Contributo di Δ_{gauge}

$$\Delta^{\mu\nu}_{gauge} = -D^2\delta^{\mu\nu} - 2i\,F^{a\mu\nu}t^a_{Ad}$$
$$= \underbrace{\Delta_{gh}}_{\text{op. come nei ghost}}\delta^{\mu\nu} - 2i\,F^{a\mu\nu}t^a_{Ad}$$

ma ora otteniamo $\delta^{\mu\nu}$ → quando tracciamo sugli indici di Lorentz, otteniamo un fattore 4.

$$\mathrm{Tr}\log\Delta_{gauge} = 4\,\mathrm{Tr}\log\Delta_{gh} + F^{\mu\nu}\text{-terms}$$

dove
$$F^{\mu\nu}\text{-terms} = \underbrace{-\frac{1}{2}(-2i)^2}_{+2}\mathrm{Tr}\left((-\partial^2)^{-1}F^{a\mu\nu}t^a_{Ad}(-\partial^2)^{-1}F^b_{\nu\mu}t^b_{Ad}\right)$$

$\int d^dx\,|x\rangle\langle x|$

$$= -2\int d^dy\,\langle y|(-\partial^2)^{-1}F^{\mu\nu a}(-\partial^2)^{-1}F^b_{\mu\nu}(-\partial^2)^{-1}|y\rangle\,\mathrm{tr}(t^a_{Ad}t^b_{Ad})$$

$$= -2\int d^dy\,d^dx\,\underline{\langle y|(-\partial^2)^{-1}|x\rangle}\,\underline{\langle x|(-\partial^2)^{-1}|y\rangle}\,F^{\mu\nu a}(x)\,F^b_{\mu\nu}(y)\,\mathrm{tr}(t^a_{Ad}t^b_{Ad})$$

$$= -2\int d^dy\,d^dx\,\frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{1}{k^2}\frac{1}{k'^2}F^{\mu\nu a}(x)\,F^b_{\mu\nu}(y)\,\mathrm{tr}(t^at^b)\,e^{ik(y-x)}e^{ik'(x-y)}$$

trasf. Fourier ↘

$$= -2\int d^dy\,d^dx\,\frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{1}{k^2}\frac{1}{k'^2}\mathrm{tr}(t^at^b)\,e^{-ipx}e^{-iqy}e^{i(k-k')y}e^{i(k'-k)x}\cdot$$
$$\cdot(-i)^2\left(p^\rho\tilde{A}^{a\sigma}(p) - p^\sigma\tilde{A}^{a\rho}(p)\right)\left(q_\rho\tilde{A}^b_\sigma(q) - q_\sigma\tilde{A}^b_\rho(q)\right)$$

$\delta(k'-k-p)$ $\delta(k-k'-q)$ → $q=-p$, $k'=k+p$

$$= -\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\tilde{A}^a_\mu(p)\,\tilde{A}^b_\nu(-p)\,\mathrm{tr}(t^at^b)\cdot\int\frac{d^dk}{(2\pi)^d}\frac{4(p^\rho\delta^{\mu\sigma}-p^\sigma\delta^{\mu\rho})(p_\rho\delta^\nu_\sigma - p_\sigma\delta^\nu_\rho)}{k^2(k+p)^2}$$

$$= 4\int \frac{d^dp}{(2\pi)^d}\, \tilde{A}^a_\mu(p)\, \tilde{A}^b_\nu(-p)\, \mathrm{tr}(t^a t^b)\,(p^\mu p^\nu - p^2\delta^{\mu\nu}) \int_0^1 d\xi\, \frac{\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-\frac{d}{2}}(4\pi)^{d/2}}$$

$$\underset{\text{div.}}{\sim} \quad \frac{4\, C(\mathrm{Adj})\,\delta^{ab}}{(4\pi)^2}\, \frac{1}{2-\omega} \int \frac{d^dp}{(2\pi)^d}\, \hat{A}^a_\mu(p)\, \hat{A}^b_\nu(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$\Downarrow$$

$$\frac{1}{2}\left(4\,\mathrm{Tr}\log\Delta_{gh} + F^{\mu\nu}\text{-terms}\right)$$

$$S_{eff}(A) = \frac{1}{2g^2}\int d^4x\, \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \underbrace{\overbrace{\frac{1}{2}\mathrm{Tr}\log\Delta_{gauge}}^{=} - \mathrm{Tr}\log\Delta_{gh}}_{\frac{C(\mathrm{Adj})}{(4\pi)^2}\left[\underbrace{\frac{1}{2}\cdot 4\left(-\frac{1}{6}\right) + \frac{1}{2}\cdot 4 + \frac{1}{6}}_{-\frac{2}{6}+\frac{1}{6}+2 = \frac{11}{6}}\right]\frac{1}{2-\omega}\int\frac{d^4p}{(2\pi)^4}\tilde{A}^a_\mu(p)\tilde{A}^a_\nu(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})}$$

$$\frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}F^{a\,\mu\nu}$$

$$- \frac{1}{2g^2_{bare}}\int \frac{d^4p}{(2\pi)^4}\, \tilde{A}^a_\mu(p)\,\tilde{A}^a_\nu(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

RINORMALIZZAZIONE

$$\Downarrow$$

$$-\frac{\mu^{4-2\omega}}{2g^2_{bare}} + \frac{1}{2}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{11}{3}\,\frac{1}{2-\omega} = -\frac{1}{2g_r^2(\mu)}$$

$$\frac{(\mu^{2-\omega})^2}{g^2_{bare}} = \frac{1}{g_r^2} + \frac{11}{3}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega} = \frac{1}{g_r^2}\left(1 + \frac{11}{3}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega}\,g_r^2\right)$$

$$\rightarrow \quad g_{bare} = g_r(\mu)\,\mu^{2-\omega}\left(1 - \frac{11}{6}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega}\,g_r^2(\mu)\right)$$

$$\beta(g) = -\frac{11}{3}\,\frac{C_2(G)}{16\pi^2}\,g^3$$

β-funct di YM

Contributo fermioni

Se abbiamo un fermione in rep. R di G, abbiamo

$$e^{-S_{eff}(A)} = \int e^{-S(A,c,\bar{c},a)} \mathcal{D}a\mathcal{D}c\mathcal{D}\bar{c}\; \underbrace{e^{-S_f(\psi,A,a)}\mathcal{D}\psi\mathcal{D}\bar\psi}_{\det i\not{D}}$$

$$\not{D} = \gamma^\mu D_\mu$$

$\to$ contributo a S_{eff} pari a

$$-\log\det(i\not{D}) \qquad \left(\det M = \sqrt{\det(M^2)}\right)$$

$$\det(i\not{D}) = \det{}^{1/2}\left(-\gamma^\mu\gamma^\nu D_\mu D_\nu\right) =$$

$$= \det{}^{1/2}\Big(-\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2\delta^{\mu\nu}\mathbb{1}_4} D_\mu D_\nu - \frac{1}{2}[\gamma^\mu,\gamma^\nu]\overbrace{D_\mu D_\nu}^{\frac{1}{2}[D_\mu,D_\nu] = -\frac{i}{2}F_{\mu\nu}}\Big) =$$

$$= \det{}^{1/2}\left(-D^2\mathbb{1}_4 + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)$$

anche su indici spinoriali

$$\Rightarrow -\log\det(i\not{D}) = -\frac{1}{2}\mathrm{Tr}\log\left(-D^2\mathbb{1}_4 + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)$$

$$= -2\,\mathrm{Tr}\log(-D^2) + [\gamma^\mu,\gamma^\nu]F_{\mu\nu}\text{-terms}$$

$$\underset{div}{\sim} +\underbrace{\frac{1}{2}\left(\frac{2}{3}-4\right)}_{=-2/3}\frac{C(R)}{(4\pi)^2}\int\frac{d^4p}{(2\pi)^4}\tilde{A}^a_\mu(p)\tilde{A}^b_\nu(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})\frac{1}{2-\omega}$$

$$\beta(g) = -\frac{1}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}C(R)\right)g^3$$

RG flow

$$\beta(g) = \mu \frac{dg}{d\mu} \quad \leftarrow g_r(\mu)$$

$$\beta(g) = \underset{\text{1-loop}}{\beta_0 g^3} + \underset{\text{higher loops}}{\dots}$$

(ordini superiori in g)

Valida in regime perturbativo ($g(\mu) \ll 1$)

$\beta_0 = -\frac{11}{3} \frac{C_2(G)}{(4\pi)^2}$ in YM

Integriamo

$$\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{\beta(g)} = \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} = \ln(\mu/\mu_0)$$

in regime perturbativo

$$\frac{1}{\beta_0} \int_{g(\mu_0)}^{g(\mu)} \frac{dg}{g^3} = \frac{1}{\beta_0} \left(-\frac{1}{2g^2} \right|_{g(\mu_0)}^{g(\mu)} = \frac{1}{2\beta_0 g^2(\mu_0)} - \frac{1}{2\beta_0 g^2(\mu)}$$

$$\Downarrow$$

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \beta_0 \ln\left(\mu_0^2/\mu^2\right)$$

$$\hookrightarrow \quad g^2(\mu) = \frac{g^2(\mu_0)}{1 + \beta_0 g^2(\mu_0) \ln\left(\mu_0^2/\mu^2\right)}$$

$$\text{YM}: \quad \beta_0 = -\frac{11}{3} \frac{C_2(G)}{(4\pi)^2} < 0 \quad \Rightarrow$$

$\Rightarrow$ all'aumentare della scala μ, $g(\mu)$ diminuisce $\rightarrow$ ASYMPTOTIC FREEDOM in UV

$\Rightarrow$ diminuendo la scala μ,

$g(\mu)$ aumenta e ad un certo punto diventa >1 $\rightarrow$ Regime di ACCOPPIAM. FORTE (strong coupling) a basse energie

Dimensional transmutation

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \beta_0 \ln\left(\mu_0^2/\mu^2\right)$$

$$\Downarrow$$

$$\frac{\mu}{\mu_0} = e^{\frac{1}{2\beta_0 g^2(\mu_0)}} \cdot e^{-\frac{1}{2\beta_0 g^2(\mu)}}$$

$$\Downarrow$$

$$\mu \, e^{\frac{1}{2\beta_0 g^2(\mu)}} = \mu_0 \, e^{\frac{1}{2\beta_0 g^2(\mu_0)}}$$

$\leftarrow$ dimensioni di una scala di ENERGIA

$$\Rightarrow \quad \Lambda(\mu) \equiv \mu \, e^{\frac{1}{2\beta_0 g^2(\mu)}}$$

è INDIPENDENTE dalla scala μ

$$\parallel$$

$$\Lambda_{QCD}$$

Λ_{QCD} è una SCALA della TEORIA "generata" dalla QFT.

Λ_{QCD} è detta SCALA DI QCD o "RG-invariant scale"

Λ_{QCD} è la scala $\bar{\mu}$ a cui $g(\bar{\mu}) \to \infty$
(perchè in questo caso $\mu = \Lambda_{QCD}$)

Λ_{QCD} è la scala sotto la quale la teoria diventa fortemente accoppiata

$\hookrightarrow$ La teoria di YM classica non ha parametri di dimensione massa $\neq 0$.
D'altra parte la QFT ha generato una scala fisica Λ_{QCD}
$\rightsquigarrow$ "DIMENSIONAL TRANSMUTATION"

Domanda: come appare la teoria di YM a scale $\ll \Lambda_{QCD}$?

Risposta parziale ottenuta da analisi di esperimenti e calcoli numerici sul reticolo:
- YM NON descrive particelle MASSLESS.
- invece, i gluoni sono in stati composti detti GLUEBALLS che sono particelle MASSIVE $m \sim \Lambda_{QCD}$

si dice che la Teoria ha un MASS GAP
(c'è un gap tra lo stato di vuoto e il 1° livello eccitato)

[Il MASS GAP non è dimostrato analiticam. in YM]