = f'_d(x_0) = f'_s(x_0)$

$$\Rightarrow \quad f'(x_0) = 0$$

$\square$

$(x^3)'\big|_{x=0} = 3x^2\big|_{x=0} = 0$

$f(x) = x^3$ su $|x| \leq 1$